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# ON THE HOMOTOPY OF CERTAIN MAPPING SPACES

Dedicated to Professor Hiroshi Toda on his 60th birthday

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## § 0. Introduction.

We denote by map $(X, Y)$ the space of continuous maps $X \to Y$ preserving base points endowed with Compact-Open topology. We are interested in studing homotopy groups $\pi_k(\mathrm{map}(X, Y), f)$. This problem has been attacked in papers [1], [2], [3], [5], and etc.. In this note we are mainly concerned with the subspace $\varepsilon(X, Y)$ consisting of homotopy equivalences.

Since $\pi_k(\varepsilon(X, Y), f)$ is isomorphic to $\pi_k(\varepsilon(X, X), 1_X)$ for any $f$ the study of $\pi_k(\varepsilon(X, X), 1_X)$, i.e. $\pi_k(\mathrm{map}(X, X), 1_X)$ is essential. Our purpose is to describe these homotopy groups in the case of $X$ being a principal bundle over a $n$-sphere $S^n$. One of difficulty for determing $\pi_k(\mathrm{map}(X, Y), f)$ arises from a choise of the base point. For example if we chose the trivial map as the base point these groups are explained as a set $\{\Sigma^k X, Y\}$, i.e. the group of homotopy classes of maps from the itterated suspension $\Sigma^k X$ to $Y$ preserving base points. Therefore, in this case, we consider the problem solved, so our purpose is in getting expression like this for our homotopy groups.

Let $p; X \to S^n$ be a principal $G$-bundle. Then we have a fibre space

$$\mathrm{map}(X, G) \longrightarrow \mathrm{map}(X, X) \longrightarrow \mathrm{map}(X, S^n),$$

and therefore a long exact sequence:

$$\pi_{k+1}(\mathrm{map}(X, S^n), p) \underset{\partial}{\longrightarrow} \pi_k(\mathrm{map}(X, G), \bar{e}_0) \longrightarrow \pi_k(\mathrm{map}(X, X), 1_X) \longrightarrow \pi_k(\mathrm{map}(X, S^n), p)$$

where $\bar{e}_0$ denote the constant map $\bar{e}_0(X) = e_0$; the unit of $G$.

Hence we want to describe the boundary homomorphism

$$\partial_{k+1}: \pi_{k+1}(\mathrm{map}(X, S^n), p) \longrightarrow \pi_k(\mathrm{map}(X, G), \bar{e}_0).$$

Of course, this needs some description of three objects:

$$\pi_{k+1}(\mathrm{map}(X, S^n), p), \quad \pi_k(\mathrm{map}(X, G), \bar{e}_0) \text{ and } \partial_{k+1}.$$

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Let $\xi: S^n \to B_G$ be the characteristic map of the bundle. Then $\partial_{k+1}$ is equivalent to the induced homomorphism

$$\xi_*: \pi_{k+1}(\mathrm{map}(X, S^n), p) \longrightarrow \pi_{k+1}(\mathrm{map}(X, B_G), \xi p),$$

and the target of $\xi_*$ is isomorphic to the group $\{\Sigma^{k+1}X, B_G\}$ because of the contractibility of $\xi p$. However, in general, we could not have a suitable description of $\pi_{k+1}(\mathrm{map}(X, S^n), p)$ compatible with the above identification. On the other hand, we noted that the space $X/G$ which is obtained from $X$ by collapsing $G$ to the base point is homeomorphic to a wedge sum $S^n \vee \Sigma^n G$, and as a fundamental lemma, we will show in §1 that there exists an isomorphism:

$$\pi_{k+1}(\mathrm{map}(X/G, Y), f) \cong \pi_{k+n+1}(Y) + \{\Sigma^{k+n+1}G, Y\}$$

having convenient properties for describing $\partial_{k+1}$.

Thus we can determine $\partial_{k+1}$ for elements contained in the image:

$$\pi_{k+1}(\mathrm{map}(X/G, S^n), \bar{p}) \longrightarrow \pi_{k+1}(\mathrm{map}(X, S^n), p),$$

and moreover this is enable us to calculate homotopy groups of certain subspaces of $\mathrm{map}(X, X)$ (Theorem 2.3 and 2.4).

Here we give an example. Let $p: S^7 \to S^3$ be the Hopf bundle and $\mathrm{map}(S^7; S^2)$ be the space of maps $(S^7, S^3) \to (S^7, S^3)$. Then we have

$$\pi_k(\mathrm{map}(S^7; S^3), 1) \cong \pi_{k+4}(S^7) + \pi_{k+7}(S^7) + \pi_{k+3}(S^3)$$

although we have $\pi_k(\mathrm{map}(S^7, S^7), 1) \cong \pi_{k+7}(S^7)$.

Through out this paper we denote by $\infty$ the base point and by $\overline{\infty}$ the constant map $X \to \infty$ for any $X$.

## § 1. A fundamental isomorphism.

Let $X$ be the space $A \cup D^n \times A$ obtained from the following identification:

$$(x, a) \equiv \xi(x, a), \quad \xi: S^{n-1} \times A \longrightarrow A, \quad \xi(\infty, a) = a,$$

and let $X/A$ be the space obtained from collapsing $A$ to $\infty$. In this section we are mainly concerned with homotopy groups

$$\pi_*(\mathrm{map}(X/A, Y), f).$$

First we note that there are the natural homeomorphism

$$j: (D^n \times A/S^{n-1} \times A, \infty) \longrightarrow (X/A, \infty)$$

and its induced homeomorphism

$$j_Y: (\mathrm{map}(X/A, Y), f) \longrightarrow (\mathrm{map}(D^n \times A/S^{n-1} \times A, Y), fj).$$

Hence our purpose is to study $\pi_*(\mathrm{map}(D^n\times A/S^{n-1}\times A, Y), fj)$. Consider the fibre space

$$r:\mathrm{map}(D^n\times A, Y)\longrightarrow \mathrm{map}(S^{n-1}\times A, Y)\quad .$$

defined by restricting the domain of maps on $S^{n-1}\times A$. Since the fibre $r^{-1}(\infty)$ is just considered as the space $\mathrm{map}(D^n\times A/S^{n-1}\times A, Y)$ we have a long exact sequence:

$$\longrightarrow \pi_*(\mathrm{map}(S^{n-1}\times A, Y), \infty)\longrightarrow \pi_*(\mathrm{map}(D^n\times A/S^{n-1}\times A, Y), fj)$$
$$\longrightarrow \pi_*(\mathrm{map}(D^n\times A, Y), fj')\longrightarrow .$$

On the other hand we have a commutative diagram

$$\begin{array}{ccc} \pi_*(\mathrm{map}(D^n\times A, Y), fj) & \longrightarrow & \pi_*(S^{n-1}\times A, Y), \overline{\infty}) \\ \downarrow & \swarrow & \\ \pi_*(\mathrm{map}(\infty\times A, Y), \overline{\infty}) & & \end{array}$$

where arrows denote homomorphism induced by maps analogus to the map $r$. Since it follows from the contractibility of the cell that the vertical arrow is isomorphic we can know the injectivity of $r_*$.

Hence from the long exact sequence we have

LEMMA 1.1. *There exists an isomorphism*

$$\pi_*(\mathrm{map}(D^n\times A/S^{n-1}\times A, Y), fj)$$
$$\cong \pi_{*+1}(\mathrm{map}(S^{n-1}\times A, Y), \overline{\infty})/\mathrm{proj.}_*\pi_{*+1}(\mathrm{map}(A, Y), \overline{\infty}).$$

And moreover we have the standard isomorphism

$$\{\Sigma^{*+1}(S^{n-1}\times A), Y\}\cong\{\Sigma^{*+1}S^{n-1}, Y\}+\{\Sigma^{*+1}A, Y\}+\{\Sigma^{*}(S^{n-1}\sharp A), Y\}$$

where $S^{n-1}\sharp A$ denotes the reduced join of $S^{n-1}$ with $A$.

Now combining these isomorphisms we have an isomorphism

$$\sigma_Y:\pi_*(\mathrm{map}(X/A, Y), f)\cong\{\Sigma^{*+1}S^{n-1}, Y\}+\{\Sigma^{*}(S^{n-1}\sharp A), Y\}$$

Thus our fundamental lemma is the following

LEMMA 1.2. (1) *$\sigma_Y$ is natural, i.e. for a map $\alpha: Y\to Z$ we have the commutative diagram*

$$\begin{array}{ccccc} \pi_*(\mathrm{map}(X/A, Y), g\alpha) & \overset{\sigma_Y}{\cong} & \{\Sigma^{*+1}S^{n-1}, Y\} & + & \{\Sigma^{*}(S^{n-1}\sharp A), Y\} \\ \alpha_*\downarrow & & \alpha_*\downarrow & & \alpha_*\downarrow \\ \pi_*(\mathrm{map}(X/A, Z), g) & \overset{\sigma_Z}{\cong} & \{\Sigma^{*+1}S^{n-1}, Z\} & + & \{\Sigma^{*}(S^{n-1}\sharp A), Z\} \end{array}$$

(2) *Let us consider the fibring*

$$\mathrm{map}(X/A, Y) \longrightarrow \mathrm{map}(X, Y) \longrightarrow \mathrm{map}(A, Y).$$

*Then there exists the following identification of the boundary*

$$\pi_{*+1}(\mathrm{map}(A, Y), \overline{\infty}) \longrightarrow \pi_*(\mathrm{map}(X/A, Y), f)$$

$$\{\Sigma^{*+1}A, Y\} \longrightarrow \{\Sigma^{*+1}S^{n-1}, Y\} + \{\Sigma^*(S^{n-1}\sharp A), Y\}$$

$$\beta \longrightarrow \beta\Sigma^{*+1}\lambda_\xi + \beta\Sigma^* c(\xi)$$

*where* $\lambda_\xi = \xi | S^{n-1}\times\infty$ *and* $c(\xi)$ *denotes the Hopf construction of* $\xi$.

*Proof.* (1) easily follows from definitions. Next consider the commutative diagram of fibrings

$$\begin{array}{ccccc} \mathrm{map}(X/A, Y) & \longrightarrow & \mathrm{map}(X, Y) & \longrightarrow & \mathrm{map}(A, Y) \\ \nwarrow & & \nwarrow & & \nwarrow \\ \mathrm{map}(D^n\times A/S^{n-1}\times A, Y) & \longrightarrow & \mathrm{map}(D^n\times A, Y) & \longrightarrow & \mathrm{map}(S^{n-1}\times A, Y). \end{array}$$

Then, back to Lemma 1.1 and using the commutativity of the homotopy exact sequences the proof is completed from expressing

$$\Sigma\xi : \Sigma(S^{n-1}\times A) \sim \Sigma S^{n-1} \vee \Sigma A \vee S^{n-1}\sharp A \longrightarrow \Sigma A$$

as $(\Sigma\lambda_\xi, \Sigma id, c(\xi))$.

Here we give a few direct consequence of Lemma 1.2. Define a subgroup of $\{\Sigma^* A, Y\}$ by

$$\Gamma_*(\xi : Y) = \{\alpha \mid \alpha\Sigma^*\lambda_\xi = 0 = \alpha\Sigma^{*-1}c(\xi)\}$$

and a quotient group of $\{S^{*+n}, Y\} + \{\Sigma^{*+n}A, Y\}$ by

$$\Delta_*(\xi : Y) = \{S^{*+n}, Y\} + \{\Sigma^{*+n}A, Y\} / \Delta\{\Sigma^{*+1}A, Y\}(\Sigma^{*+1}\lambda_\xi, \Sigma^* c(\xi))$$

where $\Delta$ is the diagonal map:

$$\{\Sigma^{*+1}A, Y\} \longrightarrow \{\Sigma^{*+1}A, Y\} + \{\Sigma^{*+1}A, Y\}$$

PROPOSITION 1.3. *If* $Y$ *is a topological group then there exists a short exact sequence*

$$\{0\} \longrightarrow \Delta_*(\xi : Y) \longrightarrow \pi_*(\mathrm{map}(X, Y), f) \longrightarrow \Gamma_*(\xi : Y) \longrightarrow \{0\}.$$

*Proof.* Define a homeomorphism $\phi : (\mathrm{map}(X, Y), f) \to (\mathrm{map}(X, Y), \overline{\infty})$ by

$$\phi(h)(x) = h(x)f(x)^{-1}.$$

Since $\phi_*$ in $\pi_*$ is an isomorphism the proof follows from applying Lemma 1.2 to the case $f = \overline{\infty}$.

Next, let $p$ be the map $X=A\cup D^n\times A\to S^n$ defined by

$$p|A=\infty \quad\text{and}\quad p|D^n\times A=(D^n/S^{n-1})(\text{proj.}\,D^n).$$

Since $p$ has a decomposition $X\to X/A\to S^n$ we have

PROPOSITION 1.4. *There exists a short exact sequence*

$$\{0\}\longrightarrow \Delta_*(\xi\,;S^n)\longrightarrow \pi_*(\text{map}(X, S^n), p)\longrightarrow \Gamma_*(\xi\,;S^n)\longrightarrow \{0\}.$$

For example, Let $X\to S^n$ be a $S^{m-1}$-bundle with its characteristic map $\xi: S^{n-1}\to 0(m)$. Then $\Gamma_*(\xi\,;S^n)$ and $\Delta_*(\xi\,;S^n)$ can be described as follows:

$$\Gamma_*(\xi\,;S^n)=\{\alpha\,|\,\alpha\Sigma^*\lambda=0=\alpha\Sigma^{*-1}J(\xi)\}\subset\pi_{*+m-1}(S^n)$$

and

$$\Delta_*(\xi\,;S^n)=\pi_{*+n}(S^n)+\pi_{*+n+m-1}(S^n)/\Delta\pi_{*+m}(S^n)(\Sigma^{*+1}\lambda,\ \Sigma^*J(\xi))$$

where $\lambda=\partial\iota_n$ is in the homotopy exact sequence of the bundle and $J$ denotes the $J$-homomorphism.

## § 2. Principal $G$-bundles over $S^n$.

Let $p: X\to S^n$ be a principal $G$-bundle with its characteristic map $\xi: S^n\to B_G$. We are interested in homotopy groups

$$\pi_*(\text{map}(X, X), 1_X), \qquad (*\geqq 0)$$

where $\pi_0$ denotes a semi-group of homotopy classes of maps $X\to X$ with the distinguished element $1_X$.

Let $\tilde{p}: \text{map}(X, X)\to\text{map}(X, S^n)$ be the fibre space associated with the bundle and let the fibre $\tilde{p}^{-1}(p)$ identify with the space $\text{map}(X, G)$ which is considered as a subspace of $\text{map}(X, X)$ by imbedding $f(x)\to x\cdot f(x)$ where $\cdot$ denotes the right action of $G$ on $X$ as usual.

Then we have the homotopy exact sequence of the fibring:

$$\longrightarrow \pi_*(\text{map}(X, G), \overline{\infty})\longrightarrow \pi_*(\text{map}(X, X), 1_X)$$

$$\longrightarrow \pi_*(\text{map}(X, S^n), p)\longrightarrow \pi_{*-1}(\text{map}(X, G), \overline{\infty})\longrightarrow$$

Here we note that $\text{map}(X, G)$ naturally is a topological group and hence $\pi_0$ also is a group. However, in this case the boundary is homomorphic but the inclusion is not homomorphic. Now we want to determine the boundary

$$\partial: \pi_{*+1}(\text{map}(X, S^n), p)\longrightarrow \pi_*(\text{map}(X, G), \overline{\infty}) \qquad *\geqq 0$$

The target can be identified with the group $\{\Sigma^*X, G\}$ but, in general, we have no expression like that about $\pi_*(\text{map}(X, S^n), p)$ (see 1.4).

However, under some conditions, this is isomorphic to a quotient group of

$\{S^{*+n}, S^n\}+\{\Sigma^{*+n}G, S^n\}$ so there is some posibility to calculate the boundary $\partial$.

Now considering following diagrams of fibrings:

$$\begin{array}{ccc} & E_G \longrightarrow B_G \\ \nearrow & \downarrow \qquad \downarrow \\ G & \\ \searrow & X \longrightarrow S^n \end{array} \quad \text{and} \quad \begin{array}{ccc} & \mathrm{map}(X, E_G) \longrightarrow \mathrm{map}(X, B_G) \\ \nearrow & \downarrow \qquad\qquad \downarrow \\ \mathrm{map}(G, X) & \\ \searrow & \mathrm{map}(X, X) \longrightarrow \mathrm{map}(X, S^n) \end{array}$$

where upper fibrings denote universal fibrings for $G$ and $\mathrm{map}(G, X)$ we know that $\partial$ is equivalent to $\xi_*^X$, i.e. we have a commutative diagram

$$\begin{array}{ccc} \pi_{*+1}(\mathrm{map}(X, S^n), p) & \xrightarrow[\partial]{} & \pi_*(\mathrm{map}(G, X), \infty) \\ & \searrow^{\xi_*} & \uparrow \partial \\ & & \pi_{*+1}(\mathrm{map}(X, B_G), \xi p). \end{array}$$

Let $q: X \to X/G$ be the map collapsing $G$ to $\infty$. Then $q$ induces a diagram

$$\begin{array}{ccc} \pi_*(\mathrm{map}(X, S^n), p) & \longrightarrow & \pi_*(\mathrm{map}(X, B_G), \xi p) \\ \uparrow q_{X/G*} & & \uparrow q_{B_G*} \\ \pi_*(\mathrm{map}(X/G, S^n), \bar{p}) & \longrightarrow & \pi_*(\mathrm{map}(X/G, B_G), \xi \bar{p}). \end{array}$$

Next, let $\mu$ be the multiplication of $G$ and let $c(\mu): G \sharp G \to G$ be the Hopf construction of $\mu$. For any space $Y$, define a homomorphism

$$\nabla_*(\xi : Y): \{\Sigma^{*+1}G, Y\} \longrightarrow \{S^{*+n}, Y\}+\{\Sigma^{*+n}G, Y\}$$

by $\nabla_*(\xi : Y)(\alpha)=\alpha\Sigma^{*+1}\lambda_\xi+\Sigma^*\alpha c(\mu)(\lambda_\xi \sharp 1_G)$ where $\lambda_\xi=\partial\xi$ for the boundary $\partial: \pi_n(B_G) \to \pi_{n-1}(G)$. Then, from Lemma 1.2 and above diagrams, we have

PROPOSITION 2.1. *Suppose that $q_{X/G*+1}$ is onto. Then the kernel of*

$$\partial_{*+1}: \pi_{*+1}(\mathrm{map}(X, S^n), p) \longrightarrow \pi_*(\mathrm{map}(X, G), \overline{\infty})$$

*is isomorphic to a subgroup of $\pi_{*+n+1}(S^n)+\{\Sigma^{*+n+1}G, S^n\}/\nabla_{*+1}(\{\Sigma^{*+2}G, S^n\})$, i.e. $(\xi_*)^{-1}\{\nabla_{*+1}(\xi : B_G)(\{\Sigma^{*+2}G, B_G\})/\nabla_{*+1}(\{\Sigma^{*+2}G, S^n\})$.*

Thus, since we have that $\pi_*(\mathrm{map}(G, S^n), \overline{\infty})=0$ $(*<n-\dim G)$ Proposition 2.1 implies

THEOREM 2.2. *There exists a short exact sequence* $(1 \leqq * < n-1-\dim G)$:

$$\{0\} \longrightarrow (\xi_{*+1})^{-1}(\nabla_{*+1}(\xi : B_G)\text{-image}) \longrightarrow \pi_{*+n+1}(S^n)+\{\Sigma^{*+1+n}G, S^n\}$$
$$\longrightarrow \{\Sigma^{*+1}X, B_G\} \longrightarrow \pi_*(\text{map}(X, X), 1_X) \longrightarrow \partial_*^{-1}(0) \longrightarrow \{0\}$$

Now we consider another fibrings. Let map$(X; G)$ be the subspace of map$(X, X)$ consisting with maps preserving $G$ into $G$, and let Fib. $X$ be the subspace of maps preserving fibres. Clearly these spaces are related with each other through fibrings as shown in the following diagram

$$\begin{array}{ccccc} & & \text{map}(X, G) & & \\ & & \downarrow & & \\ \text{Fib. } X & \longrightarrow & \text{map}(X, X) & \longleftarrow & \text{map}(X; G) \\ \downarrow & & \downarrow & & \downarrow \\ \text{map}(S^n, S^n) & \underset{\circ p}{\longrightarrow} & \text{map}(X, S^n) & \underset{q}{\longleftarrow} & \text{map}(X/G, S^n) \end{array}$$

Using the commutativity of the diagram, first we obtain from Lemma 1.2

THEOREM 2.3. *For* $*\geqq 1$ *there exist a short exact sequence*:

$$\{0\} \longrightarrow (\xi_{*+1})^{-1}(\nabla_{*+1}(\xi : B_G)\text{-image}) \longrightarrow \pi_{*+n+1}(S^n)+\{\Sigma^{*+n+1}G, S^n\}$$
$$\longrightarrow \{\Sigma^{*+1}X, B_G\} \longrightarrow \pi_*(\text{map}(X; G), 1_X)$$
$$\longrightarrow (\xi_*)^{-1}(\nabla_*(\xi ; B_G)\text{-image}) \longrightarrow \{0\}$$

Secondly we investigate the fibring:

$$\text{map}(X, G) \longrightarrow \text{Fib. } X \longrightarrow \text{map}(S^n, S^n).$$

We note that the fibre map $\circ p$ is decomposed as follows:

$$\text{map}(S^n, S^n) \longrightarrow \text{map}(X/G, S^n) \longrightarrow \text{map}(X, S^n)$$

Let $\phi_n$ be a map: $(D^n, S^{n-1})\rightarrow(S^n, \infty)$ of degree 1. Since we may consider that $\pi_*(\text{map}(S^n, S^n), 1_{S^n})$ is isomorphic to $\pi_*(\text{map}(D^n/S^{n-1}, S^n), \phi_n)$ the homomorphism $(\circ p)_*$ may be considered as the $(\text{proj.})_*$, i.e. we have a commutative diagram

$$\begin{array}{ccc} \pi_*(\text{map}(D^n/S^{n-1}, S^n), \phi_n) & \underset{\phi_{n*}}{\longrightarrow} & \pi_*(\text{map}(S^n, S^n), 1_{S^n}) \\ \downarrow (\text{proj.})_* & & \downarrow (\circ p)_* \\ \pi_*(\text{map}(D^n\times G/S^{n-1}\times G, S^n), \overline{\infty}) & \underset{j_*}{\longrightarrow} & \pi_*(\text{map}(X/G, S^n), \bar{p}) \end{array}$$

Hence, considering the fibre map:

$$\begin{array}{ccccc}
(\mathrm{map}(S^{n-1}, S^n), \overline{\infty}) & \longrightarrow & (\mathrm{map}(D^n, S^n), \phi_n) & \longrightarrow & (\mathrm{map}(D^n/S^{n-1}, S^n), \phi_n) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathrm{map}(S^{n-1}\times G, S^n), \overline{\infty}) & \longrightarrow & (\mathrm{map}(D^n\times G, S^n), \overline{\infty}) & \longrightarrow & (\mathrm{map}(D^n\times G/S^{n-1}\times G, S^n), \bar{p})
\end{array}$$

we have identifications:

$$\begin{array}{ccc}
\pi_*(\mathrm{map}(X/G, S^n), \bar{p}) & \underset{\bar{p}_*}{\longleftarrow} & \pi_*(\mathrm{map}(S^n, S^n), 1_{S^n}) \\
\cong\downarrow & & \cong\downarrow \\
\pi_*(\mathrm{map}(D^n\times G/S^{n-1}\times G, S^n), \bar{p}) & \underset{\mathrm{proj.}_*}{\longleftarrow} & \pi_*(\mathrm{map}(D^n/S^{n-1}, S^n), \phi) \\
\cong\downarrow & & \cong\downarrow \\
\{\Sigma^{*+1}S^{n-1}, S^n\}+\{\Sigma^{*+1}(S^{n-1}\# G), S^n\} & \longrightarrow & \{\Sigma^{*+1}S^{n-1}, S^n\}
\end{array}$$

Now we define a subgroup $T_*(\xi)$ of $\pi_{*+n}(B_G)$ by

$$T_*(\xi)=\{h\circ\Sigma^{*+1}\lambda_\xi \mid h\in\{\Sigma^{*+1}G, B_G\},\ h\circ\Sigma^*c(\mu)(\lambda_\xi\# 1_G)=0\}.$$

THEOREM 2.4. *There exists a short exact sequence* $(S_*(\xi)=\xi_*^{-1}T_*(\xi))$:

$$\{0\}\longrightarrow\pi_{*+n+1}(S^n)/S_{*+1}(\xi)\longrightarrow\{\Sigma^{*+1}X, B_G\}\longrightarrow\pi_*(\mathrm{Fib.}\ X, 1_X)\longrightarrow S_*(\xi)\longrightarrow\{0\}.$$

*Proof.* The proof is clearly completed from describing the boundary

$$\partial_*: \pi_*(\mathrm{map}(S^n, S^n), 1_{S^n})\longrightarrow\pi_{*-1}(\mathrm{map}(X, G), \overline{\infty})$$

and this can be read off in the following diagram with Lemma 1.2.

$$\begin{array}{ccccc}
\pi_*(\mathrm{Fib.}\ X, 1_X) \longrightarrow & \pi_*(\mathrm{map}(S^n, S^n), 1_{S^n}) & \underset{\partial_*}{\longrightarrow} & \pi_{*-1}(\mathrm{map}(X, G), \overline{\infty}) & \\
\swarrow\cong & \downarrow & & \downarrow & \uparrow \\
\pi_{*+n}(S^n)+\{\Sigma^{*+n}G, S^n\}\cong & \pi_*(\mathrm{map}(X/G, S^n), \bar{p}) & \longrightarrow & \pi_*(\mathrm{map}(X, S^n), p) & \cong \\
 & \downarrow\xi_* & & \downarrow\xi_* & | \\
\pi_{*+n}(B_G)+\{\Sigma^{*+n}G, B_G\}\cong & \pi_*(\mathrm{map}(X/G, B_G), \xi\bar{p}) & \longrightarrow & \pi_*(\mathrm{map}(X, B_G), \xi p). &
\end{array}$$

## § 3. Examples.

Let $p: X_\xi\to S^n$ be the $S^3$-principal bundle whose characteristic map is $\xi: S^n\to BS^3$, and we call it a $N$-suspension trivial bundle if the $N$-fold suspension of $\partial\xi$ is trivial.

If $X_\xi$ is $N$-suspension trivial it is known in [4] that $\Sigma^{N-4}X$ has the homotopy type of $S^{N+n-4}\vee S^{N-1}\vee S^{N+n-1}$. Hence from Theorem 2.3 we have

*Example* 3.1. If $p: X_\xi \to S^n$ is $N$-suspension trivial then there exists an exact sequence ($k \geqq N-1$)

$$\{0\} \longrightarrow \xi_{k+1}^{-1}(0) \longrightarrow \begin{matrix} \pi_{k+n+1}(S^n) \\ + \\ \pi_{k+n+4}(S^n) \end{matrix} \longrightarrow \begin{matrix} \pi_{k+n+1}(BS^3) \\ + \\ \pi_{k+n+4}(BS^3) \\ + \\ \pi_{k+4}(BS^3) \end{matrix} \longrightarrow \pi_k(\mathrm{map}\,(X_m\,;\,S^3),1) \longrightarrow \xi_k^{-1}(0) \to \{0\}$$

where $\xi_*$ is the induced homomorphism by $\xi$, i.e.

$$\xi_*: \pi_{*+n}(S^n) + \pi_{*+n+3}(S^n) \longrightarrow \pi_{*+n}(BS^3) + \pi_{*+n+3}(BS^3)\,.$$

*Remark.* In the case $N=1$ we have $\xi=0$.

Next, let $p_m: X_m \to S^2$ be the $S^1$-principal bundle with its characteristic map $S^1 \to S^1$ of degree $m$. Since $\{\Sigma^{*+1}X_m, BS^1\} = 0 = \pi_{*+2}(BS^1)$ for $* \geqq 1$ Theorem 2.4 gives

*Example* 3.2. $\pi_*(\mathrm{Fib.}\ X_m, 1_{X_m}) \cong \pi_{*+2}(S^2)$ $(* \geqq 1)$.

At last, considering the $S^1$-principal bundle $SO(3) \to S^2$, Proposition 1.3 gives

*Example* 3.3. There exists a short exact sequence

$$\{0\} \longrightarrow H(*, \eta) \longrightarrow \pi_*(\mathrm{map}\,(SO(3), SO(3)), 1) \longrightarrow G(*, \eta) \longrightarrow \{0\}$$

where $H(*, \eta)$ and $G(*, \eta)$ are defined respectively as follows:

$$H(*, \eta) = \pi_{*+2}(S^2) + \pi_{n+*+1}(S^2) / \{2\alpha + \alpha\Sigma\,\eta,\ \alpha\,\pi_{*+2}(S^2)\}$$

$$G(*, \eta) = \{2\alpha = 0 = \alpha\Sigma^{*-1}\eta\}\pi_{*+1}(S^2)\,.$$

Let $p: S^7 \to S^3$ be the Hopf bundle, in which we have $\partial\xi = \lambda_\xi = \iota: S^3 \to S^3$. Hence the followings follows from Theorem 2.3 and 2.4 respectively.

*Example* 3.4.

$$\pi_*(\mathrm{map}\,(S^7\,;\,S^3), 1_{S^7}) \cong \pi_{*+4}(S^7) + \pi_{*+7}(S^7) + \pi_{*+3}(S^3)$$

*Example* 3.5. For the Hopf map $\nu: S^7 \to S^3$ we define the homomorphism

$$(\Sigma^*\nu): \pi_{*+4}(S^3) \longrightarrow \pi_{*+7}(S^3)$$

by $(\Sigma^*\nu)(\alpha) = \alpha\Sigma^*\nu$. Then we have an exact sequence

$$\{0\} \longrightarrow \pi_{*+4}(S^3)/(\Sigma^*\nu)^{-1}(0) \longrightarrow \pi_{*+7}(S^3) \longrightarrow \pi_*(\mathrm{Fib.}\ S^7, 1)$$
$$\longrightarrow \pi_{*+4}(S^7) + (\Sigma^{*-1}\nu)^{-1}(0) \longrightarrow \{0\}.$$

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