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HYPERELLIPTIC COMPACT NON-ORIENTABLE KLEIN SURFACES WITHOUT BOUNDARY

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By a compact non-orientable Klein surface (KS) X, we shall mean a compact non-orientable surface together with a dianalytic structure on X [3].

A dianalytic homeomorphism of X onto itself will be called an automorphism. We say that a compact non-orientable KS without boundary X is q-hyperelliptic if and only is there exists an involution Φ of X such that $X/\langle\Phi\rangle$ has algebraic genus q (if $q=0$, X is hyperelliptic, if $q=1$, then X is elliptic-hyperelliptic).

In this paper we characterize the q-hyperellipticity of compact non-orientable KS without boundary by means of non-Euclidean crystallographic groups (NEC groups). Similar characterizations for compact orientable KS without boundary (i.e. Riemann surfaces) or for compact KS with boundary have been obtained in [11], [6], [7], [8]. As a consequence from these results, it is obtained: The bound 84 $(p-1)$ for a group of automorphisms of a non orientable KS without boundary can be reduced to 12 $(p-1)$ for most of q-hyperelliptic KS.

1. Preliminary

In this paper we characterize KS by means of NEC groups. Such a group is a discrete subgroup of the group G of all isometries of the hyperbolic plane, including orientation-reversing ones, with compact quotient space (see [10], [14]).

Each NEC group has a signature, that is

$$(g\,;\,\pm\,;\,[m_1,\cdots,m_t]\,;\,\{(n_{i1}\cdots n_{is_i})i=1\cdots k\}). \tag{1}$$

The numbers m_i are the proper periods, the brachets $(n_{i1}\cdots n_{is_i})$ are the period-cycles and the n_{ij} are the periods of the period-cycles. This signature determines a presentation of the group. Generators:

i) x_i $i=1,\cdots,r$
ii) e_i $i=1,\cdots,k$
iii) c_{ij} $i=1,\cdots,k\,;\ j=0,\cdots,s_i$
iv) (if sign '+') a_i, b_i $i=1,\cdots,g$
(if sign '−') d_i $i=1,\cdots,g$

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relations:

i) $x_i^{m_i}=1 \quad i=1, \cdots, r$
ii) $e_i^{-1}c_{i0}e_i c_{is_i}=1 \quad i=1, \cdots, k$
iii) $c_{ij-1}^2=c_{ij}^2=(c_{ij-1}c_{ij})^{n_{ij}}=1 \quad i=1, \cdots, k; j=1, \cdots, s_i$
iv) (if sign '+'): $x_1 \cdots x_r e_1 \cdots e_k a_1^{-1} b_1^{-1} a_1 b_1 \cdots a_g^{-1} b_g^{-1} a_g b_g=1$
v) (if sign '−'): $x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2=1$.

An NEC group Γ with signature (1) has an associated area

$$|\Gamma|=2\pi(\alpha g+k-2+\sum_{i=1}^{r}(1-1/m_i)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1-1/n_{ij})), \tag{2}$$

where $\alpha=1$ if the sign is '−' and $\alpha=2$ if sign is '+'.

Let X be a KS of topological genus g with k boundary components, and algebraic genus $p=\alpha g+k-1$ ($\alpha=2$ if X is orientable and $\alpha=1$ if X is non-orientable). Then X may be expressed as D/Γ, D being the hyperbolic plane and Γ an NEC group with signature

$$(g; \pm; [-]; \{(-)^k\}),$$

where the sign is '+' if X is orientable and '−' if X is non-orientable.

May [12] proved that a group of automorphism of the surface D/Γ, may be expressed as Γ'/Γ, Γ' being another NEC group such that $\Gamma \lhd \Gamma'$. The full group of automorphism of D/Γ is $N_G(\Gamma)/\Gamma$, where $N_G(\Gamma)$ is the normalizer of Γ in G. When Γ' and Γ are two NEC groups such that $[\Gamma':\Gamma]=N$ then $N|\Gamma'|=|\Gamma|$.

An NEC group Γ has the canonical Fuchsian subgroup Γ^+ associated [13]. The Riemann surface D/Γ^+ is canonically associated to the KS D/Γ.

From now on, a KS is assumed to have algebraic genus not smaller than 2.

2. Non-orientable Q-hyperelliptic surfaces.

In this Section we characterize by means of NEC groups the compact q-hyperelliptic KS.

PROPOSITION 2.1. *Let $X=D/\Gamma$ be a non-orientable KS without boundary. Then X is q-hyperelliptic, if and only if there exists an NEC group Γ_1 of algebraic genus q, containg Γ as a subgroup of index 2. (The group Γ_1 is said to be a q-hyperellipticity group).*

Proof. Let us suppose that an NEC group Γ_1 with algebraic genus q, such that $[\Gamma_1:\Gamma]=2$ exists. Then Γ_1/Γ is a group of automorphisms of D/Γ of order two, and D/Γ_1 has genus q. Thus D/Γ is q-hyperelliptic.

If X is q-hyperelliptic then the exists an involution Φ of D/Γ such that $X'=X/\langle\Phi\rangle$ has genus q, moreover $\langle\Phi\rangle\cong\Gamma_1/\Gamma$, for a certain NEC group Γ_1

such that $[\Gamma_1:\Gamma]=2$. Since $X'=X/\langle\Phi\rangle\cong(D/\Gamma)/(\Gamma_1/\Gamma)\cong D/\Gamma_1$ and the genus of X' is q, the algebraic genus of Γ_1 is q.

THEOREM 2.2. *Let $X=D/\Gamma$ be a non-orientable KS without boundary with algebraic genus p. Then X is q-hyperelliptic if and only if there exists an NEC group Γ_1, containing Γ as a subgroup of index 2 and having the signature of one of the following types.*

(a) $(h\,;+\,;[2\overset{(p+1)-2q}{\cdots\cdots\cdots}2]\,;\{(-)^{q-2h+1}\}]\,;\ 0\leqq h\leqq q/2$

(b) $(h\,;-\,;[2\overset{(p+1)-2q}{\cdots\cdots\cdots}2]\,;\{(-)^{q-h+1}\})\,;\ 0\leqq h\leqq q$ *if p is even*, $1\leqq h\leqq q+1$ *if p is odd. Moreover*

1) *For each $0\leqq h\leqq q/2$ there exists a non-orientable q-hyperelliptic KS without boundary D/Γ whose q-hyperellipticity group has signature* (a).

2) *For each $1\leqq h\leqq q$ (if p is even) and for each $1\leqq h\leqq q+1$ (if p is odd) there exists a non-orientable q-hyperelliptic KS without boundary D/Γ whose q-hyperellipticity group has signature* (b).

Proof. Sufficiency is a consequence of Proposition 2.1.

We are going to see neccesity. Since $X=D/\Gamma$ is a non-orientable q-hyperelliptic KS without boundary, there exists an involution Φ of X, such that the algebraic genus of $X/\langle\Phi\rangle$ is q. There is an NEC group Γ_1 such that $\langle\Phi\rangle\cong\Gamma_1/\Gamma$.

Having in mind that the signature of Γ is $(p+1\,;-\,;[-]\,;\{-\})$ we are going to determine the possible signatures of Γ_1. Since the algebraic genus of Γ_1 is q, thus by theorem 1 [5] the possible signatures of Γ_1 will have the following form,

$$\sigma(\Gamma_1)=(h\,;\pm\,;[2\overset{t}{\cdots\cdots}2]\,;\{(-)^{q-\alpha h+1}\}),$$

where $\alpha=2$ if the sign is '+' and $\alpha=1$ if sign is '$-$'.

From the relation between the areas of the fundamental regions of Γ and Γ_1 we have, $|\Gamma|=2|\Gamma_1|$. Thus $p-1=2(q-1+t/2)$, and hence $t=p+1-2q$. So, the signature of Γ_1 has the form,

(*) $(h\,;+\,;[2\overset{(p+1)-2q}{\cdots\cdots\cdots}2]\,;\{(-)^{q-2h+1}\})$

(**) $(h\,;-\,;[2\overset{(p+1)-2q}{\cdots\cdots\cdots}2]\,;\{(-)^{q-h+1}\})$.

If Γ_1 has a signature (*), then $q-2h+1\geqq 1$; otherwise Γ_1 only has orientation preserving elements. As a result Γ_1 could not contain the subgroup Γ, since the last one contains orientation reversing elements. So $0\leqq h\leqq q/2$.

If Γ_1 has signature (**), then $1\leqq h\leqq q+1$. If its happen that $h=q+1$, then Γ_1 has a signature of the form

$$(q+1\,;-\,;[2\overset{(p+1)-2h}{\cdots\cdots\cdots}2])$$

and so the q-hyperellipticity homeomorphism $\theta: \Gamma_1 \to Z_2$ verifies $\theta(x_i)=\bar{1}$ for $i=1, \cdots, (p+1)-2q$. Thus since θ preserves the relations of Γ_1 then

$$\theta(d_1^2 \cdots d_{q+1}^2 x_1 \cdots x_{(p+1)-2q})=\bar{0}$$

consequently $\theta(x_1 \cdots x_{(p+1)-2q})=\bar{0}$; and it is impossible if p is even. And so if p is even $h \leqq q$, and if p is odd $h \leqq q+1$.

Now, we are going to see that 1) is satisfied. Given h, $0 \leqq h \leqq q/2$ and an NEC group Γ_1 with signature (a), we define the epimorphism $\theta: \Gamma_1 \to Z_2$, by means of

$$\theta(a_i)=\theta(b_i)=\bar{1} \quad i=1, \cdots, h$$
$$\theta(x_i)=\bar{1} \quad i=1, \cdots, p+1-2q$$
$$\theta(e_i)=\theta(c_i)=\bar{1} \quad i=1, \cdots, q-2h+1$$

where a_i, b_i, x_i, e_i, c_i are the canonical generators of Γ_1. Then, by [9], [5] and [4], $\ker\theta$ is an NEC groups with signature $(p+1; -; [-]; \{-\})$. So, $D/\ker\theta$ is a non-orientable q-hyperelliptic KS without boundary with algebraic genus p and its q-hyperellipticity group Γ_1 has signature (a).

Let us see 2). Given $1 \leqq h \leqq q$ (if p is even) or $1 \leqq h \leqq q+1$ (if p is odd) and an NEC group Γ_1 with signature (b), we define the epimorphism $\theta: \Gamma_1 \to Z_2$ by means of

$$\theta(d_i)=\bar{0} \quad i=1, \cdots, h$$
$$\theta(x_i)=\bar{1} \quad i=1, \cdots, (p+1)-2q$$
$$\theta(e_i)=\theta(c_i)=\bar{1} \quad i=1, \cdots, p-h+1$$

where d_i, x_i, e_i, c_i are the canonical generators of Γ_1. As in the previous case one can show that $D/\ker\theta$ is a surface we have looked for.

PROPOSITION 2.3. *Let $X=D/\Gamma$ be a non-orientable q-hyperelliptic KS without boundary, with algebraic genus $p>4q+1$. Then the automorphism of the hyperellipticity Φ is central and unique.*

Proof. First we will show the uniqueness.

Let us suppose that there are two q-hyperellipticity groups Γ_1 and Γ_1'. Then $[\Gamma_1:\Gamma]=2$ and $[\Gamma_1':\Gamma]=2$, and so $[\Gamma_1^+:\Gamma^+]=2$ and $[\Gamma_1'^+:\Gamma^+]=2$. Thus Γ_1^+/Γ^+ and $\Gamma_1'^+/\Gamma^+$ are groups of automorphisms of order two of $X^+=D/\Gamma^+$. Moreover both D/Γ_1^+ and $D/\Gamma_1'^+$ have genus q. By hypothesis $p>4q+1$, thus (see [1] and [2]) $\Gamma_1^+=\Gamma_1'^+$. Since Γ has reversing-orientation elements, $\Gamma=\Gamma^+ \cup g\Gamma^+$, where $g \in (\Gamma-\Gamma^+)$.

Since $\Gamma \subset \Gamma_1$ and $\Gamma \subset \Gamma_1'$, we have $g \in \Gamma_1$ and $g \in \Gamma_1'$. Thus $\Gamma_1=\Gamma_1^+ \cup g\Gamma_1^+$ and $\Gamma_1'=\Gamma_1'^+ \cup g\Gamma_1'^+$. Hence $\Gamma_1=\Gamma_1'$.

Now, we are going to see that Φ is central.

Let $\langle\Phi\rangle \cong \Gamma_1/\Gamma$, where Γ_1 is the group of the q-hyperellipticity of D/Γ, and let $\beta \in N_G(\Gamma)/\Gamma$. Then there exists $g \in N_G(\Gamma)$ such that, $\beta=g\Gamma$. From

$\langle\Phi\rangle\cong\Gamma_1/\Gamma=\{\Gamma, l\Gamma\}$, we obtain

$$\langle\beta\Phi\beta^{-1}\rangle=\{g\Gamma\Gamma\Gamma g^{-1}, g\Gamma l\Gamma\Gamma g^{-1}\}=\{\Gamma, glg^{-1}\Gamma\}=(g\Gamma_1 g^{-1})/\Gamma.$$

Thus since Γ_1 is isomorphic to $g\Gamma_1 g^{-1}$ and Γ is contained in both groups, the uniqueness of Γ_1 implies that $\Gamma_1=g\Gamma_1 g^{-1}$ and so $\langle\beta\Phi\beta^{-1}\rangle=\langle\Phi\rangle$. Here by $\beta\Phi\beta^{-1}=\Phi$ and hence Φ is central in the full group of automorphism of X.

3. Bounds for the order of the group of automorphisms.

Let $X=D/\Gamma$ be a non-orientable q-hyperelliptic KS without boundary, and let Φ be an automorphism of the q-hyperellipticity. Then $\langle\Phi\rangle\cong\Gamma_1/\Gamma$ for a certain NEC groups Γ_1 and by (2.2) $X/\langle\Phi\rangle\cong D/\Gamma_1$ is an orientable KS with boundary or a non-orientable KS (with or without boundary).

In the next Theorem we are going to see that the bound $84(g-1)$ for the order of the group of the automorphisms of a non-orientable KS without boundary X can be improved when X is q-hyperelliptic and $X/\langle\Phi\rangle$ has non empty boundary.

THEOREM 3.1. *Let $X=D/\Gamma$ be a non-orientable q-hyperelliptic KS without boundary of algebraic genus $p>4q+1$.*

Let Φ be the automorphism of the q-hyperellipticity. If $X/\langle\Phi\rangle$ is a surface with boundary then,

1) $|\mathrm{Aut}X|\leqq 12(p-1)$ (*where* $\mathrm{Aut}X$ *is the full group of automorphism of* X).
2) *If* $|\mathrm{Aut}X|=12(p-1)$, *then X is a hyperelliptic surface of algebraic genus* 2 *and* $\mathrm{Aut}X\cong D_6$.

Proof. Since $X=D/\Gamma$ is a non-orientable KS without boundary of algebraic genus p, the signature of Γ must be $(p+1;-;[-];\{-\})$. A group of automorphisms of X can be represented as Γ''/Γ where $\Gamma\lhd\Gamma''$.

X is q-hyperelliptic and $p>4q+1$. Thus by (2.2) and (2.3) there exists an unique NEC groups Γ_1 with signature

(a) $(h;+;[2\overset{p+1-2q}{\cdots\cdots\cdots}2];\{(-)^{q-2h+1}\})$, $\quad 0\leqq h\leqq q/2$

or

(b) $(h;-;[2\overset{p+1-2q}{\cdots\cdots\cdots}2];\{(-)^{q-h+1}\})$, $\quad 1\leqq h\leqq q$ if p is even
$1\leqq h\leqq q+1$ if p is odd,

such that $\Gamma\lhd\Gamma_1\lhd\Gamma''$ and $\langle\Phi\rangle\cong\Gamma_1/\Gamma$. Since $X/\langle\Phi\rangle$ has boundary, $h\neq q+1$ in (b), and thus there are empty period cycles in the signature of Γ_1.

We are going to show that there is either an empty period cycles or a period cycle with two consecutive periods equals to '2' in the signature of Γ''.

If this were not the case then from the normality of Γ_1 a reflection c of canonical generators of Γ'' would belong to Γ_1.

If this reflection c belongs to a period cycle with unique period n, then the two reflections c and c' that generate this period are conjugated. Thus by the normality the reflection c' also belongs to Γ_1. As a result the period n must appear in a period cycle of Γ_1. This is a contradiction, because all period cycles of Γ_1 are empty. Thus we can assume that the reflection c belongs to a period cycle having two consecutive periods n' and n'' one of which is different than 2. Without loss generality we can assume that this periods are associated to reflections c, c', c'', i.e.

$$(c'c)^{n'}=(cc'')^{n''}=1.$$

If c' or c'' belong to Γ_1 then a period n' or n'' respectively appear in a period cycle of Γ_1. It is a contradiction as above.

If neither c' nor c'' belong to Γ_1 then by Theorem 1 of [5] the values $n'/2$ and $n''/2$ must appear in period cycle of Γ_1. A contradiction appears once more.

Since $|\mathrm{Aut}\,X|=|\Gamma|/|\Gamma''|\geqq 12(p-1)$ and $|\Gamma|=2(p-1)$, $|\Gamma''|\leqq\pi/6$. From Section 1 (2) we have that the unique signature that has an empty period-cycle or a period-cycle with two consecutive periods equal to 2 and area smaller than $\pi/6$ is

$$(0\,;\,+\,;\,[-]\,;\,\{(2,2,2,3)\}). \qquad (*)$$

Thus $|\mathrm{Aut}\,X|\leqq 12(p-1)$ and if $|\mathrm{Aut}\,X|=12(p-1)$ Γ'' has the signature (*).

Now, we are going to prove that if $|\mathrm{Aut}\,X|=12(p-1)$, then $p=2$ and $\mathrm{Aut}\,X$ is D_6.

As $\Gamma_1\lhd\Gamma''$, there exists the epimorphism $\theta:\Gamma''\to\Gamma''/\Gamma_1$ with $\ker\theta=\Gamma_1$. If c_1, c_2, c_3, c_4, c_5 are the generators of Γ'' by [5] we have

$$\theta(c_1)=x$$
$$\theta(c_2)=y$$
$$\theta(c_3)=1$$
$$\theta(c_4)=z$$
$$\theta(c_5)=x\,,$$

where x, y, z are elements of order 2 in Γ''/Γ_1 and one of the following relations must be satisfied

i) $(xy)^2=(zx)^3=1$
ii) $x=y$ and $(zx)^3=1$.

If i) is satisfied, then by [4] there are no proper-periods in the signature of Γ_1, but $p>4q+1$ and then the number of '2' in the proper-periods of (a) of (b) is $p+1-2q>2q+2>2$ and so i) can not be fulfiled.

Now assume that ii) is satisfied. Since $[\Gamma'':\Gamma_1]=6(p-1)$, by [4] the number of '2' in the proper-periods of Γ_1 is $6(p-1)/2$, and so $3(p-1)=p+1-2q$. This equality is possible only in the case $p=2$ and $q=0$. Hence $h=0$ and Γ_1

would have a single cycle-period.

As Γ'/Γ_1 has order 6 and it is a groups generated by y, z with the relations $y^2=z^2=(yz)^3=1$, we have that $\Gamma''/\Gamma_1 \cong D_3$; so that Γ''/Γ would have order 12 and its quotient by a subgroups generated by a central element must be D_3. Thus Γ''/Γ is D_6.

Finally, we need to prove that there exists a non-orientable hyperelliptic KS without boundary of algebraic genus 2 whose group of automorphism is D_6.

Let Γ'' be an NEC group with signature (*) and let θ be the epimorphism $\theta : \Gamma'' \to D_6 = \langle x, y \mid x^2=y^2=(xy)^6 \rangle$ defined by

$$\theta(c_1)=x$$

$$\theta(c_2)=x(yx)^3$$

$$\theta(c_3)=(xy)^3$$

$$\theta(c_4)=x(yx)^2$$

$$\theta(c_5)=x\,.$$

Then by [9] and [4], $\ker\theta$ is an NEC group with signature $(3; -; [-]; \{-\})$. Moreover, it is easy to check that $(xy)^3$ is a central element of D_6 and c_3, c_1c_2, $(c_4c_5)^3$ and $(c_2c_4)^3$ belong to $\theta^{-1}(\langle (xy)^3 \rangle)$. By [9], [4] and [5] $\theta^{-1}(\langle (xy)^3 \rangle)$ is an NEC group with signature $(0; +; [2, 2, 2]; \{(-)\})$. Let us denote $\ker\theta$ by Γ and $\theta^{-1}(\langle (xy)^3 \rangle)$ by Γ_1. We have $\Gamma \lhd \Gamma_1 \lhd \Gamma''$ and so D/Γ is a non-orientable hyperelliptic KS without boundary of algebraic genus 2 having D_6 as the group of automorphisms.

If p is even, then by Theorem 2.2 $X/\langle \Phi \rangle$ is a surface with nonempty boundary and thus we have the following Corollary.

COROLLARY 3.2. *Let* $X=D/\Gamma$ *a non-orientable q-hyperelliptic KS without boundary with algebraic genus* $p>4q+1$ *and* p *even, then:*

1) $|\mathrm{Aut}\,X| \leqq 12(p-1)$
2) *If* $|\mathrm{Aut}\,X| = 12(p-1)$, *then* X *is a hyperelliptic surface of algebraic genus* 2 *and* $\mathrm{Aut}\,X \approx D_6$.

Remark. By the functorial equivalence between the category of KS and the category of algebraic curves, established by Alling and greenleaf in [3], there is a correspondence between the non-orientable KS without boundary and the purely imaginary real curves (for example $x^2+y^2=-1$ corresponds to real projective plane). We say that a purely imaginary real curve C is said to be q-hyperelliptic if and only if it admits an involution Φ, such that if $\Phi^\sim$ is the extension of Φ to the complexification $C^\sim$ of C, then $C^\sim/\Phi^\sim$ has genus q. So all the results obtained above may be expressed in terms of purely imaginary real curves.

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