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ON THE ZERO-ONE-POLE SET OF A MEROMORPHIC FUNCTION

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Introduction.

Let $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ be three disjoint sequences with no finite limit points. If it is possible to construct a meromorphic function f in the plane C whose zeros, one points and poles are exactly $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively, where their multiplicities are taken into consideration, then the given triad $(\{a_n\}, \{b_n\}, \{p_n\})$ is called a zero-one-pole set. In general an arbitrary triad $(\{a_n\}, \{b_n\}, \{p_n\})$ is not a zero-one-pole set. Further if there exists only one meromorphic function f whose zero-one-pole set is just the given trias, then the triad is called unique. It is well known that unicity in this sense does not hold in general.

Our first result of this note is the following

THEOREM 1. *Suppose that* $(\{a_n\}, \{b_n\}, \{p_n\})$ $(\{a_n\}\cup\{b_n\}\cup\{p_n\}\neq\emptyset)$ *is a zero-one-pole set which is not unique. Let* $\{c_n\}$, $\{d_n\}$ *and* $\{q_n\}$ *be the subsequences of* $\{a_n\}$, $\{b_n\}$ *and* $\{p_n\}$ *respectively such that* $\{c_n\}\cup\{d_n\}\cup\{q_n\}\neq\emptyset$ *and such that* $\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}+\sum_{q_n\neq 0}|q_n|^{-1}<+\infty$. *Then* $(\{a_n\}\setminus\{c_n\}, \{b_n\}\setminus\{d_n\}, \{p_n\}\setminus\{q_n\})$ *is not a zero-one-pole set of any nonconstant meromorphic function.*

Ozawa has proved this result for $\{p_n\}=\{q_n\}=\{c_n\}=\emptyset$ and $1\leqq\#\{d_n\}<+\infty$. See the second supplement in [3, p. 315]. The assumption $\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}+\sum_{q_n\neq 0}|q_n|^{-1}<+\infty$ cannot be omitted. For example, let us consider

$$N(z)=\frac{e^z-1}{e^z-c}, \qquad c\neq 0, 1$$

and

$$g(z)=\frac{e^z}{e^z-c}.$$

Clearly $N(z)\neq 1$, c^{-1}, and so the zero-one-pole set $(\{a_n\}, \phi, \{p_n\})$ $(\{a_n\}\neq\emptyset)$ of N is not unique. On the other hand, the zero-one-pole set of g is $(\phi, \phi, \{p_n\})$ and $\sum_{a_n\neq 0}|a_n|^{-1}=(2\pi)^{-1}\sum_{k\neq 0}|k|^{-1}=+\infty$.

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Theorem 1 does not hold in general in the case that $(\{a_n\}, \{b_n\}, \{p_n\})$ is unique. This is shown by the following example: Let $P(z)$ and $Q(z)$ be two nonconstant entire functions of order <1 with no common zeros, and consider $g(z)=P(z)/Q(z)$ and $N(z)=\{g(z)\}^2$. For further examples, see Ozawa [3, pp. 313, 314]. But there exist unique zero-one-pole sets for which a similar result as Theorem 1 holds.

Our second result is the following

THEOREM 2. *Suppose that $f(z)$ is meromorphic of finite nonintegral order $\rho>1$. Let $(\{a_n\}, \{b_n\}, \{p_n\})$ be the zero-one-pole set of $f(z)$ and let $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ be the subsequences of $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively such that $\{c_n\}\cup\{d_n\}\cup\{q_n\}\neq\emptyset$ and such that $\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}+\sum_{q_n\neq 0}|q_n|^{-1}<+\infty$. Then $(\{a_n\}\backslash\{c_n\}, \{b_n\}\backslash\{d_n\}, \{p_n\}\backslash\{q_n\})$ is not a zero-one-pole set.*

It is well known that the zero-one-pole set of f in Theorem 2 is unique. See [4]. Further, we prove

THEOREM 3. *Suppose that $f(z)$ is meromorphic, and satisfies $\limsup_{r\to\infty} T(r, f)/r>0$ and $\liminf_{r\to\infty} \bar{N}_1(r, 0, f)/T(r, f)>0$. Let $(\{a_n\}, \{b_n\}, \{p_n\})$ be the zero-one-pole set of $f(z)$ and let $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ be the subsequences of $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively such that $\{c_n\}\cup\{d_n\}\cup\{q_n\}\neq\emptyset$ and such that $\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}+\sum_{q_n\neq 0}|q_n|^{-1}<+\infty$. Then $(\{a_n\}\backslash\{c_n\}, \{b_n\}\backslash\{d_n\}, \{p_n\}\backslash\{q_n\})$ is not a zero-one-pole set of any nonconstant meromorphic function.*

We remark that the zero-one-pole set of $f(z)$ in Theorem 3 is unique. The proof is substantially contained in Theorem 1′ of [2, p. 414]. The assumption $\limsup_{r\to\infty} T(r, f)/r>0$ is necessary. If $f=S^{2m}$ and $g=S^m$, where m is a positive integer and $S=\prod_{\nu=1}^{\infty}(1-z/a_\nu)$ with $a_\nu=-[\nu]^{1/\rho}$ $(1/2<\rho<1)$, then $\lim_{r\to\infty} m(r, f)/r=0$, $\lim_{r\to\infty} \bar{N}_1(r, 0, f)/m(r, f)=(\sin\pi\rho)/2m>0$ and

$$\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}=\int_0^\infty \frac{N(r, 0, S^m)+N(r, -1, S^m)}{r^2}\,dr$$

$$=m\int_0^\infty \frac{(1+\sin\pi\rho)r^\rho(1+\varepsilon(r))}{(\rho\sin\pi\rho)r^2}\,dr<+\infty$$

($\varepsilon(r)\to 0$ as $r\to\infty$). (Cf. [5, pp. 18, 19].) Also the assumption $\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}+\sum_{q_n\neq 0}|q_n|^{-1}<+\infty$ cannot be omitted. For example, we set $g=(\cos z-1)/2$ and $f=g^2$. In this case, $\lim_{r\to\infty} m(r, f)/r=4\pi^{-1}$, $\lim_{r\to\infty} \bar{N}_1(r, 0, f)/m(r, f)=1/4$ and $\sum|d_n|^{-1}=2\pi^{-1}\sum|2k+1|^{-1}=+\infty$.

Our final result of this note is

THEOREM 4. *Let $f(z)$ be a meromorphic function whose zero-one-pole set is $(\{a_n\}, \{b_n\}, \{p_n\})$, where neither $\{a_n\}$ nor $\{p_n\}$ is empty. Let $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ be the subsequences of $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively such that $\{c_n\}\cup\{d_n\}\cup\{q_n\} \neq \emptyset$, $\{c_n\}\neq\{a_n\}$ and $\{q_n\}\neq\{p_n\}$. Further suppose that*

$$(*) \qquad \limsup_{r\to\infty} \frac{\bar{N}(r, 0, f)+N(r, \infty, f)+\bar{N}(r, \{c_n\}\cup\{d_n\}\cup\{q_n\})}{T(r, f)} < 1/2$$

holds. Then $(\{a_n\}\backslash\{c_n\}, \{b_n\}\backslash\{d_n\}, \{p_n\}\backslash\{q_n\})$ is not a zero-one-pole set.

The unicity of zero-one-pole set of f in Theorem 4 has been proved in Theorem 2 of [7]. The assumption $(*)$ is necessary. Let $P(z)$ and $Q(z)$ be the canonical products with no common zeros, and let $L(z)$ be a transcendental entire function. If we set $g=(P/Q)e^L$ and $f=g^2$, then

$$T(r, f)=2T(r, g)\sim 2m(r, e^L) \qquad (r\to\infty),$$

$$\bar{N}(r, 0, f)=\bar{N}(r, 0, P)=o(m(r, e^L)) \qquad (r\to\infty),$$

$$N(r, \infty, f)=2N(r, 0, Q)=o(m(r, e^L)) \qquad (r\to\infty),$$

$$\begin{aligned}\bar{N}(r, \{c_n\}\cup\{d_n\}\cup\{q_n\})&=\bar{N}(r, 0, P)+\bar{N}(r, 0, Q)+\bar{N}(r, -1, (P/Q)e^L)\\ &=(1+o(1))m(r, e^L) \qquad (r\notin E,\ r\to\infty),\end{aligned}$$

and so that

$$\frac{\bar{N}(r, 0, f)+N(r, \infty, f)+\bar{N}(r, \{c_n\}\cup\{d_n\}\cup\{q_n\})}{T(r, f)}\to 1/2 \quad (r\notin E,\ r\to\infty).$$

(Throughout this note, the letter E will denote sets of finite linear measure which will not necessarily be the same at each occurrence.)

1. Proof of Theorem 1.

Let N and f be two distinct nonconstant meromorphic functions whose zero-one-pole sets are $(\{a_n\}, \{b_n\}, \{p_n\})$, and suppose that $(\{a_n\}\backslash\{c_n\}, \{b_n\}\backslash\{d_n\}, \{p_n\}\backslash\{q_n\})$ is the zero-one-pole set of a nonconstant meromorphic function g. If P, R and Q are the entire functions of genus 0 whose zeros are $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ respectively (where for example, we put $P\equiv 1$ if $\{c_n\}=\emptyset$), then there exist four entire functions α, β, γ and δ such that

$$(1.1) \qquad f=Ne^{\alpha},$$

$$(1.2) \qquad f-1=(N-1)e^{\beta},$$

$$(1.3) \qquad gP/Q=Ne^{\gamma},$$

$$(1.4) \qquad (g-1)(R/Q)=(N-1)e^{\delta}.$$

Eliminating f from (1.1) and (1.2), we have

$$(e^{\alpha}-e^{\beta})N=1-e^{\beta}. \tag{1.5}$$

Similarly, from (1.3) and (1.4) we obtain

$$((R/P)e^{\gamma}-e^{\delta})N=R/Q-e^{\delta}. \tag{1.6}$$

Combining (1.5) and (1.6), and noting that $N\not\equiv 0$, we get

$$PR(e^{\alpha}-e^{\beta})=QPe^{\delta}(e^{\alpha}-1)+QRe^{\gamma}(1-e^{\beta}). \tag{1.7}$$

Since $N\not\equiv f$, each of e^{α}, e^{β} and $e^{\alpha-\beta}$ is not identically equal to one.

In what follows we frequently use the following form of the impossibility of Borel's identity.

Let $P_0, P_1, \cdots, P_n$ $(P_j\not\equiv 0,\ 0\leqq j\leqq n,\ n\geqq 1)$ *be entire functions satisfying* $m(r, P_j)=o(r)$ $(r\to\infty)$, *and let* $g_1, g_2, \cdots, g_n$ *be nonconstant entire functions. Then an identity of the following form*

$$\sum_{i=1}^{n} P_i e^{g_i}=P_0$$

is impossible.

This is an easy consequence of Lemma 1 in [1, p. 283].

(I) Assume first that $e^{\alpha}\equiv c(\neq 1)$. In this case (1.5) implies that e^{β} is not a constant, and from (1.7) we deduce that $R\equiv 1$ and that

$$(c-1)PQe^{\delta}+Qe^{\gamma}-Qe^{\beta+\gamma}+Pe^{\beta}=cP. \tag{1.8}$$

Since $P(\not\equiv 0)$ and $Q(\not\equiv 0)$ are entire functions of genus 0, as we have stated above, at least one of e^{δ}, e^{γ} and $e^{\beta+\gamma}$ must be a constant, so we discuss the following three cases: (I_1) e^{δ}, (I_2) e^{γ}, or (I_3) $e^{\beta+\gamma}$ is a constant x.

For Case (I_1), (1.8) gives

$$Qe^{\gamma}-Qe^{\beta+\gamma}+Pe^{\beta}=\{c-x(c-1)Q\}P. \tag{1.9}$$

If $c-x(c-1)Q\equiv 0$, then $Q\equiv 1$ and $e^{\gamma}-e^{-\beta+\gamma}=P\not\equiv$const., a contradiction. Hence $c-x(c-1)Q\not\equiv 0$, which implies that at least one of e^{γ} and $e^{\beta+\gamma}$ is a constant y. First suppose that $e^{\gamma}\equiv y$. Then (1.9) becomes

$$(P-yQ)e^{\beta}=\{c-x(c-1)Q\}P-yQ. \tag{1.10}$$

Clearly $P-yQ\not\equiv 0$ and $\{c-x(c-1)Q\}P-yQ\not\equiv 0$, which combined with the fact of the nonconstancy of e^{β} imply that (1.10) is impossible. Thus we conclude that e^{γ} is not a constant in the present case (I_1). Next suppose that $e^{\beta+\gamma}\equiv y$. Then (1.9) becomes

$$Qe^{\gamma}+yPe^{-\gamma}=\{c-x(c-1)Q\}P+yQ\ (\not\equiv 0).$$

Since e^{γ} is not a constant, this is untenable. Therefore Case (I $_1$) is impossible in (I).

For Case (I $_2$), (1.8) gives

$$(c-1)PQe^{\delta}+(P-xQ)e^{\beta}=cP-xQ\ (\not\equiv 0). \tag{1.11}$$

Here neither e^{δ} nor e^{β} is a constant, thus (1.11) is absurd.

For Case (I $_3$), (1.8) yields

$$(c-1)PQe^{\delta}+Qe^{\gamma}+xPe^{-\gamma}=cP+xQ\ (\not\equiv 0),$$

which is also impossible since we may assume that neither e^{δ} nor e^{γ} is a constant in the present case.

These observations lead us to conclude that e^{α} is not a constant.

(II) Assume next that $e^{\beta}\equiv c(\neq 1)$. From (1.7) it follows that $P\equiv 1$ and that

$$Re^{\alpha}-Qe^{\alpha+\delta}+Qe^{\delta}+(c-1)QRe^{\gamma}=cR, \tag{1.12}$$

and thus at least one of $e^{\alpha+\delta}$, e^{δ} and e^{γ} is a constant.

(II$_1$) $e^{\alpha+\delta}$ is a constant x. By (1.12)

$$Re^{\alpha}+xQe^{-\alpha}+(c-1)QRe^{\gamma}=cR+xQ\ (\not\equiv 0), \tag{1.13}$$

so that e^{γ} must be a constant y. Then (1.13) becomes

$$Re^{\alpha}+xQe^{-\alpha}=cR+xQ+y(1-c)QR\ (\not\equiv 0),$$

which is impossible. Thus we conclude that $e^{\alpha+\delta}$ is not a constant.

(II$_2$) e^{δ} is a constant x. By (1.12)

$$(R-xQ)e^{\alpha}+(c-1)QRe^{\gamma}=cR-xQ\ (\not\equiv 0), \tag{1.14}$$

so that e^{γ} must be a constant y. Then (1.14) becomes

$$(R-xQ)e^{\alpha}=cR-xQ+y(1-c)QR\ (\not\equiv 0),$$

which is absurd. Thus also e^{δ} is not a constant.

(II$_3$) e^{γ} is a constant x. By (1.12)

$$Re^{\alpha}-Qe^{\alpha+\delta}+Qe^{\delta}=\{c-x(c-1)Q\}R. \tag{1.15}$$

Taking into account the fact that all of e^{α}, $e^{\alpha+\delta}$ and e^{δ} are not constants, (1.15) implies $c-x(c-1)Q\equiv 0$. Hence $Q\equiv 1$ and $e^{\alpha}-Re^{\alpha-\delta}=1$. This is impossible.

Thus we conclude that also e^{β} is not a constant.

(III) Assume thirdly that $e^{\gamma}\equiv c$. In this case we have by (1.7)

$$PRe^{\alpha}+(cQ-P)Re^{\beta}+QPe^{\delta}-QPe^{\delta+\alpha}=cQR\,, \tag{1.16}$$

so that either e^{δ} or $e^{\delta+\alpha}$ is a constant.

(III$_1$) e^{δ} is a constant x. By (1.16)

$$P(R-xQ)e^{\alpha}+(cQ-P)Re^{\beta}=(cR-xP)Q\,, \tag{1.17}$$

which implies $cR-xP\equiv 0$. Hence $R\equiv P\equiv 1$ and $c=x$. From these and (1.17) it follows that $e^{\alpha-\beta}\equiv 1$, a contradiction. Thus e^{δ} is not a constant.

(III$_2$) $e^{\delta+\alpha}$ is a constant x. By (1.16)

$$PRe^{\alpha}+(cQ-P)Re^{\beta}+xQPe^{-\alpha}=(cR+xP)Q\,, \tag{1.18}$$

which yields $cR+xP\equiv 0$. Hence $R\equiv P\equiv 1$ and $c+x=0$. Substituting these into (1.18) we have $e^{2\alpha}+(cQ-1)e^{\beta+\alpha}=cQ$. This is untenable.

Thus we are led to the conclusion that e^{γ} is not a constant.

(IV) Assume fourthly that $e^{\delta}\equiv c$. In this case (1.7) becomes

$$P(R-cQ)e^{\alpha}-PRe^{\beta}-QRe^{\gamma}+QRe^{\gamma+\beta}=-cQP, \tag{1.19}$$

which implies that $e^{\gamma+\beta}$ is a constant x. From this and (1.19) it follows that $P(R-cQ)e^{\alpha}-PRe^{\beta}-QRe^{\gamma}=-(cP+xR)Q$, so that $cP+xR\equiv 0$ i.e. $P\equiv R\equiv 1$ and $c+x=0$. Hence $(1-cQ)e^{\alpha-\beta}-Qe^{\gamma-\beta}=1$. This leads us to conclude that $e^{\alpha-\beta}\equiv 1$, which is impossible. Thus we see that e^{δ} is not a constant.

(V) Assume fifthly that $e^{\alpha-\beta}\equiv c(\neq 1)$. By (1.7)

$$QRe^{\gamma}-QRe^{\gamma-\beta}+QPe^{\delta-\beta}-cQPe^{\delta}=(1-c)PR\,, \tag{1.20}$$

so that (V$_1$) $e^{\gamma-\beta}$ or (V$_2$) $e^{\delta-\beta}$ is a constant x.

For Case (V$_1$), (1.20) gives

$$xQRe^{\beta}+QPe^{\delta-\beta}-cQPe^{\delta}=\{(1-c)P+xQ\}R\,. \tag{1.21}$$

If $(1-c)P+xQ\equiv 0$, then $P\equiv Q\equiv 1$ and $1-c+x=0$, so that (1.21) yields $(c-1)Re^{\beta-\delta}+e^{-\beta}=c$. This is impossible. If $(1-c)P+xQ\not\equiv 0$, then we deduce from (1.21) that $e^{\delta-\beta}$ is a constant y and that $(xR-cyP)Qe^{\beta}=\{(1-c)P+xQ\}R-yQP$. This identity yields $c=1$, a contradiction.

For Case (V$_2$), (1.20) becomes

$$QRe^{\gamma}-QRe^{\gamma-\beta}-cQPe^{\delta}=\{(1-c)R-xQ\}P. \tag{1.22}$$

It is easily seen that $Q\equiv 1$, and thus (1.22) yields

$$Re^{\gamma}-Re^{\gamma-\beta}-cPe^{\delta}=\{(1-c)R-x\}P, \tag{1.23}$$

which implies $(1-c)R-x\equiv 0$. Hence $R\equiv 1$ and $1-c=x$. From these and (1.23) it follows that $cPe^{\delta-\gamma}+e^{-\beta}=1$. This is impossible.

Thus we conclude that $e^{\alpha-\beta}$ is not a constant.

(VI) Assume sixthly that $e^{\gamma-\beta}\equiv c$. In this case (1.7) gives

$$cQRe^{\beta}+PRe^{\alpha-\beta}+QPe^{\delta-\beta}-QPe^{\delta+\alpha-\beta}=(P+cQ)R. \tag{1.24}$$

Suppose first that $P+cQ\equiv 0$, so that $P\equiv Q\equiv 1$ and $c=-1$. Substituting these into (1.24), we have

$$Re^{2\beta-\alpha}-e^{\delta-\alpha}+e^{\delta}=R, \tag{1.25}$$

which implies that $e^{2\beta-\alpha}$ or $e^{\delta-\alpha}$ is a constant. If $e^{2\beta-\alpha}$ is a constant x, then (1.25) yields that $-e^{\delta-\alpha}+e^{\delta}=(1-x)R$. Since $e^{\alpha}\not\equiv 1$, $x\neq 1$, so that $e^{\delta-\alpha}$ must be a constant y. Hence we have $ye^{\alpha}=(1-x)R+y$. This is impossible. If $e^{\delta-\alpha}$ is a constant x and $e^{2\beta-\alpha}$ is not, then (1.25) gives $Re^{2\beta-\alpha}+xe^{\alpha}=R+x$ $(\not\equiv 0)$, which is also impossible. Thus we may suppose that $P+cQ\not\equiv 0$, so that we deduce from (1.24) that either $e^{\delta-\beta}$ or $e^{\delta+\alpha-\beta}$ is a constant.

(VI_1) $e^{\delta-\beta}$ is a constant x. By (1.24)

$$cQRe^{\beta}+PRe^{\alpha-\beta}-xQPe^{\alpha}=(P+cQ)R-xQP,$$

which implies that $(P+cQ)R-xQP\equiv 0$. This is impossible because $\{c_n\}\cup\{d_n\}\cup\{q_n\}$ is not empty.

(VI_2) $e^{\delta+\alpha-\beta}$ is a constant x. Then by (1.24)

$$cQRe^{\beta}+PRe^{\alpha-\beta}+xQPe^{-\alpha}=(P+cQ)R+xQP,$$

which implies that $(P+cQ)R+xQP\equiv 0$, but this is impossible.

Thus $e^{\gamma-\beta}$ is not a constant.

(VII) Assume seventhly that $e^{\delta-\beta}\equiv c$. In this case we have by (1.7)

$$QRe^{\gamma}+PRe^{\alpha-\beta}-QRe^{\gamma-\beta}-cQPe^{\alpha}=(R-cQ)P, \tag{1.26}$$

from which $R-cQ\equiv 0$ i.e. $R\equiv Q\equiv 1$ and $c=1$. From these and (1.26) it follows that

$$-Pe^{\alpha-\beta-\gamma}+e^{-\beta}+Pe^{\alpha-\gamma}=1, \tag{1.27}$$

which implies that either $e^{\alpha-\beta-\gamma}$ or $e^{\alpha-\gamma}$ is a constant. If $e^{\alpha-\gamma}$ is a constant x, then (1.27) gives $(1-xP)e^{-\beta}=1-xP\not\equiv 0$, a contradiction. Similarly also the constancy of $e^{\alpha-\beta-\gamma}$ gives a contradiction.

Thus $e^{\delta-\beta}$ is not a constant.

(VIII) Assume eighthly that $e^{\delta+\alpha-\beta}\equiv c$. In view of (1.7)

$$QRe^{\gamma}+PRe^{\alpha-\beta}-QRe^{\gamma-\beta}+cQPe^{-\alpha}=(R+cQ)P, \tag{1.28}$$

which implies that $R+cQ\equiv 0$. Hence $R\equiv Q\equiv 1$ and $c=-1$, so that (1.28) gives

$$(1.29) \qquad e^{-\beta}-Pe^{\alpha-\beta-\gamma}+Pe^{-\alpha-\gamma}=1.$$

From (1.29) it follows that (VIII_1) $e^{\alpha-\beta-\gamma}$ or (VIII_2) $e^{-\alpha-\gamma}$ is a constant x.

For Case (VIII_1), (1.29) becomes $e^{-\beta}+xPe^{\beta-2\alpha}=1+xP(\not\equiv 0)$, which is impossible. For Case (VIII_2), (1.29) becomes $e^{-\beta}-xPe^{2\alpha-\beta}=1-xP$ $(\not\equiv 0)$, which is also impossible.

Thus $e^{\delta+\alpha-\beta}$ is not a constant.

(IX) Assume finally that all of e^{γ}, $e^{\alpha-\beta}$, $e^{\gamma-\beta}$, $e^{\delta-\beta}$ and $e^{\delta+\alpha-\beta}$ are nonconstant. In view of (1.7)

$$QRe^{\gamma}+PRe^{\alpha-\beta}-QRe^{\gamma-\beta}+QPe^{\delta-\beta}-QPe^{\delta+\alpha-\beta}=PR,$$

which is clearly impossible.

All the above arguments (I)-(IX) are combined to show that $(\{a_n\}\setminus\{c_n\}$, $\{b_n\}\setminus\{d_n\}$, $\{p_n\}\setminus\{q_n\})$ is not a zero-one-pole set of any nonconstant meromorphic function.

2. Proof of Theorem 2 and 3.

2.1. Proof of Theorem 2. Suppose that $(\{a_n\}\setminus\{c_n\}$, $\{b_n\}\setminus\{d_n\}$, $\{p_n\}\setminus\{q_n\})$ is the zero-one-pole set of a meromorphic function $g(z)$.

In this case we first note that $g(z)$ is not a constant. In fact, if $g(z)$ is a constant, then $\{a_n\}=\{c_n\}$, $\{b_n\}=\{d_n\}$ and $\{p_n\}=\{q_n\}$ hold. This implies that $N(r, 0, f)+N(r, 1, f)+N(r, \infty, f)=N(r, \{c_n\}\cup\{d_n\}\cup\{q_n\})=o(r)$ $(r\to\infty)$. On the other hand, by the second fundamental theorem $(1-o(1))T(r, f)\leqq N(r, 0, f)+N(r, 1, f)+N(r, \infty, f)$ $(r\to\infty)$, and so that $T(r, f)=o(r)$ $(r\to\infty)$, which contradicts the assumption $\rho>1$.

Secondly we note that $g(z)$ is of finite order. In order to show this, we may note that $N(r, 0, g)+N(r, 1, g)+N(r, \infty, g)\leqq N(r, 0, f)+N(r, 1, f)+N(r, \infty, f)$ $\leqq 3T(r, f)+O(1)\leqq O(r^{\rho+\varepsilon})$ $(\varepsilon>0,\ r\to\infty)$, and recall the fact that a nonconstant meromorphic function has at most two Borel deficient values.

Let $P(z)$, $R(z)$ and $Q(z)$ be the entire functions of genus 0 whose zeros are $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ respectively. For example, we set $P(z)\equiv 1$ if $\{c_n\}$ is empty. Then we have

$$(2.1) \qquad gP/Q=fe^{\alpha},$$

$$(2.2) \qquad (g-1)R/Q=(f-1)e^{\beta}$$

with two polynomials α and β. Eliminating g from (2.1) and (2.2), we obtain

$$(2.3) \qquad ((R/P)e^{\alpha}-e^{\beta})f=(R/Q)-e^{\beta}.$$

If $(R/P)e^{\alpha}-e^{\beta}\equiv 0$, then $R\equiv P\equiv 1$, and so by (2.3) $Qe^{\beta}\equiv 1$. This is a contradiction. Thus

$$f=\frac{(R/Q)-e^{\beta}}{(R/P)e^{\alpha}-e^{\beta}}. \tag{2.4}$$

If both e^{α} and e^{β} are constants, we deduce from (2.4) that $\rho\leqq 1$, a contradiction. Hence e^{α} or e^{β} is not a constant. In this case by (2.4) we obtain

$$\begin{aligned} T(r, f)&\leqq 2m(r, R)+m(r, Q)+m(r, P)+2m(r, e^{\beta})+m(r, e^{\alpha})+O(1) \\ &\leqq(2+o(1))\{m(r, e^{\alpha})+m(r, e^{\beta})\} \qquad (r\to\infty). \end{aligned} \tag{2.5}$$

Next, we proceed to estimate $T(r, f)$ from below. Using the second fundamental theorem we easily deduce $T(r, g)\leqq(3+o(1))T(r, f)$ $(r\to\infty)$, and so that by (2.1) and (2.2)

$$\begin{aligned} m(r, e^{\alpha})+m(r, e^{\beta})&\leqq m(r, P)+m(r, R)+2m(R, Q)+2T(r, g) \\ &\quad +2T(r, f)+O(1)\leqq(8+o(1))T(r, f) \qquad (r\to\infty). \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we conclude that $\rho=\max(\deg\alpha, \deg\beta)$, which contradicts the nonintegrity of ρ.

2.2. Proof of Theorem 3. In this case we have (2.1)-(2.5) with two entire functions α and β (where e^{α} or e^{β} is not a constant), and (2.6) as $r\notin E$, $r\to\infty$. Differentiating (2.1) and (2.2), we obtain

$$g'P/Q+g(P/Q)'=(f'+f\alpha')e^{\alpha}, \tag{2.7}$$

$$g'R/Q+(g-1)(R/Q)'=(f'+f\beta'-\beta')e^{\beta}. \tag{2.8}$$

If a is a multiple zero of f, then by (2.1) and (2.7) at least one of $g(a)=g'(a)=0$, $g(a)=P(a)=0$ and $P(a)=P'(a)=0$ holds. In particular by (2.2) and (2.8) $g(a)=g'(a)=0$ implies that a is a zero of $(R/Q)'-(R/Q)\beta'$. Hence if β is not a constant, then

$$\begin{aligned} \overline{N}_1(r, 0, f)&\leqq N(r, 0, (R/Q)'-(R/Q)\beta')+N(r, 0, P) \\ &\leqq T(r, (R/Q)')+T(r, R/Q)+m(r, \beta')+m(r, P)+O(1) \\ &\leqq(3+o(1))T(r, R/Q)+m(r, (e^{\beta})'/e^{\beta})+m(r, P)+O(1) \\ &\leqq(3+o(1))\{m(r, R)+m(r, Q)+m(r, P)\}+o(m(r, e^{\beta})) \\ &=o(m(r, e^{\beta}))=o(T(r, f)) \qquad (r\notin E,\ r\to\infty), \end{aligned}$$

which is impossible. If β is a constant, then

$$\begin{aligned} \overline{N}_1(r, 0, f)&\leqq N(r, 0, (R/Q)')+N(r, 0, P) \\ &\leqq T(r, (R/Q)')+m(r, P)\leqq(2+o(1))T(r, R/Q)+m(r, P) \\ &\leqq(2+o(1))\{m(r, R)+m(r, Q)\}+m(r, P)=o(r) \qquad (r\to\infty) \end{aligned} \tag{2.9}$$

Since $\bar{N}_1(r, 0, f) > \kappa T(r, f)$ with a suitable $\kappa > 0$ and $r \geqq r_0$, (2.9) implies $T(r, f) = o(r)$ $(r \to \infty)$, which contradicts the assumption $\limsup_{r\to\infty} T(r, f)/r > 0$.

3. Proof of Theorem 4.

Let $(\{a_n\}, \{b_n\}, \{p_n\})$ be the zero-one-pole set of a nonconstant meromorphic function $f(z)$. Suppose that $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ are subsequences of $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively such that $\{c_n\} \cup \{d_n\} \cup \{q_n\} \neq \emptyset$ and such that $(\{a_n\} \setminus \{c_n\}, \{b_n\} \setminus \{d_n\}, \{p_n\} \setminus \{q_n\})$ is the zero-one-pole set of a nonconstant meromorphic function $g(z)$.

If $P(z)$, $R(z)$ and $Q(z)$ are entire functions whose zeros are $\{c_n\}$, $\{d_n\}$ and $\{q_n\}$ respectively (where for example, if $\{c_n\}$ is empty, we set $P(z) \equiv 1$), then we have

$$gP/Q = fe^{\alpha}, \tag{3.1}$$

$$(g-1)R/Q = (f-1)e^{\beta} \tag{3.2}$$

with two entire functions α and β. Eliminating g from (3.1) and (3.2), we obtain

$$f - fSe^{\gamma} + Te^{-\beta} = 1, \tag{3.3}$$

where $S = R/P$, $T = R/Q$ and $\gamma = \alpha - \beta$. Put $\phi_1 = f$, $\phi_2 = -fSe^{\gamma}$ and $\phi_3 = Te^{-\beta}$. Then by (3.3)

$$\phi_1 + \phi_2 + \phi_3 = 1, \quad \phi_1^{(n)} + \phi_2^{(n)} + \phi_3^{(n)} = 0 \qquad (n = 1, 2). \tag{3.4}$$

Further put

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \phi_1'/\phi_1 & \phi_2'/\phi_2 & \phi_3'/\phi_3 \\ \phi_1''/\phi_1 & \phi_2''/\phi_2 & \phi_3''/\phi_3 \end{vmatrix}, \quad \Delta' = \begin{vmatrix} \phi_2'/\phi_2 & \phi_3'/\phi_3 \\ \phi_2''/\phi_2 & \phi_3''/\phi_3 \end{vmatrix}. \tag{3.5}$$

Here we state two lemmas.

LEMMA 1. *If Δ vanishes identically, then at least one of $\{a_n\} \setminus \{c_n\}$ and $\{p_n\} \setminus \{q_n\}$ is empty.*

Proof. Since Δ vanishes identically, we deduce from (3.4) and (3.5) that

$$0 = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{vmatrix} = \begin{vmatrix} \phi_1 & \phi_2 & 1 \\ \phi_1' & \phi_2' & 0 \\ \phi_1'' & \phi_2'' & 0 \end{vmatrix} = \begin{vmatrix} \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \end{vmatrix},$$

which implies with two constants C and D

$$-fSe^{\gamma} = \phi_2 = C\phi_1 + D = Cf + D, \tag{3.6}$$

$$Te^{-\beta}=\phi_3=1-\phi_1-\phi_2=1-D-(C+1)f. \tag{3.7}$$

Combining (3.6) and (3.7), we easily get

$$R\{CPe^{-\beta}+Re^{\gamma-\beta}+(D-1)Qe^{\gamma}\}=(C+D)PQ. \tag{3.8}$$

Case 1. Assume first that $\{d_n\}$ is not empty. Since $R(d_n)=0\neq P(d_n)Q(d_n)$, (3.8) gives $C+D=0$ and $(C+1)Qe^{\gamma+\beta}-Re^{\gamma}=CP$. From the latter identity, we easily see that $C\neq 0, -1$. Then by (3.6) and (3.7)

$$f=\frac{CP}{CP+Re^{\gamma}}=\frac{(1+C)Q-Re^{-\beta}}{(1+C)Q}, \quad f-1=\frac{-Re^{\gamma}}{CP+Re^{\gamma}},$$

which show that $\{c_n\}=\{a_n\}$, $\{d_n\}=\{b_n\}$ and $\{q_n\}=\{p_n\}$. This contradicts the assumption that $g(z)$ is a nonconstant meromorphic function.

Case 2. Assume next that $\{d_n\}$ is empty. Since $R(z)\equiv 1$, (3.6) gives $(CP+e^{\gamma})f=-DP$. If $D=0$, we have $CP+e^{\gamma}=0$, which implies $P(z)\equiv 1$. In this case $\{q_n\}$ is not empty, and hence by (3.7) $C\neq -1$ and

$$f=\frac{Q-e^{-\beta}}{(C+1)Q},$$

which implies that g has no poles. If $D\neq 0$, (3.6) gives

$$f=\frac{-DP}{CP+e^{\gamma}},$$

which implies that g has no zeros.

LEMMA 2. *Assume that Δ is not identically equal to zero. Then we have*

$$\begin{aligned}(3.9)\qquad &(1-o(1))T(r, f)\\ &\leqq 2\{\overline{N}(r, 0, f)+N(r, \infty, f)+\overline{N}(r, 0, P)+\overline{N}(r, 0, Q)+\overline{N}(r, 0, R)\}\\ &\quad +O(\log^+ m(r, P)+\log^+ m(r, Q)+\log^+ m(r, R)) \qquad (r\notin E,\ r\to\infty).\end{aligned}$$

Proof. Easy computations give

$$\begin{aligned}&\phi_2'=(f'/f+S'/S+\gamma')\phi_2,\\ &\phi_2''=\{f''/f+S''/S+2(f'/f)(S'/S+\gamma')+2(S'/S)\gamma'+\gamma''+(\gamma')^2\}\phi_2,\\ &\phi_3'=(T'/T-\beta')\phi_3,\\ &\phi_3''=\{T''/T-2(T'/T)\beta'-\beta''+(\beta')^2\}\phi_3.\end{aligned}$$

Hence

$$\begin{aligned}(3.10)\qquad \Delta=&(S'/S+\gamma')\{2(f'/f)^2-f''/f\}\\ &-\{2(T'/T-\beta')(S'/S+\gamma')-S''/S-2(S'/S)\gamma'-\gamma''-(\gamma')^2\}(f'/f)\\ &+(S'/S)(T''/T)-(T'/T)(S''/S)+(S'/S)\{(\beta')^2-2(T'/T)\beta'\}\end{aligned}$$

$$-(T'/T)\{2(S'/S)\gamma'+\gamma''+(\gamma')^2\}+\gamma'\{(T''/T)-2(T'/T)\beta'+(\beta')^2\}$$
$$+\beta'\{(S''/S)+2(S'/S)\gamma'+\gamma''+(\gamma')^2\}-\beta''(S'/S+\gamma').$$

As we have shown in [7, Lemma 2],

$$N(r, \infty, 2(f'/f)^2-f''/f)\leqq 2\bar{N}(r, 0, f)+N(r, \infty, f). \tag{3.11}$$

Also the term $(S'/S)(T''/T)-(T'/T)(S''/S)$ requires attention. A direct computation gives

$$\begin{aligned}&(S'/S)(T''/T)-(T'/T)(S''/S)\\&=(R'/R)\{(P''/P)-2(P'/P)^2+2(Q'/Q)^2-(Q''/Q)+2(R'/R)(P'/P-Q'/Q)\}\\&\quad+(Q'/Q-P'/P)(R''/R)+(P'/P)(Q''/Q)-(Q'/Q)(P''/P)\\&\quad-2(P'/P)(Q'/Q)(Q'/Q-P'/P),\end{aligned}$$

which implies

$$\begin{aligned}&N(r, \infty, (S'/S)(T''/T)-(T'/T)(S''/S))\\&\leqq 2\{\bar{N}(r, 0, P)+\bar{N}(r, 0, Q)+\bar{N}(r, 0, R)\}.\end{aligned} \tag{3.12}$$

Thus (3.10) combined with (3.11) and (3.12) yields

$$\begin{aligned}N(r, \infty, \Delta)\leqq{}& 2\bar{N}(r, 0, f)+N(r, \infty, f)\\&+2\{\bar{N}(r, 0, P)+\bar{N}(r, 0, Q)+\bar{N}(r, 0, R)\}.\end{aligned} \tag{3.13}$$

Now, $\Delta\not\equiv 0$ and (3.5) give $f=\Delta'/\Delta$, and so

$$\begin{aligned}m(r, f)&\leqq m(r, \Delta')+m(r, \Delta^{-1})\\&\leqq m(r, \Delta')+m(r, \Delta)+N(r, \infty, \Delta)+O(1).\end{aligned} \tag{3.14}$$

Here we estimate $m(r, \Delta')$ and $m(r, \Delta)$. By (3.1) and (3.2)

$$\begin{aligned}m(r, e^{\alpha})&\leqq m(r, g)+m(r, f^{-1})+m(r, P)+m(r, Q^{-1})\\&\leqq T(r, g)+T(r, f)+m(r, P)+m(r, Q)+O(1)\\&\leqq(4+o(1))T(r, f)+m(r, P)+m(r, Q)\qquad (r\notin E,\ r\to\infty),\end{aligned}$$

$$\begin{aligned}m(r, e^{\beta})&\leqq m(r, g-1)+m(r, (f-1)^{-1})+m(r, R)+m(r, Q^{-1})\\&\leqq T(r, g)+T(r, f)+m(r, R)+m(r, Q)+O(1)\\&\leqq(4+o(1))T(r, f)+m(r, R)+m(r, Q)\qquad (r\notin E,\ r\to\infty).\end{aligned}$$

Hence

$$\begin{aligned}T(r, \phi_3)&\leqq T(r, R/Q)+m(r, e^{-\beta})\\&\leqq m(r, R)+m(r, Q)+m(r, e^{\beta})+O(1)\\&\leqq(4+o(1))T(r, f)+2\{m(r, R)+m(r, Q)\}\qquad (r\notin E,\ r\to\infty).\end{aligned}$$

$$
\begin{aligned}
T(r, \phi_2) &\leqq T(r, f)+T(r, R/P)+m(r, e^{\gamma}) \\
&\leqq T(r, f)+m(r, R)+m(r, P)+m(r, e^{\alpha})+m(r, e^{\beta})+O(1) \\
&\leqq (9+o(1))T(r, f)+2\{m(r, P)+m(r, Q)+m(r, R)\} \\
&\qquad\qquad (r \notin E,\ r\to\infty),
\end{aligned}
$$

so that

$$
\begin{aligned}
m(r, \Delta'),\ m(r, \Delta) &\leqq O(\log r+\log T(r, \phi_1)+\log T(r, \phi_2)+\log T(r, \phi_3)) \\
&= O(\log r+\log T(r, f)+\log^{+} m(r, P) \\
&\qquad +\log^{+} m(r, Q)+\log^{+} m(r, R)) \qquad (r \notin E,\ r\to\infty).
\end{aligned} \tag{3.15}
$$

After (3.13) and (3.15) are taken into account, (3.14) gives (3.9).

We are now in position to prove our Theorem 4. Using Lemma 4 in [6], we can choose $P(z)$, $R(z)$ and $Q(z)$ such that

$$
\frac{\log^{+} m(r, P)+\log^{+} m(r, R)+\log^{+} m(r, Q)}{N(r, 0, P)+N(r, 0, R)+N(r, 0, Q)} \to 0 \qquad \text{as } r \in \Omega,\ r\to\infty \tag{3.16}
$$

for a suitable set $\Omega \subset [1, \infty)$ of infinite linear measure. Clearly

$$
\begin{aligned}
& N(r, 0, P)+N(r, 0, R)+N(r, 0, Q) \\
& \leqq N(r, 0, f)+N(r, 1, f)+N(r, \infty, f) \leqq 3T(r, f)+O(1).
\end{aligned} \tag{3.17}
$$

Assume now that $(\{a_n\}\backslash\{c_n\}, \{b_n\}\backslash\{d_n\}, \{p_n\}\backslash\{q_n\})$ is the zero-one-pole set of a meromorphic function $g(z)$. By assumption $g(z)$ has zeros and poles, so we deduce from Lemma 1 that Δ is not identically equal to zero. On combining (3.9) in Lemma 2 with (3.16) and (3.17) we obtain

$$
\begin{aligned}
(1-o(1))T(r, f) &\leqq 2\{\bar{N}(r, 0, f)+N(r, \infty, f)+\bar{N}(r, 0, P) \\
&\qquad +\bar{N}(r, 0, R)+\bar{N}(r, 0, Q)\} \qquad (r \in \Omega\backslash E,\ r\to\infty),
\end{aligned}
$$

which contradicts the condition (*).

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