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CHANGE OF THE BASE POINT IN THE HOMOTOPY EXACT SEQUENCE OF A FIBRATION

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§1. Introduction.

The purpose of this note is to show that the homotopy exact sequence of a kind of principal quasifibration in the sense of Dold-Lashof [2], which shall be called a principal fibration, does not depend on the choice of base points (Theorem 5.1). A typical example of the principal fibration in Example 3.4 is also a principal fibre space in the sense of Peterson-Thomas [9].

The object of this note is classical and may be known to experts but we have not seen any fully general description of it in reference except related results in [10] and [3] which assert that the homotopy exact sequence of a Serre fibration does not depend on the choice of base points and will be proved in a general form in this note ((8) of Lemma 4.1).

Although we can work in the usual category of topological spaces under some restrictions, we shall deal mainly with more convenient category *WHK*, the category of weak Hausdorff k-spaces [7], [8].

We are grateful to Y. Hirashima for bringing our attention to [3] and [10].

§2. Notations.

Continuous functions are called "maps".

We recall from [7] some notions. A space X is weak Hausdorff if $f(K)$ are closed for all compact Hausdorff spaces K and all maps $f: K \to X$. A subset A of the space X is compactly closed if for every map $f: K \to X$, where K is a compact Hausdorff space, $f^{-1}(A)$ is closed in K. A space X is a k-space if every compactly closed subset of X is closed. The k-ification $k(X)$ of a space X is the space whose underlying set is that of X and whose closed sets are the compactly closed subsets of X. If X, Y are spaces, let $X \otimes Y = k(X \times Y)$, where $\times$ denotes the usual cartesian product, and let $\mathrm{Map}(X, Y) = kC(X, Y)$, where $C(X, Y)$ denotes the space of maps from X to Y with the compact-open topology. In *WHK*, we have the exponential law: $\mathrm{Map}(X \otimes Y, Z) = \mathrm{Map}(X, \mathrm{Map}(Y, Z))$. See [8] and [12].

A space X with a distinguished point $*$ is called a based space. A based space X is well based if $(X, *)$ is an NDR-pair. Recall from [6] and [13] that

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a pair of spaces (X, A) is cofibred if and only if (X, A) has the homotopy extension property (HEP) for all spaces, and that (X, A) is an NDR-pair if and only if A is closed in X and (X, A) is cofibred, that is, a closed cofibred pair. Notice that if (X, A) is cofibred and $X \in WHK$ then A is closed in X and so (X, A) is an NDR-pair.

We shall use the following notations:

WHK=the category of weak Hausdorff k-spaces;

CW=the category of CW complexes;

FCW=the category of finite CW complexes;

WHK_*=the category of well based weak Hausdorff k-spaces;

CW_*=the category of CW complexes with a distinguished vertex;

FCW_*=the category of finite CW complexes with a distinguished vertex;

HEP=homotopy extension property;

CHP=covering homotopy property;

CHEP=covering homotopy extension property;

$c(x)=c_x$=the constant function into the point x;

$I=[0, 1]$, the unit interval;

$C(X, X_1, \cdots, X_n; Y, Y_1, \cdots, Y_n)$=the subspace of $C(X, Y)$ of maps f with $f(X_i) \subset Y_i$ for all i;

$\mathrm{Map}(X, X_1, \cdots, X_n; Y, Y_1, \cdots, Y_n)=kC(X, X_1, \cdots, X_n; Y, Y_1, \cdots, Y_n)$;

$[X, X_1, \cdots, X_n; Y, Y_1, \cdots, Y_n]$=the set of homotopy classes of maps $f: X \to Y$ with $f(X_i) \subset Y_i$ for all i, where homotopy ϕ_t satisfies $\phi_t(X_i) \subset Y_i$;

$\Sigma X = X \times I/(X \times \{0, 1\} \cup \{*\} \times I)$, the reduced suspension of $(X, *)$;

$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$;

$\pi_j: X_1 \times \cdots \times X_n \to X_j$, the projection to the j-th component.

§ 3. Principal fibration over an H-space.

DEFINITION 3.1. An H-space is a weak Hausdorff k-space G together with a continuous multiplication

$$G \otimes G \longrightarrow G, \qquad (g, g') \longrightarrow gg'$$

with two sided unit e.

DEFINITION 3.2. Let G be an H-space. A Serre (resp. Hurewicz) fibration over G is a quintuplet (E, p, B, μ, G), usually triplet (E, p, B) is sufficient notation, satisfying the following conditions.

(1) $E \in WHK$ and $p: E \to B$ is a map having the CHP for spaces in FCW (resp. WHK), and

(2) $$\mu: E \otimes G \longrightarrow E, \qquad \mu(x, g) = xg$$

is a map such that

(2-1) $xe = x$,

(2-2) $xG=\mu(x, G)\subset F_x=p^{-1}(p(x))=$fibre through x,

(2-3) $\kappa(x)_*: [Z, *; G, e]\cong[Z, *; F_x, x]$ for all $x\in E$ and all $Z\in FCW_*$ (resp. WHK_*), where

$$\kappa(x): G \longrightarrow F_x, \qquad \kappa(x)(g)=xg.$$

Remarks. (1) $\kappa(x)$ factorizes into the composition of maps

$$G \xrightarrow{\kappa(x)'} k(F_x) \xrightarrow{k} F_x$$

where k is the canonical identity map, and (2-3) can be replaced by

(2-3)′ $\kappa(x)'_*: [Z, *; G, e]\cong[Z, *; k(F_x), x]$ for all $x\in E$ and all $Z\in FCW_*$ (resp. WHK_*).

(2) As is well known, FCW_* can be replaced by CW_* in (2-3) and (2-3)′.

(3) Recall from p. 63 of [4], p. 154 of [5] and 6.44 of [6] that if $p: E\to B$ has the CHP for spaces in FCW (resp. WHK), then p has the CHEP for relative CW pairs (resp. NDR-pairs in WHK).

If (E, p, B) is a Serre (resp. Hurewicz) fibration over G, then for any $Z\in CW_*$ (resp. WHK_*) and any $x\in E$, the function

$$\chi(x)=\kappa(x)_*^{-1}\Delta: [\Sigma Z, *; B, p(x)] \longrightarrow [Z, *; F_x, x]\cong[Z, *; G, e]$$

is called the *characteristic homomorphism* for the fibration with respect to x, where Δ is the connecting (boundary) function. See the proof of Lemma 4.1(8) given below or the chapter 4, especially Theorem 1.25 and Example 1.6, of [5] for the definition of Δ.

DEFINITION 3.3. Let G be an associative H-space, that is, a monoid in WHK. A Serre (resp. Hurewicz) fibration (E, p, B) over G is principal if

$$x(gg')=(xg)g' \qquad \text{for every} \quad x\in E,\ g,\ g'\in G.$$

Remarks. (1) Any principal Serre fibration over G is a principal quasi-fibration over G in the sense of Dold-Lashof [2] except the lack of an assumption about the left translations of G.

(2) A principal fibration over G, (E, p, B), is not necessarily locally trivial nor the translation function

$$\tau: E^*=\{(x, xg)\mid x\in E,\ g\in G\} \longrightarrow G, \qquad \tau(x, xg)=g,$$

can be well-defined.

(3) A numerable principal T-bundle in WHK, where T is a topological group in WHK (see [1] and [8]), is a principal Hurewicz fibration over T.

(4) If we work in the usual category of topological spaces and in addition

we assume that the translation function $\tau: E^* \to G$ can be defined and is continuous, then the principal Serre fibration over G is a principal fibre space in the sense of Peterson-Thomas [9].

Example 3.4. Let T be a topological group in WHK, that is, let T be a group and a space in WHK having the continuous multiplication $T \otimes T \to T$ and the continuous inversion $T \to T$. Let $q: Y \to Y/T$ be a numerable principal T-bundle ([1], [8]) and let $X \in WHK$. Then $G = \mathrm{Map}(X, T)$ becomes naturally a topological group in WHK. Moreover we have a free action

$$\mathrm{Map}(X, Y) \otimes \mathrm{Map}(X, T) \longrightarrow \mathrm{Map}(X, Y), \qquad (fg)(x) = f(x)g(x),$$

and the map

$$p = \mathrm{Map}(id_X, q): E = \mathrm{Map}(X, Y) \longrightarrow B = \mathrm{Map}(X, Y/T)$$

is a principal fibre space in the sense of Peterson-Thomas [9] and also a principal Hurewicz fibration over $\mathrm{Map}(X, T)$. Indeed, q has the CHP for k-spaces by 4.8 of [1], hence so does p by the exponential law for the functor "Map" (see 3.6 of [12] or 4.11 of [8]). By 6.27 of Ōshima [8], we have

(1) $xG = F_x$,

(2) $\kappa(x)': G \to k(F_x)$ is a homeomorphism, and

(3) the translation function $\tau: k(E^*) \to G$ is well defined and continuous.

§ 4. Operations induced by a path.

Case 1. Let (Z, A, A') be a cofibred triplet, that is, (Z, A) and (A, A') are cofibred pairs. Let (X, C) be a pair of spaces and let $\omega: (I, 0, 1) \to (C, x_0, x_1)$ be a path. We define

$$\omega^{\#}: [Z, A, A'; X, C, x_0] \longrightarrow [Z, A, A'; X, C, x_1]$$

as follows. Given a map $f: (Z, A, A') \to (X, C, x_0)$, we have a map $L: I \times A \to C$ extending $f\pi_2 \cup \omega\pi_1: \{0\} \times A \cup I \times A' \to C$. The map $f\pi_2 \cup L: \{0\} \times Z \cup I \times A \to X$ has an extension $H: I \times Z \to X$. We define $\omega^{\#}[f] = [H_1]$.

The above case can be seen in texts such as [4], [5] and [11]. Next we consider less familiar case of which the first $\omega^{\#}$ was essentially treated in Lemma B of [10] and the chapter 5 of [3]. See also Example 1.9 of [5; p. 235].

Case 2. Let $p: E \to B$ have the CHP for spaces in FCW (resp. WHK), $\omega: (I, 0, 1) \to (E, x_0, x_1)$ a path, (Z, A, A') a triplet of CW complexes (resp. a cofibred triplet in WHK). Set $F_\iota = p^{-1}(p(x_\iota))$. We shall define operators

$$\omega^{\#}: [A, A'; F_0, x_0] \longrightarrow [A, A'; F_1, x_1],$$

$$\omega^{\#}: [Z, A, A'; E, F_0, x_0] \longrightarrow [Z, A, A'; E, F_1, x_1].$$

Notice that A and A' are closed in Z and so (Z, A, A') is an NDR-triplet. Given a map $f:(A, A')\to(F_0, x_0)$, there exists a map $H: I\times A\to E$ extending $f\pi_2\cup\omega\pi_1: \{0\}\times A\cup I\times A'\to E$ and lifting $p\omega\pi_1: I\times A\to B$, by the CHEP. Hence we have a map $H_1=H(1,\ ):(A, A')\to(F_1, x_1)$, and we define $\omega^{\#}[f]=[H_1]$. Given a map $g:(Z, A, A')\to(E, F_0, x_0)$, we have a map $g': I\times A\to E$ lifting $p\omega\pi_1: I\times A\to B$ and extending $g\pi_2\cup\omega\pi_1: \{0\}\times A\cup I\times A'\to E$, by the CHEP. We have also a map $L: I\times Z\to E$ extending $g\pi_2\cup g': \{0\}\times Z\cup I\times A\to E$, by the HEP. Hence we have a map $L_1:(Z, A, A')\to(E, F_1, x_1)$, and we define $\omega^{\#}[g]=[L_1]$.

LEMMA 4.1. (1) *$\omega^{\#}$ is well defined.*

(2) *If $\omega\simeq\omega'$ rel $\{0, 1\}$, then $\omega^{\#}=\omega'^{\#}$.*

(3) *If $\omega=c(x)$, the constant path to x, then $\omega^{\#}$ is the identical automorphism.*

(4) *If $\omega(1)=\tau(0)$, then $(\omega+\tau)^{\#}=\tau^{\#}\omega^{\#}$.*

(5) *$\omega^{\#}$ is a bijection.*

(6) *In Case 2, the square*

$$\begin{array}{ccc}[Z, A, A'; E, F_0, x_0] & \xrightarrow{\omega^{\#}} & [Z, A, A'; E, F_1, x_1]\\ \downarrow & & \downarrow\\ [A, A'; F_0, x_0] & \xrightarrow[\omega^{\#}]{} & [A, A'; F_1, x_1]\end{array}$$

is commutative, where the verticals are the restrictions. Similar assertion holds in Case 1.

(7) *In Case 2, $\omega^{\#}$ is natural with respect to bundle maps, that is, if $p': E'\to B'$ is an other fibration of the same type with p and there is a commutative square*

$$\begin{array}{ccc}E & \xrightarrow{f} & E'\\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'}\\ B & \xrightarrow[f']{} & B'\end{array}$$

where f and f' are continuous, then the squares

$$\begin{array}{ccc}[A, A'; F_0, x_0] & \xrightarrow{f_*} & [A, A'; F'_0, x'_0]\\ {\scriptstyle \omega^{\#}}\downarrow & & \downarrow{\scriptstyle (f\omega)^{\#}}\\ [A, A'; F_1, x_1] & \xrightarrow[f_*]{} & [A, A'; F'_1, x'_1]\end{array}$$

$$\begin{array}{ccc}[Z, A, A'; E, F_0, x_0] & \xrightarrow{f_*} & [Z, A, A'; E', F'_0, x'_0]\\ {\scriptstyle \omega^{\#}}\downarrow & & \downarrow{\scriptstyle (f\omega)^{\#}}\\ [Z, A, A'; E, F_1, x_1] & \xrightarrow[f_*]{} & [Z, A, A'; E', F'_1, x'_1]\end{array}$$

are commutative, where $x'_\iota=f(x_\iota)$ *and* $F'_\iota=p'^{-1}(p'(x'_\iota))$.

(8) ([3], [10]). *In Case 2, the following ladder of homotopy exact sequences of the fibration is commutative*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & [\Sigma Z, *; B, b_0] & \longrightarrow & [Z, *; F_0, x_0] & \longrightarrow & [Z, *; E, x_0] & \longrightarrow & [Z, *; B, b_0] \\
& \downarrow{\scriptstyle (p\omega)^\#} & ① & \downarrow{\scriptstyle \omega^\#} & ② & \downarrow{\scriptstyle \omega^\#} & ③ & \downarrow{\scriptstyle (p\omega)^\#} \\
\cdots \longrightarrow & [\Sigma Z, *; B, b_1] & \longrightarrow & [Z, *; F_1, x_1] & \longrightarrow & [Z, *; E, x_1] & \longrightarrow & [Z, *; B, b_1]
\end{array}$$

where $Z\in CW_*$ *(resp.* WHK_**) and* $b_i=p(x_\iota)$.

Remark. In Case 1, the family of sets $\{[Z, A, A'; X, C, x] \mid x\in C\}$ and operations $\{\omega^\# \mid \omega\in C(I, C)\}$ forms a local system of sets in C, by (1)$\sim$(5). In Case 2, similar assertion holds.

Proof of Lemma 4.1. Let $\pi: I\times Y\times I\to I\times I$ be the projection for any Y.

To prove (1) of Case 1, let $f:(Z, A, A')\times I\to(X, C, x_0)$ be a map. Let $f_\iota': I\times A\times\{i\}\to C$ $(i=0, 1)$ be any extension of $f_i\cup\omega\pi_1$: $\{0\}\times A\times\{i\}\cup I\times A'\times\{i\}\to C$. Let $L^\iota: I\times Z\times\{i\}\to X$ be any extension of $f_i\cup f_\iota'$: $\{0\}\times Z\times\{i\}\cup I\times A\times\{i\}\to X$. Let also $g: I\times A\times I\to C$ be any extension of $f\cup(f_0'\cup f_1')\cup\omega\pi_1$: $\{0\}\times A\times I\cup I\times A\times\{0, 1\}\cup I\times A'\times I\to C$. Since $(Z, A)\times(I, \{0, 1\})$ is cofibred, the map $f\cup(L^0\cup L^1)\cup g$: $\{0\}\times Z\times I\cup I\times Z\times\{0, 1\}\cup I\times A\times I\to X$ extends to a map $H: I\times Z\times I\to X$. Then H: $\{1\}\times(Z, A, A')\times I\to(X, C, x_1)$ is a desired homotopy of $L^0{}_1$ to $L^1{}_1$.

To prove (1) of Case 2, let $\phi:(A, A')\times I\to(F_0, x_0)$ be a homotopy, and let $H^i: I\times Z\times\{i\}\to E$ $(i=0, 1)$ be a map lifting $p\omega\pi_1$ and extending $\phi_i\pi_2\cup\omega\pi_1$: $\{0\}\times A\times\{i\}\cup I\times A'\times\{i\}\to E$. Then $p\omega\pi_1 \mid \{0\}\times A\times I\cup I\times A'\times I\cup I\times A\times\{0, 1\}=p(\phi\cup\omega\pi_1\cup(H^0\cup H^1))$. Since $(A, A')\times(I, \{0, 1\})$ is an NDR-pair by 6.12 of [6], there is a map $L: I\times A\times I\to E$ lifting $p\omega\pi_1$ and extending $\phi\cup\omega\pi_1\cup(H^0\cup H^1)$, by the CHEP, hence the restriction L: $\{1\}\times(A, A')\times I\to(F_1, x_1)$ is a desired homotopy of $H^0{}_1$ to $H^1{}_1$.

Let $\phi:(Z, A, A')\times I\to(E, F_0, x_0)$ be a homotopy. We then have $\phi_i': I\times A\times\{i\}\to E$ $(i=0, 1)$ lifting $p\omega\pi_1$ and extending $\phi_i\cup\omega\pi_1$: $\{0\}\times A\times\{i\}\cup I\times A'\times\{i\}\to E$. Extend $\phi_i\cup\phi_i'$: $\{0\}\times Z\times\{i\}\cup I\times A\times\{i\}\to E$ to a map $H^i: I\times Z\times\{i\}\to E$. We have $L: I\times A\times I\to E$ lifting $p\omega\pi_1: I\times A\times I\to B$ and extending $\phi\cup\omega\pi_1\cup(\phi_0'\cup\phi_1')$: $\{0\}\times A\times I\cup I\times A'\times I\cup I\times A\times\{0, 1\}\to E$. Since $(Z, A)\times(I, \{0, 1\})$ is cofibred by 6.12 of [6], the map $\phi\cup(H^0\cup H^1\cup L)$: $\{0\}\times Z\times I\cup I\times(Z\times\{0, 1\}\cup A\times I)\to E$ extends to $H: I\times Z\times I\to E$. Then the restriction H: $\{1\}\times(Z, A, A')\times I\to(E, F_1, x_1)$ is a desired homotopy of $H^0{}_1$ to $H^1{}_1$.

To prove (2) of Case 1, let $f:(Z, A, A')\to(X, C, x_0)$ and $\theta:(I, 0, 1)\times I\to(C, x_0, x_1)$ be maps. The map $g\pi_2\cup\theta\pi$: $\{0\}\times A\times\{i\}\cup I\times A'\times\{i\}\to C$ extends to a map $L^\iota: I\times A\times\{i\}\to C$. The map $g\pi_2\cup L^\iota$: $\{0\}\times Z\times\{i\}\cup I\times A\times\{i\}\to X$ extends to a map $K^\iota: I\times Z\times\{i\}\to X$. The map $g\pi_2\cup(L^0\cup L^1)\cup\theta\pi$: $\{0\}\times A\times I\cup I\times A\times\{0, 1\}\cup I\times A'\times I\to C$ extends to a map $h: I\times A\times I\to C$, since $(A, A')\times(I, \{0, 1\})$ is cofibred. Since $(Z, A)\times(I, \{0, 1\})$ is cofibred, the map $g\pi_2\cup$

$(K^0\cup K^1)\cup h:\{0\}\times Z\times I\cup I\times Z\times\{0,1\}\cup I\times A\times I\to X$ extends to a map $H: I\times Z\times I\to X$. Then the restriction $H:\{1\}\times(Z, A, A')\times I\to(X, C, x_1)$ is a desired homotopy of $K^0{}_1$ to $K^1{}_1$.

To prove (2) of Case 2, let $f:(A, A')\to(F_0, x_0)$ and $\theta:(I, 0, 1)\times I\to(E, x_0, x_1)$ be maps. There exists a map $H^i: I\times A\times\{i\}\to E$ $(i=0, 1)$ extending $f\pi_2\cup\theta\pi$: $\{0\}\times A\times\{i\}\cup I\times A'\times\{i\}\to E$ and lifting $p\theta\pi: I\times Z\times\{i\}\to B$. Since $(A, A')\times(I, \{0, 1\})$ is an NDR-pair, there exists a map $H: I\times A\times I\to E$ extending $f\pi_2\cup(H^0\cup H^1)\cup\theta\pi:\{0\}\times Z\times I\cup I\times Z\times\{0,1\}\cup I\times\{*\}\times I\to E$ and lifting $p\theta\pi: I\times Z\times I\to B$. Then the restriction $H:\{1\}\times(A, A')\times I\to(F_1, x_1)$ is a desired homotopy of $H^0{}_1$ to $H^1{}_1$.

Let $f:(Z, A, A')\to(E, F_0, x_0)$ and $\theta:(I, 0, 1)\times I\to(E, x_0, x_1)$ be maps. There exists a map $L^i: I\times A\times\{i\}\to E$ extending $f\cup\theta\pi:\{0\}\times A\times\{i\}\cup I\times\{*\}\times\{i\}\to E$ and lifting $p\theta\pi: I\times A\times\{i\}\to B$. The map $f\pi_2\cup L^i:\{0\}\times Z\times\{i\}\cup I\times A\times\{i\}\to E$ extends to a map $K^i: I\times Z\times\{i\}\to E$. Since $(A, A')\times(I, \{0, 1\})$ is an NDR-pair, there exists a map $L: I\times A\times I\to E$ extending the map $f\pi_2\cup(L^0\cup L^1)\cup\theta\pi:\{0\}\times A\times I\cup I\times A\times\{0,1\}\cup I\times A'\times I\to E$ and lifting $p\theta\pi: I\times A\times I\to B$. Since $(Z, A)\times(I, \{0, 1\})$ is cofibred, the map $f\pi_2\cup(K^0\cup K^1)\cup L:\{0\}\times Z\times I\cup I\times Z\times\{0,1\}\cup I\times A\times I\to E$ extends to a map $H: I\times Z\times I\to E$. Then the restriction $H:\{1\}\times(Z, A, A')\times I\to(E, F_1, x_1)$ is a desired homotopy of $K^0{}_1$ to $K^1{}_1$.

To prove (3) of Case 1, let $f:(Z, A, A')\to(X, C, x)$ be a map. Since $f\pi_2\cup c_x\pi_1=f\pi_2:\{0\}\times A\cup I\times A'\to C$, it has an extension $f\pi_2: I\times A\to C$. Thus $f\pi_2\cup f\pi_2:\{0\}\times Z\cup I\times A\to X$ has an extension $f\pi_2: I\times Z\to X$. Therefore $c_x{}^\#[f]=[f]$.

To prove (3) of Case 2, let $f:(A, A')\to(F_x, x)$ be a map. The map $f\pi_2$: $I\times A\to E$ extends $f\pi_2\cup c_x\pi_1:\{0\}\times A\cup I\times A'\to E$ and lifts $pc_x\pi_1: I\times A\to C$. Thus $c_x{}^\#[f]=[f]$. Let $g:(Z, A, A')\to(E, F_x, x)$ be a map. Then the map $g\pi_2: I\times A\to E$ extends $g\pi_2\cup c_x\pi_1:\{0\}\times A\cup I\times A'\to E$ and lifts $pc_x\pi_1: I\times A\to B$. The map $g\pi_2\cup g\pi_2:\{0\}\times Z\cup I\times A\to E$ has an extension $g\pi_2: I\times Z\to E$. Thus $c_x{}^\#[g]=[g]$.

To prove (4), let ω and τ be paths in an appropriate space such that $\omega(0)=x_0$, $\omega(1)=\tau(0)=x_1$ and $\tau(1)=x_2$.

To prove (4) of Case 1, let $f:(Z, A, A')\to(X, C, x_0)$ be a map. Let H': $I\times A\to C$ be an extension of $f\pi_2\cup\omega\pi_1:\{0\}\times A\cup I\times A'\to C$, and let $L': I\times A\to C$ be an extension of $H_1\pi_2\cup\tau\pi_1:\{0\}\times A\cup I\times A'\to C$. Let $H: I\times Z\to X$ be an extension of $f\pi_2\cup H':\{0\}\times Z\cup I\times A\to X$, and let $L: I\times Z\to X$ be an extension of $H_1\pi_2\cup L':\{0\}\times Z\cup I\times A\to X$. Then $[H_1]=\omega^\#[f]$ and $[L_1]=\tau^\#[H_1]=\tau^\#\omega^\#[f]$, by definition. Define $K: I\times Z\to X$ by

$$K(t, z)=\begin{cases} H(2t, z) & \text{if } 0\leqq t\leqq 1/2\\ L(2t-1, z) & \text{if } 1/2\leqq t\leqq 1.\end{cases}$$

Then $K|\{0\}\times A\cup I\times A'=f\pi_2\cup(\omega+\tau)\pi_1$ and $K|\{0\}\times Z\cup I\times A=f\pi_2\cup K|I\times A$. Thus $[L_1]=[K_1]=(\omega+\tau)^\#[f]$. Therefore $\tau^\#\omega^\#[f]=(\omega+\tau)^\#[f]$.

To prove (4) of Case 2, let $f:(A, A')\to(F_0, x_0)$ be a map. Let $H: I\times A\to E$ be a map extending $f\pi_2\cup\omega\pi_1:\{0\}\times A\cup I\times A'\to E$ and lifting $p\omega\pi_1: I\times A\to B$. Let $L: I\times A\to E$ be a map extending $H_1\pi_2\cup\tau\pi_1:\{0\}\times A\cup I\times A'\to E$ and lifting $p\tau\pi_1: I\times A\to B$. Then $[H_1]=\omega^\#[f]$ and $[L_1]=\tau^\#[H_1]=\tau^\#\omega^\#[f]$. Define $K: I\times A$

$\to E$ by

$$K(t, z)=\begin{cases} H(2t, z) & \text{if } 0\leqq t\leqq 1/2 \\ L(2t-1, z) & \text{if } 1/2\leqq t\leqq 1. \end{cases}$$

Then K is an extension of $f\pi_2\cup(\omega+\tau)\pi_1:\{0\}\times A\cup I\times A'\to E$ and a lifting of $p(\omega+\tau)\pi_1: I\times A\to B$. Thus $[L_1]=[K_1]=(\omega+\tau)^{\#}[f]$ and so $\tau^{\#}\omega^{\#}[f]=(\omega+\tau)^{\#}[f]$. Let $g:(Z, A, A')\to(E, F_0, x_0)$ be a map. Let $h: I\times A\to E$ be a map extending $g\pi_2\cup\omega\pi_1:\{0\}\times A\cup I\times A'\to E$ and lifting $p\omega\pi_1: I\times A\to B$, and let $H': I\times Z\to E$ be an extension of $g\pi_2\cup h:\{0\}\times Z\cup I\times A\to E$. Let $h': I\times A\to E$ be a map extending $H'_1\pi_2\cup\tau\pi_1:\{0\}\times A\cup I\times A'\to E$ and lifting $p\tau\pi_1: I\times A\to B$, and let L': $I\times Z\to E$ be an extension of $H'_1\pi_2\cup h':\{0\}\times Z\cup I\times A\to E$. Then $[H'_1]=\omega^{\#}[g]$ and $[L'_1]=\tau^{\#}[H'_1]=\tau^{\#}\omega^{\#}[g]$. Define $K': I\times Z\to E$ by

$$K'(t, z)=\begin{cases} H'(2t, z) & \text{if } 0\leqq t\leqq 1/2 \\ L'(2t-1, z) & \text{if } 1/2\leqq t\leqq 1. \end{cases}$$

Then $K'|\{0\}\times A\cup I\times A'=g\pi_2\cup(\omega+\tau)\pi_1$ and $K'|\{0\}\times Z\cup I\times A=g\pi_2\cup K'|I\times A$. Thus $[L'_1]=[K'_1]=(\omega+\tau)^{\#}[g]$ and so $\tau^{\#}\omega^{\#}[g]=(\omega+\tau)^{\#}[g]$.

To prove (5), let ω be any path in an appropriate space with $\omega(0)=x_0$ and $\omega(1)=x_1$. Since $\omega+\omega^{-1}\simeq c(x_0)$ rel $\{0, 1\}$ and $\omega^{-1}+\omega\simeq c(x_1)$ rel $\{0, 1\}$, $\omega^{\#}$ is a bijection of which the inverse is $\omega^{-1\#}$, by (2), (3) and (4).

(6) and (7) can be proved easily by definition. So we omit their proofs.

Let us prove (8). The commutativity of ② and ③ is trivial by definition. Consider the following diagram:

$$\begin{tikzcd}
{[\Sigma Z, *; B, b_0]} \arrow[r, "q^*"] \arrow[d, "\omega^{\#}"'] \arrow[dr, phantom, "④"] & {[CZ, Z; B, b_0]} \arrow[r, equal] \arrow[d, "(p\omega)^{\#}"] & {[CZ, Z, *; B, b_0, b_0]} \\
{[\Sigma Z, *; B, b_1]} \arrow[r, "q^*"'] & {[CZ, Z; B, b_1]} \arrow[r, equal] & {[CZ, Z, *; B, b_1, b_1]}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[dr, phantom, "⑤"] & {[CZ, Z, *; E, F_0, x_0]} \arrow[l, "p_*"', "\cong"] \arrow[r, "\partial"] \arrow[d, "\omega^{\#}"] \arrow[dr, phantom, "⑥"] & {[Z, *; F_0, x_0]} \arrow[d, "\omega^{\#}"] \\
{} & {[CZ, Z, *; E, F_1, x_1]} \arrow[l, "p_*", "\cong"'] \arrow[r, "\partial"'] & {[Z, *; F_1, x_1]}
\end{tikzcd}$$

where $CZ=Z\times I/(Z\times\{1\}\cup\{*\}\times I)$ is the reduced cone of Z, $q:(CZ, Z, *)\to(\Sigma Z, *, *)$ is the quotient map, and $\partial[f]=[f|Z]$. If this diagram is commutative, then so is ①, since $\Delta=\partial p_*^{-1}q^*$.

By definition, ⑥ is commutative.

Given any map $f:(CZ, Z, *)\to(E, F_0, x_0)$, there exist maps $f': I\times Z\to E$ lifting $p\omega\pi_1: I\times Z\to B$ and extending $f\pi_2\cup\omega\pi_1:\{0\}\times Z\cup I\times\{*\}\to E$, and $H: I\times CZ\to E$ extending $f\pi_2\cup f':\{0\}\times CZ\cup I\times Z\to E$, so $[H_1]=\omega^{\#}[f]$, hence $p_*\omega^{\#}[f]=[pH_1]$. Since $pf': I\times Z\to B$ and $pH: I\times CZ\to B$ extend $pf\pi_2\cup p\omega\pi_1:\{0\}\times Z\cup$

$I\times\{*\}\to B$ and $pf\pi_2\cup pf'$: $\{0\}\times CZ\cup I\times Z\to B$, respectively, it follows that $(p\omega)^{\#}p_*[f]=[pH_1]$, so ⑤ is commutative.

Given any map $g:(\Sigma Z, *)\to(B, b_0)$, there exists a map $L: I\times\Sigma Z\to B$ extending $g\cup p\omega$: $\{0\}\times\Sigma Z\cup I\times\{*\}\to B$, so $[L_1]=(p\omega)^{\#}[g]$ and $q^*(p\omega)^{\#}[g]=[L_1q]$. Since $p\omega\pi_1: I\times Z\to B$ extends $p\omega\pi_1\cup c(b_0): I\times\{*\}\cup\{0\}\times Z\to B$, and since $L(id\times q)$: $I\times CZ\to B$ extends $gq\pi_2\cup p\omega\pi_1$: $\{0\}\times CZ\cup I\times Z\to B$, it follows that $(p\omega)^{\#}q^*[g]=[L_1q]=q^*(p\omega)^{\#}[g]$. Thus ④ is commutative, hence so is ①, and this proves (8). This completes the proof of Lemma 4.1. ■

We shall show that $\omega^{\#}=h_*: [A, A'; F_0, x_0]\to[A, A'; F_1, x_1]$ for some map $h:(F_0, x_0)\to(F_1, x_1)$ in some cases.

LEMMA 4.2. *Let $p: E\to B$ be a map, $\omega:(I, 0, 1)\to(E, x_0, x_1)$ a path, and set $F_i=p^{-1}(p(x_i))$. Suppose that*

(1) *(F_0, x_0) is a closed cofibred pair,*

(2) *p has the CHP for $I\times F_0$, and*

(3) *p has the CHP for $I\times F_0\times I$.*

Then there exists a map $h:(F_0, x_0)\to(F_1, x_1)$ which is unique up to homotopy with respect to the following property: there exists a map $H: I\times F_0\to E$ extending $\omega\pi_1\cup i_0\pi_2\cup h\pi_2: I\times\{x_0\}\cup\{0\}\times F_0\cup\{1\}\times F_0\to E$ and lifting $p\omega\pi_1: I\times F_0\to B$, where $i_j: F_j\to E$ is the inclusion.

Proof. By (1), (2) and the CHEP, there exists a map $H: I\times F_0\to E$ extending $\omega\pi_1\cup i_0\pi_2: I\times\{x_0\}\cup\{0\}\times F_0\to E$ and lifting $p\omega\pi_1: I\times F_0\to B$. Set $h=H_1:(F_0, x_0)\to(F_1, x_1)$. To prove the uniqueness of h, suppose that there is a map H^i: $I\times F_0\times\{i\}\to E$ $(i=0, 1)$ extending $i_0\pi_2\cup\omega\pi_1$: $\{0\}\times F_0\times\{i\}\cup I\times\{x_0\}\times\{i\}\to E$ and lifting $p\omega\pi_1: I\times F_0\times\{i\}\to B$. Since $(F_0, x_0)\times(I, \{0, 1\})$ is a closed cofibred pair, it follows from (3) and the CHEP that there exists a map $H: I\times F_0\times I\to E$ extending $(i_0\pi_2)\cup(\omega\pi_1)\cup(H^0\cup H^1)$: $\{0\}\times F_0\times I\cup I\times\{x_0\}\times I\cup I\times F_0\times\{0, 1\}\to E$ and lifting $p\omega\pi_1: I\times F_0\times I\to B$. Then $H|\{1\}\times F_0\times I$ defines a homotopy $(F_0, x_0)\times I\to(F_1, x_1)$ of $H^0{}_1$ to $H^1{}_1$. ■

PROPOSITION 4.3. *Let $p: E\to B$ have the CHP for spaces in FCW (resp. WHK), $\omega:(I, 0, 1)\to(E, x_0, x_1)$ a path, and set $F_i=p^{-1}(p(x_i))$. Suppose that $(F_0, x_0)\in CW_*$ (resp. WHK_*). Then there exists a map $h:(F_0, x_0)\to(F_1, x_1)$ such that*

$$\omega^{\#}=h_*: [A, A'; F_0, x_0]\longrightarrow[A, A'; F_1, x_1]$$

for every CW pair (resp. NDR-pair in WHK) (A, A'). Moreover h is unique in the same sense as 4.2.

Proof. By 4.2, there exists a map $H: I\times F_0\to E$ extending $i_0\pi_2\cup\omega\pi_1$: $\{0\}\times F_0\cup I\times\{x_0\}\to E$ and lifting $p\omega\pi_1: I\times F_0\to B$. Set $h=H_1:(F_0, x_0)\to(F_1, x_1)$. Given any map $f:(A, A')\to(F_0, x_0)$, the map $G=H(id\times f): I\times A\to E$ extends $f\pi_2\cup\omega\pi_1$: $\{0\}\times A\cup I\times A'\to E$ and lifts $p\omega\pi_1: I\times A\to B$. It follows from the definition that

$\omega^{\#}[f]=[G_1]=h_*[f]$. The uniqueness of h was proved in 4.2. ■

Remarks. (1) If $p: E \to B$ has the CHP for weak Hausdorff k-spaces and every fibre F_x is a topological manifold, then $(F_x, x) \in WHK_*$ and hence the assertion of 4.3 holds (for every NDR-pair in WHK).

(2) The maps h of 4.2 and 4.3 are called $p\omega^{-1}$-admissible in [13; p. 185].

§ 5. Main result.

Our main result is

THEOREM 5.1. *Let G be an associative H-space, and let (E, p, B) be a principal Serre (resp. Hurewicz) fibration over G. Suppose that*

(1) *$x_0, x_1 \in E$, $g_0, g_1 \in G$, and g_0, g_1 have inverses,*

(2) *$\omega: (I, 0, 1) \to (E, x_0 g_0, x_1 g_1)$ is a path, and*

(3) *$Z \in CW_*$ (resp. WHK_*).*

Set $b_i = p(x_i)$ and $F_i = p^{-1}(b_i)$. Then the square

$$\begin{tikzcd}
{[Z, *; G, e]} \arrow[r, "\kappa(x_0)_*"] \arrow[dddd, "(g_0 g_1^{-1})^{\#}"'] & {[Z, *; F_0, x_0]} \arrow[d, "R(g_0)_*"] \\
 & {[Z, *; F_0, x_0 g_0]} \arrow[d, "\omega^{\#}"] \\
 & {[Z, *; F_1, x_1 g_1]} \arrow[d, "R(g_1^{-1})_*"] \\
 & \\
{[Z, *; G, e]} \arrow[r, "\kappa(x_1)_*"] & {[Z, *; F_1, x_1]}
\end{tikzcd}$$

is commutative, where $R(g)(y) = yg$ and $g^{\#}$ is the homomorphism induced by the inner automorphism $h \to g^{-1} h g$ of G. Moreover the following diagram is also commutative

$$\begin{tikzcd}
\cdots \arrow[r, "p_*"] & {[\Sigma Z, *; B, b_0]} \arrow[r, "\chi(x_0)"] \arrow[dddd, "(p\omega)^{\#}"'] & {[Z, *; G, e]} \arrow[r, "j_{0*}"] \arrow[dddd, "(g_0 g_1^{-1})^{\#}"'] & {[Z, *; E, x_0]} \arrow[r, "p_*"] \arrow[d, "R(g_0)_*"] & {[Z, *; B, b_0]} \arrow[dddd, "(p\omega)^{\#}"] \\
 & & & {[Z, *; E, x_0 g_0]} \arrow[d, "\omega^{\#}"] & \\
 & & & {[Z, *; E, x_1 g_1]} \arrow[dd, "R(g_1^{-1})_*"] & \\
 & & & & \\
\cdots \arrow[r, "p_*"'] & {[\Sigma Z, *; B, b_1]} \arrow[r, "\chi(x_1)"'] & {[Z, *; G, e]} \arrow[r, "j_{1*}"'] & {[Z, *; E, x_1]} \arrow[r, "p_*"'] & {[Z, *; B, b_1]}
\end{tikzcd}$$

where $j_i: (G, e)\to(F_i, x_i)\to(E, x_i)$ *is the composition of* $\kappa(x_i)$ *and the inclusion.*

COROLLARY 5.2. *Suppose the following data*: $X\in WHK$; $Z\in WHK_*$; T *a topological group in* WHK; (Y, p, B) *a numerable principal* T*-bundle*; $f_0, f_1\in \mathrm{Map}(X, Y)$; $\lambda_0, \lambda_1\in \mathrm{Map}(X, T)$; $\omega: (I, 0, 1)\to(\mathrm{Map}(X, Y), f_0\lambda_0, f_1\lambda_1)$ *a path. Then the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & {[\Sigma Z, *; \mathrm{Map}(X, B), pf_0]} \arrow[r] \arrow[d, "(p\omega)^{\#}"] & {[Z, *; \mathrm{Map}(X, T), c_e]} \arrow[d, "(\lambda_0\lambda_1{}^{-1})^{\#}"] \\
\cdots \arrow[r] & {[\Sigma Z, *; \mathrm{Map}(X, B), pf_1]} \arrow[r] & {[Z, *; \mathrm{Map}(X, T), c_e]}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & {[Z, *; \mathrm{Map}(X, Y), f_0]} \arrow[r] \arrow[d, "R(\lambda_1{}^{-1})_*\omega^{\#}R(\lambda_0)_*"] & {[Z, *; \mathrm{Map}(X, B), pf_0]} \arrow[d, "(p\omega)^{\#}"] \\
{} \arrow[r] & {[Z, *; \mathrm{Map}(X, Y), f_1]} \arrow[r] & {[Z, *; \mathrm{Map}(X, B), pf_1]}
\end{tikzcd}$$

is commutative, where the horizontal sequences are homotopy exact sequences of the principal Hurewicz fibration, $p_*: \mathrm{Map}(X, Y)\to\mathrm{Map}(X, B)$, *over* $\mathrm{Map}(X, T)$ *given in Example* 3.4. *If in addition* X *and* Y *are well based, then the statement replacing* $\mathrm{Map}(,)$ *by* $\mathrm{Map}(, *; , *)$ *is true.*

Corollary 5.2 follows immediately from Theorem 5.1 which will be proved in the next section. Now we prove

PROPOSITION 5.3 (cf. Theorem 7.1 of [5; p. 305]). *Let* G *be an associative* H*-space,* (E, p, B) *a principal Serre* (*resp. Hurewicz*) *fibration over* G, *and let* $Z\in CW_*$ (*resp.* WHK_*).

(1) $\chi(x): [\Sigma Z, *; B, p(x)]\to[Z, *; G, e]$ *is a homomorphism with respect to the semi-group structures derived from* ΣZ *and* G.

(2) *Let* G' *be an associative* H*-space with unit* e'; (E', p', B') *a principal Serre* (*resp. Hurewicz*) *fibration over* G'; $\rho: G\to G'$ *a homomorphism, that is,* ρ *is continuous,* $\rho(gh)=\rho(g)\rho(h)$ $(g, h\in G)$ *and* $\rho(e)=e'$; *and suppose a commutative square*

$$\begin{tikzcd}
E \arrow[r, "\Psi"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\
B \arrow[r, "\phi"'] & B'
\end{tikzcd}$$

where Ψ, ϕ *are maps and* $\Psi(xg)=\Psi(x)\rho(g)$ $(x\in E, g\in G)$. *Then the square*

$$\begin{tikzcd} [\Sigma Z, *; B, p(x)] \arrow[r, "\chi(x)"] \arrow[d, "\phi_*"'] & [Z, *; G, e] \arrow[d, "\rho_*"] \\ [\Sigma Z, *; B', p'(x')] \arrow[r, "\chi(x')"'] & [Z, *; G', e'] \end{tikzcd}$$

is commutative for all $x \in E$, *where* $x' = \Psi(x)$.

Proof. Let $h^i: (\Sigma Z, *) \to (B, p(x))$ be maps ($i=1, 2$). Let $q: Z \times I \to \Sigma Z$ be the canonical quotient map. Then $h^i q | \{*\} \times I \cup Z \times \{1\} = pc_x$, and hence there are maps $H^i: Z \times I \to E$ lifting $h^i q$ and extending $c(x)$, by the CHEP. We then have $\Delta[h^i] = [H^i{}_0]$, by Example 1.6 of [5; p. 229]. Set $\chi(x)[h^i] = [h'^i]$, where h'^i: $(Z, *) \to (G, e)$. Since $(Z, *) \times (I, \{0, 1\})$ is an NDR-pair and since $p(c_x \cup \kappa(x) h'^i \pi_1)$: $[\{*\} \times I \cup Z \times \{1\}] \cup Z \times \{0\} \to B$ is the restriction of $hq: Z \times I \to B$, it follows that there exists a map $H': Z \times I \to E$ lifting $h^i q$ and extending $c_x \cup \kappa(x) h'^i \pi_1$. Then

$$\Delta[h^i] = [H'^i{}_0];$$

$$H'^i(z, 0) = x h'^i(z) \qquad \text{for all} \quad z \in Z.$$

Define $H: Z \times I \to E$ by

$$H(z, t) = \begin{cases} H'^1(z, 2t) h'^2(z) & \text{for} \quad 0 \leqq t \leqq 1/2 \\ H'^2(z, 2t-1) & \text{for} \quad 1/2 \leqq t \leqq 1. \end{cases}$$

Then

$$p(H(z, t)) = \left.\begin{cases} h^1(z, 2t) & \text{for } 0 \leqq t \leqq 1/2 \\ h^2(z, 2t-1) & \text{for } 1/2 \leqq t \leqq 1 \end{cases}\right\} = (h^1 + h^2)(z, t);$$

$$H(z, 1) = H'^2(z, 1) = x; \qquad H(*, t) = x,$$

so $[H_0] = \Delta[h^1 + h^2]$. Also $H(z, 0) = H'^1(z, 0) h'^2(z) = (x h'^1(z)) h'^2(z) = x(h'^1(z) h'^2(z))$, so $\Delta[h'^1 + h'^2] = [H_0] = \kappa(x)_*([h'^1][h'^2])$ and hence $\chi(x)[h^1 + h^2] = \chi(x)[h^1] + \chi(x)[h^2]$. This proves (1).

To prove (2), let $h: (\Sigma Z, *) \to (B, p(x))$ be a map. Since $hq = pc_x$ on $\{*\} \times I \cup Z \times \{1\}$, there is a map $H': Z \times I \to E$ lifting hq and extending $c(x)$, by the CHEP. It follows that $H'_0 \simeq \kappa(x) u$ rel $*$ for some $u \in C(Z, *; G, e)$. Since $(Z, *) \times (I, \{0, 1\})$ is an NDR-pair, there exists a map $H: Z \times I \to E$ lifting hq and extending $c(x) \cup \kappa(x) u \pi: [\{*\} \times I \cup Z \times \{1\}] \cup Z \times \{0\} \to E$. Then $\Delta[h] = [H_0]$. Also $H(z, 0) = \kappa(x) u(z)$ and hence $\chi(x)[h] = [u]$. We have

$$\Psi H(z, 0) = \Psi(x u(z)) = \Psi(x) \rho u(z) = \kappa(x')(\rho u(z))$$

hence $\Delta' \phi_*[h] = \Psi_* \Delta[h] = \kappa(x')_* \rho_*[u]$, so $\chi(x') \phi_*[h] = \rho_*[u] = \rho_*(\chi(x)[h])$. This proves (2). ∎

§6. Proof of Theorem 5.1.

THEOREM 6.1. *Suppose the following data*: G *an associative H-space*; (E, p, B) *a principal Serre (resp. Hurewicz) fibration over* G; $g_0, g_1 \in G$; $\omega: (I, 0, 1) \to (E, x_0 g_0, x_1 g_1)$ *a path*; $Z \in CW_*$ *(resp.* WHK_**). Set* $F_i = p^{-1}(p(x_i))$.

(i) *If* g_0 *and* g_1 *have inverses, then the next two diagrams are commutative in which* $j_i: G \to F_i \to E$ *is the composition of* $\kappa(x_i)$ *and the inclusion, and* $L(g)(y) = gy$.

$$
\begin{tikzcd}
{[\Sigma Z, *; B, p(x_0)]} \arrow[r, "\Delta"] \arrow[d, "="'] \arrow[dr, phantom, "7"] \arrow[rr, bend left, "\chi(x_0)_*"] & {[Z, *; F_0, x_0]} \arrow[d, "R(g_0)_*"] & {[Z, *; G, e]} \arrow[l, "\kappa(x_0)_*"'] \arrow[d, "R(g_0)_*"] \\
{[\Sigma Z, *; B, p(x_0)]} \arrow[r] \arrow[d, "="'] & {[Z, *; F_0, x_0 g_0]} \arrow[d, "="] & {[Z, *; G, g_0]} \arrow[l, "\kappa(x_0)_*"'] \arrow[d, "L(g_0^{-1})_*"] \\
{[\Sigma Z, *; B, p(x_0)]} \arrow[r, "\Delta"] \arrow[d, "(p\omega)^{\#}"'] \arrow[dr, phantom, "8"] & {[Z, *; F_0, x_0 g_0]} \arrow[d, "\omega^{\#}"] \arrow[dr, phantom, "9"] & {[Z, *; G, e]} \arrow[l, "\kappa(x_0 g_0)_*"'] \arrow[d, "="] \\
{[\Sigma Z, *; B, p(x_1)]} \arrow[r, "\Delta"] \arrow[d, "="'] & {[Z, *; F_1, x_1 g_1]} \arrow[d, "="] & {[Z, *; G, e]} \arrow[l, "\kappa(x_1 g_1)_*"'] \arrow[d, "L(g_1)_*"] \\
{[\Sigma Z, *; B, p(x_1)]} \arrow[r] \arrow[d, "="'] \arrow[dr, phantom, "7"] & {[Z, *; F_1, x_1 g_1]} \arrow[d, "R(g_1^{-1})_*"] & {[Z, *; G, g_1]} \arrow[l, "\kappa(x_1)_*"'] \arrow[d, "R(g_1^{-1})_*"] \\
{[\Sigma Z, *; B, p(x_1)]} \arrow[r] \arrow[rr, bend right, "\chi(x_1)_*"'] & {[Z, *; F_1, x_1]} & {[Z, *; G, e]} \arrow[l, "\kappa(x_1)_*"']
\end{tikzcd}
$$

$$\begin{tikzcd}
{[Z, *; G, e]} \arrow[r, "\kappa(x_0)_*"] \arrow[d, "R(g_0)_*"'] \arrow[rr, bend left, "j_{0*}"] & {[Z, *; F_0, x_0]} \arrow[r] \arrow[d, "R(g_0)_*"] & {[Z, *; E, x_0]} \arrow[d, "R(g_0)_*"] \\
{[Z, *; G, g_0]} \arrow[r, "\kappa(x_0)_*"] \arrow[d, "L(g_0^{-1})_*"'] & {[Z, *; F_0, x_0g_0]} \arrow[r] \arrow[d, "="] & {[Z, *; E, x_0g_0]} \arrow[d, "="] \\
{[Z, *; G, e]} \arrow[r, "\kappa(x_0g_0)_*"] \arrow[d, "="'] \arrow[dr, phantom, "⑨"] & {[Z, *; F_0, x_0g_0]} \arrow[r] \arrow[d, "\omega^\#"] \arrow[dr, phantom, "⑩"] & {[Z, *; E, x_0g_0]} \arrow[d, "\omega^\#"] \\
{[Z, *; G, e]} \arrow[r, "\kappa(x_1g_1)_*"] \arrow[d, "L(g_1)_*"'] & {[Z, *; F_1, x_1g_1]} \arrow[r] \arrow[d, "="] & {[Z, *; E, x_1g_1]} \arrow[d, "="] \\
{[Z, *; G, g_1]} \arrow[r, "\kappa(x_1)_*"] \arrow[d, "R(g_1^{-1})_*"'] & {[Z, *; F_1, x_1g_1]} \arrow[r] \arrow[d, "R(g_1^{-1})_*"] & {[Z, *; E, x_1g_1]} \arrow[d, "R(g_1^{-1})_*"] \\
{[Z, *; G, e]} \arrow[r, "\kappa(x_1)_*"] \arrow[rr, bend right, "j_{1*}"'] & {[Z, *; F_1, x_1]} \arrow[r] & {[Z, *; E, x_1]}
\end{tikzcd}$$

(ii) *If* $g_0=g_1=e$, *then the square*

$$\begin{tikzcd}
{[Z, *; G, e]} \arrow[r, "\kappa(x_0)_*"] \arrow[d, "="'] & {[Z, *; F_0, x_0]} \arrow[d, "\omega^\#"] \\
{[Z, *; G, e]} \arrow[r, "\kappa(x_1)_*"'] & {[Z, *; F_1, x_1]}
\end{tikzcd}$$

is commutative.

Proof. (ii) is a corollary to (i). We shall prove (i). The commutativity of

$$\begin{tikzcd}
{[\Sigma Z, *; B, p(x_\iota)]} \arrow[r, phantom, "\overset{p_*}{\cong}"] \arrow[d, "="'] & {[CZ, Z, *; E, F_\iota, x_\iota]} \arrow[r, "\partial"] \arrow[d, "R(g)_*"] & {[Z, *; F_\iota, x_\iota]} \arrow[d, "R(g)_*"] \\
{[\Sigma Z, *; B, p(x_\iota)]} \arrow[r, phantom, "\overset{p_*}{\cong}"] & {[CZ, Z, *; E, F_\iota, x_\iota g]} \arrow[r, "\partial"] & {[Z, *; F_\iota, x_\iota g]}
\end{tikzcd}$$

implies the commutativity of ⑦.

⑧ and ⑩ are commutative by (8) of Lemma 4.1.

To prove the commutativity of ⑨, let $h:(Z, *)\to(G, e)$ be a map and define a map $H: I\times Z\to E$ by $H(t, z)=\omega(t)h(z)$. Then

$$p(H(t, z))=p(\omega(t));$$

$$H(t, *)=\omega(t);$$

$$H(0, z)=x_0gh(z)=(\kappa(x_0g)h)(z);$$

$$H(1, z)=x_1h(z)=(\kappa(x_1)h)(z);$$

hence $\omega^{\#}\kappa(x_0g)_*[h]=\kappa(x_1)_*[h]$ by the definition of $\omega^{\#}$, so ⑨ is commutative. We can easily prove the commutativity of other squares and triangles of the diagrams in (i). ■

The first diagram of Theorem 5.1 and the left square of the second diagram of Theorem 5.1 are commutative by the commutativity of the first diagram of (i) of Theorem 6.1. The middle square of the second diagram of Theorem 5.1 is commutative by the commutativity of the second diagram of (i) of Theorem 6.1. The last square of the second diagram of Theorem 5.1 is commutative by (8) of Lemma 4.1. This proves Theorem 5.1.

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