

## ON THE QUANTITY $\delta_s(g(z), f)$ OF GAPPY ENTIRE FUNCTIONS

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### 1. Introduction.

Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be a transcendental entire function. We denote by  $\lambda_f = \{\lambda_k\}$ ,  $M_f = \{u_k\}$ ,  $k=1, 2, \dots$  the sequences of exponents  $n$  for which  $a_n \neq 0$  and  $a_n = 0$  respectively, arranged in increasing order. Also let  $g(z)$  be an arbitrary meromorphic function in the plane growing slowly compared with the function  $f(z)$ , i. e.,  $T(r, g) = o\{T(r, f)\}$  as  $r \rightarrow \infty$ .

If  $f(z)$  has finite order we define

$$\delta_s(g(z), f) = 1 - \lim_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}.$$

If  $f(z)$  has infinite order, let  $E$  be any set in  $(1, \infty)$  having finite length. We define

$$\delta_s(g(z), f) = 1 - \sup_E \lim_{r \rightarrow \infty, r \in E} \frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)} = \inf_E \overline{\lim}_{r \rightarrow \infty, r \in E} \frac{m\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}.$$

Obviously

$$\delta(g(z), f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)} \leq \delta_s(g(z), f).$$

In [1], we obtained the following

**THEOREM A.** *Let  $d_n$  be the highest common factor of all the numbers  $\lambda_{m+1} - \lambda_m$  for  $m \geq n$  and suppose that*

$$d_n \rightarrow \infty \quad \text{as } n \rightarrow \infty. \tag{1.1}$$

Then

$$\delta_s(g(z), f) = 0 \tag{1.2}$$

for every entire function  $g(z)$  satisfying  $T(r, g) = o\{T(r, f)\}$  as  $r \rightarrow \infty$ .

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In this paper we further prove that (1.2) is also valid for every meromorphic function  $g(z)$  satisfying  $T(r, g) = o\{T(r, f)\}$  as  $r \rightarrow \infty$ . That is, we shall prove

**THEOREM 1.** *With the hypotheses of Theorem A, we have*

$$\delta_s(g(z), f) = 0$$

for every meromorphic function  $g(z)$  satisfying  $T(r, g) = o\{T(r, f)\}$  as  $r \rightarrow \infty$ .

From (1.1), we see that  $A_f$  is very ‘thin’ in a sense. For the gappy entire function  $f(z)$  with ‘thickish’  $A_f$ , we have

**THEOREM 2.** *Suppose that  $b(\geq 0)$ ,  $d(\geq 2)$  are integers such that  $b + kd \in M_f$  for  $k \geq 1$ . Then for every meromorphic function  $g(z)$  satisfying  $T(r, g) = o\{T(r, f)\}$  as  $r \rightarrow \infty$ , we have*

$$\delta_s(g(z), f) \leq 1 - \frac{1}{d} \tag{1.3}$$

except possibly if, for  $0 < |z| < \epsilon$ ,  $g(z) = \frac{1}{z^m} \sum_{n=0}^{\infty} C_n z^n$  and  $C_{b+m} = a_b$ , where  $m(\geq 0)$  is an integer and  $\epsilon$  is a positive number.

*Remark 1.* (1.3) can break down if  $g(z) = \frac{1}{z^m} \sum_{n=0}^{\infty} C_n z^n$  and  $C_{b+m} = a_b$ . For

$$f(z) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{z^{2n}}{(n+3)!}$$

satisfies the hypotheses of Theorem 2 with  $b=1$  and  $d=2$ . But for  $g(z) = -(1/z^6)(1+z^2+z^4/2)$ , with  $m=6$  and  $C_{b+m} = a_b = 0$ , we have

$$f(z) - g(z) = \frac{1}{z^6} \left( 1 + z^2 + \frac{z^4}{2} + \frac{z^6}{6} + \sum_{n=1}^{\infty} \frac{z^{2(n+3)}}{(n+3)!} \right) = \frac{e^{z^2}}{z^6}$$

and so

$$\delta_s(g(z), f) = 1.$$

Also for

$$f(z) = z + e^{z^2}$$

satisfies the hypotheses of Theorem 2 with  $b=1$  and  $d=2$ . But for  $g(z) = z$ , with  $m=0$  and  $C_{b+m} = 1 = a_b$ , we have

$$\delta_s(z, f) = 1.$$

*Remark 2.* The inequality (1.3) is sharp. For  $f(z) = -\frac{\sqrt{\pi}}{2} + \int_0^z e^{-t^2} dt$  satisfies the hypotheses of Theorem 2 with  $b=0$  and  $d=2$ . Hence we have  $\delta_s(0, f) \leq 1 - 1/2 = 1/2$ . Also as is shown by Nevenlinna [5], we have  $\delta(0, f) = 1/2$ . Thus

$\delta_s(0, f) \geq \delta(0, f) = 1/2$ . Therefore  $\delta_s(0, f) = 1/2$ .

As a consequence of Theorem 2, we have

**COROLLARY 1.** *Suppose that  $b(>0)$ ,  $d(\geq 2)$  are integers such that  $b \in A_f$  but  $b+kd \in M_f$  for  $k \geq 1$ . Then*

$$\sum_{a \neq \infty} \delta(a, f) \leq 1 - \frac{1}{d}. \tag{1.4}$$

*Remark 3.* (1.4) is obviously better than the corresponding result of Hayman in [3, p. 330]:  $\sum_{a \neq \infty} \delta(a, f) \leq 1 - \frac{1}{d(d-1)}$  with the hypotheses of Corollary 1.

**2. Proof of Theorems.**

From now on we denote by  $S(r, F)$  any term which satisfies  $S(r, F) = o\{T(r, F)\}$  as  $r \in E$ ,  $r \rightarrow \infty$ , where  $E$  is a set in  $(1, +\infty)$  having finite length and in particular  $E$  is an empty set if  $F(z)$  is of finite order.

In proving our theorems, we shall quote the following lemmas.

**LEMMA 1.** [2]: *If  $F(z)$  is a transcendental entire function and  $g_1(z), g_2(z), \dots, g_q(z)$  are distinct meromorphic functions satisfying  $T(r, g_j) = o\{T(r, F)\}$  as  $r \rightarrow \infty$  ( $j=1, 2, \dots, q$ ), then*

$$\sum_{j=1}^q m\left(r, \frac{1}{F-g_j(z)}\right) \leq T(r, F) + S(r, F).$$

**LEMMA 2** [4]: *If  $a_1, \dots, a_q$  are distinct finite complex numbers, then whenever  $F(z)$  is transcendental entire function we have*

$$\sum_{j=1}^q m\left(r, \frac{1}{F-a_j}\right) \leq m\left(r, \frac{1}{F'}\right) + S(r, F).$$

*Proof of Theorem 1.*

By our Theorem A, we may assume that  $g(z)$  is a meromorphic function which has at least a pole and satisfies

$$T(r, g) = o\{T(r, f)\} \quad \text{as } r \rightarrow \infty \tag{2.1}$$

Let's discuss two cases separately.

Case (a).  $g(z)$  has a pole  $Z_0 \neq 0$ .

Let  $q(\geq 2)$  be an arbitrary integer. We write

$$\begin{aligned} W_n &= \exp\left(\frac{2\pi i}{d_n}\right), \\ P(z) &= \sum_{v=0}^{n-1} a_{\lambda v} z^{\lambda v}, \end{aligned} \tag{2.2}$$

$$F(z) = \sum_{v=n}^{\infty} a_{\lambda_v} z^{\lambda_v}. \quad (2.3)$$

Obviously,  $\lim_{n \rightarrow \infty} (Wn)^j Z_0 = Z_0$  ( $1 \leq j \leq q$ ). Hence there exists a larger  $n$  such that  $(Wn)^j Z_0$  ( $1 \leq j \leq q$ ) is not a pole of  $g(z)$ .

Noticing  $d_n | (\lambda_{m+1} - \lambda_m)$  for  $m \geq n$ , we have

$$d_n | (\lambda_v - \lambda_n) \quad \text{for } v \geq n,$$

where " $a|b$ " denotes that  $a$  is a factor of  $b$ . Thus for  $1 \leq j \leq q$  we have, in view of (2.2) and (2.3),

$$\begin{aligned} f(W_n^j z) - g(W_n^j z) &= p(W_n^j z) - g(W_n^j z) + F(W_n^j z) \\ &= p(W_n^j z) - g(W_n^j z) + W_n^{j\lambda_n} F(z) \\ &= W_n^{j\lambda_n} \{F(z) - W_n^{-j\lambda_n} (g(W_n^j z) - p(W_n^j z))\} \\ &= W_n^{j\lambda_n} \{F(z) - g_j(z)\}, \end{aligned} \quad (2.4)$$

say, where  $g_j(z) = W_n^{-j\lambda_n} (g(W_n^j z) - p(W_n^j z))$ .

Clearly, by (2.1) we have

$$T(r, g_j(z)) = o\{T(r, f)\} \quad \text{as } r \rightarrow \infty. \quad (2.5)$$

We assert that  $g_1(z), g_2(z), \dots, g_q(z)$  are distinct from each other. Otherwise, there must exist  $i, j$  ( $1 \leq i < j \leq q$ ) such that

$$\Delta_{ij}(z) = Wn^{-j\lambda_n} (g(Wn^j z) - p(Wn^j z)) - Wn^{-i\lambda_n} (g(Wn^i z) - p(Wn^i z)) \equiv 0. \quad (2.6)$$

But by the definition of the integer  $n$  we easily conclude that

$$\Delta_{ij}(Wn^{-i} z_0) = Wn^{-j\lambda_n} (g(Wn^{j-i} z_0) - p(Wn^{j-i} z_0)) - Wn^{i\lambda_n} (g(z_0) - p(z_0)) = \infty.$$

This contradicts (2.6).

Now using Lemma 1 to our functions  $g_1(z), g_2(z), \dots, g_q(z)$  and  $F(z)$  and noticing (2.4) and (2.5), we deduce that

$$\sum_{j=1}^q m\left(r, \frac{1}{F - g_j(z)}\right) \leq T(r, F) + S(r, F)$$

and so that

$$\begin{aligned} qm\left(r, \frac{1}{f - g(z)}\right) &\leq T(r, F) + S(r, F) \\ &\leq T(r, f) + S(r, f). \end{aligned}$$

Thus we have

$$\delta_s(g(z), f) \leq \frac{1}{q}.$$

But  $q$  can be assumed any large. Hence we obtain

$$\delta_s(g(z), f) = 0.$$

Case (b).  $Z_0 = 0$  is an only pole of  $g(z)$ .

It is easily seen that there exists an integer  $m \geq 1$  such that

$$g(z) = \frac{1}{z^m} g^*(z),$$

where  $g^*(z)$  is an entire function satisfying

$$T(r, g^*(z)) = o\{T(r, f)\} \quad \text{as } r \rightarrow \infty. \tag{2.7}$$

we set

$$\begin{aligned} f^*(z) &= z^m f(z) \\ &= z^m \sum_{n=1}^{\infty} a_n z^{\lambda_n} = \sum_{n=1}^{\infty} a_{\lambda_n} z^{\eta_n}, \end{aligned}$$

say, where  $\eta_n = \lambda_n + m$ .

By (2.7) we have

$$T(r, g^*(z)) = o\{T(r, f^*)\} \quad \text{as } r \rightarrow \infty. \tag{2.8}$$

Let  $dn^*$  be the highest common factor of all the numbers  $\eta_{s+1} - \eta_s$  for  $s \geq n$ . Apparently we have

$$dn^* \rightarrow \infty \quad \text{as } n \rightarrow \infty. \tag{2.9}$$

Using Theorem A to our functions  $f^*(z)$  and  $g^*(z)$  and noticing (2.8) and (2.9), we get

$$\delta_s(g^*(z), f^*(z)) = 0.$$

But

$$f(z) - g(z) = \frac{1}{z^m} (f^*(z) - g^*(z)).$$

Therefore

$$\delta_s(g(z), f) = \delta_s(g^*(z), f^*) = 0.$$

This completes the proof of Theorem 1.

*Proof of Theorem 2.*

Suppose that  $g(z)$  is any meromorphic function satisfying

$$T(r, g) = o\{T(r, f)\} \quad \text{as } r \rightarrow \infty \tag{2.10}$$

we write

$$W = \exp\left(\frac{2\pi i}{d}\right),$$

$$h_v(z) = W^{-vb} \{f(W^v z) - g(W^v z)\} \quad (1 \leq v \leq d), \tag{2.11}$$

$$h(z) = \sum_{v=1}^d h_v(z). \tag{2.12}$$

We then have, for any integer  $n(\geq 0)$ , that  $\sum_{v=1}^d W^{v(n-b)}=d$  if  $d|(n-b)$  and  $\sum_{v=1}^d W^{v(n-b)}=0$  if  $d \nmid (n-b)$ .

So by  $b+kd \in M_f$  for  $k \geq 1$  we deduce that

$$\begin{aligned} h(z) &= \sum_{v=1}^d h_v(z) \\ &= \sum_{v=1}^d W^{-vb} \sum_{n=0}^{\infty} a_n W^{vn} z^n - \sum_{v=1}^d W^{-vb} g(W^v z) \\ &= d \sum_{k=k_0}^0 a_{b+kd} z^{b+kd} - \sum_{v=1}^d W^{-vb} g(W^v z), \end{aligned}$$

where  $k_0$  is the minimum of integers  $k$  ( $-\infty < k \leq 0$ ) satisfying  $b+kd \geq 0$ .

By (2.10), it is easily deduced that

$$T(r, h(z)) = o\{T(r, f)\} \quad \text{as } r \rightarrow \infty. \tag{2.13}$$

We claim that

$$h(z) \neq 0, \tag{2.14}$$

if the exceptional case mentioned in Theorem 2 doesn't occur.

In fact, we may write

$$g(z) = \frac{1}{z^m} \sum_{n=0}^{\infty} C_n z^n \quad \text{as } 0 < |z| < \varepsilon,$$

where  $m(\geq 0)$  is an integer and  $\varepsilon$  is a small positive number.

Hence for  $0 < |z| < \varepsilon$  we have

$$\begin{aligned} h(z) &= d \sum_{k=k_0}^0 a_{b+kd} z^{b+kd} - \sum_{n=0}^{\infty} C_n z^{n-m} \sum_{v=1}^d W^{v(n-m-b)} \\ &= d \left( \sum_{k=k_0}^0 a_{b+kd} z^{b+kd} - \sum_{k=k_1}^{\infty} C_{m+b+kd} z^{b+kd} \right), \end{aligned} \tag{2.15}$$

where  $k_1$  is the minimum of integer  $k$  ( $-\infty < k \leq 0$ ) satisfying  $m+b+kd \geq 0$ .

In (2.15), if  $z=0$  is a pole of  $h(z)$ , then (2.14) is already correct. Or  $h(z)$  is holomorphic in  $\{|z| < \varepsilon\}$  and (2.15) is its power-series expansion in  $|z| < \varepsilon$ . But the coefficient of  $z^b$  in  $h(z)$  is  $d(a_b - C_{m+b}) \neq 0$ . Thus (2.14) is also correct.

Next we assume without loss of generality that  $h_1(z), h_2(z), \dots, h_t(z)$  ( $t \leq d$ ) are linearly independent. Then there exist complex numbers  $e_1, e_2, \dots, e_t$  such that

$$\begin{aligned} h(z) &= e_1 h_1(z) + e_2 h_2(z) + \dots + e_t h_t(z) \\ &= H_1(z) + H_2(z) + \dots + H_t(z), \end{aligned} \tag{2.16}$$

say, where  $H_j(z) = e_j h_j(z)$  ( $1 \leq j \leq t$ ).

Also we may assume that  $e_j \neq 0$  for  $1 \leq j \leq t$ , which doesn't, lose generality.

Hence if we set

$$\Delta = \begin{vmatrix} H_1(z) & H_2(z) & \cdots & H_t(z) \\ H_1'(z) & H_2'(z) & \cdots & H_t'(z) \\ \cdots & \cdots & \cdots & \cdots \\ H_1^{(t-1)}(z) & H_2^{(t-1)}(z) & \cdots & H_t^{(t-1)}(z) \end{vmatrix},$$

then we have  $\Delta \neq 0$ .

By (2.16), we conclude that

$$\begin{aligned} \Delta &= \begin{vmatrix} h(z) & H_2(z) & \cdots & H_t(z) \\ h'(z) & H_2'(z) & \cdots & H_t'(z) \\ \cdots & \cdots & \cdots & \cdots \\ h^{(t-1)}(z) & H_2^{(t-1)}(z) & \cdots & H_t^{(t-1)}(z) \end{vmatrix} \\ &= h(z)\Delta_0(z) + h'\Delta_1(z) + \cdots + h^{(t-1)}(z)\Delta_{t-1}(z), \end{aligned} \tag{2.17}$$

where  $\Delta_j(z)$  ( $0 \leq j \leq t-1$ ) is the algebraic complement with respect to  $h^{(j)}(z)$  ( $h^{(0)}(z) \equiv h(z)$ ).

It is well known that  $m(r, F^{(k)}/F) = S(r, F)$  for  $k \geq 1$  whenever  $F(z)$  ( $\neq$  constant) is a meromorphic function [4]. Hence we deduce that, in view of (2.11), (2.17) and (2.13),

$$\begin{aligned} T(r, f) &\leq T(r, H_1(z)) + o\{T(r, f)\} \\ &\leq N(r, H_1(z)) + m\left(r, \frac{\sum_{v=0}^{t-1} h^{(v)} \Delta_v}{\prod_{v=2}^t H_v}\right) + m\left(r, \frac{\prod_{v=1}^t H_v}{\Delta}\right) + o\{T(r, f)\} \\ &\leq N(r, H_1(z)) + \sum_{v=0}^{t-1} m(r, h^{(v)}) + \sum_{v=0}^{t-1} m\left(r, \frac{\Delta_v}{\prod_{j=2}^t H_j}\right) + m\left(r, \frac{\Delta}{\prod_{v=1}^t H_v}\right) \\ &\quad + N\left(r, \frac{\Delta}{\prod_{v=1}^t H_v}\right) + o\{T(r, f)\} \\ &\leq N(r, H_1(z)) + \sum_{v=0}^{t-1} m\left(r, \frac{h^{(v)}}{h}\right) + tm(r, h) + \sum_{v=0}^{t-1} m\left(r, \frac{\Delta_v}{\prod_{j=2}^t H_j}\right) \\ &\quad + m\left(r, \frac{\Delta}{\prod_{v=1}^t H_v}\right) + N(r, \Delta) + \sum_{v=1}^t N\left(r, \frac{1}{H_v}\right) + o\{T(r, f)\} \\ &\leq o\{T(r, f)\} + S(r, h) + o\{T(r, f)\} + \sum_{j=2}^t S(r, H_j) + \sum_{j=1}^t S(r, H_j) \\ &\quad + o\{T(r, f)\} + tN\left(r, \frac{1}{f-g(z)}\right) + o\{T(r, f)\} \end{aligned}$$

$$\leq tN\left(r, \frac{1}{f-g(z)}\right) + S(r, f).$$

That is,

$$T(r, f) \leq tN\left(r, \frac{1}{f-g(z)}\right) + S(r, f).$$

Therefore we conclude that

$$\delta_s(g(z), f) \leq 1 - \frac{1}{t} \leq 1 - \frac{1}{d}.$$

The proof of Theorem 2 is thus completed.

Using Theorem 2 to the functions  $f'(z)$  and  $g(z)=0$  and Lemma 2 to the function  $f$ , we easily get the conclusion of Corollary 1. We here omit its details.

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