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THE CONDITION FOR AN APPROXIMATION OF POISSON DISTRIBUTION TO BERNOULLI SUMS IN MULTIVARIATE DISTRIBUTION

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§ 1. Summary.

K. Kawamura [1] has discussed that a condition is shown and it plays as sufficient condition for an approximation of Poisson distribution to the sum of Bernoulli sequences and he has investigated the structure of Poisson distribution in multivariate case. C. Liu [2] also has discussed an approximation to the sum of variable (non-identically distributed) Bernoulli sequences.

In this paper the converse assertion is discussed, that is, the condition is essential for the approximation of Poisson distribution to the sum of independent Bernoulli sequences in multivariate case. The notations and discussion will prepare the break through in the case of variable Bernoulli sequences.

§ 2. Notations and definitions.

$k=(k_1, k_2, \cdots, k_n)$
where coordinates k_j $(j=1, 2, \cdots, n)$ are non-negative integers,
$0=(0, 0, \cdots, 0)$; zero-vector,
$E_0=\{0, 1\}^n$, $E=\{0, 1\}^n-0$, $i\in E_0$,
$\#k$; the number of positive coordinates in a vector k.

An ordering for $i\in E_0$ in 3-dimensional case $(n=3)$;

$$i=\begin{cases}(0,0,0)=000 & \#i=0,\\ (1,0,0)=100 & \\ (0,1,0)=010 & \#i=1,\\ (0,0,1)=001 & \\ (1,1,0)=110 & \\ (1,0,1)=101 & \#i=2,\\ (0,1,1)=011 & \\ (1,1,1)=111 & \#i=3.\end{cases} \tag{2.1}$$

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We use the ordering for $i\in E_0$, like above, in multivariate case, and also the ordering for $i\in E$.

$P(\lambda)$: n-variate Poisson distribution

$\lambda=(\lambda_{10\cdots0}, \lambda_{010\cdots0}, \cdots, \lambda_{0\cdots01}, \cdots, \lambda_{11\cdots1})$: 2^n-1 dimensional parameter each coordinate λ_k is non-negative parameter where suffix vectors $k=100\cdots0, 010\cdots0, \cdots, 00\cdots01, \cdots, 11\cdots1$ are ordered by a given ordering like above (2.1). And also we put

$p=(p_{00\cdots0}, p_{10\cdots0}, p_{010\cdots0}, \cdots, p_{0\cdots01}, \cdots, p_{11\cdots1})$: 2^n-dimensional vector.

§ 3. Main result.

THEOREM 1. *For given independent Bernoulli sequences $X_1, X_2, \cdots, X_N$ each having a distribution $B(1, p)$, the sum $X=\sum_{j=1}^{N} X_j$ has a binomial distribution $B(N, p)$. The distribution is expressed for $k\in\{0, 1, \cdots, N\}^n$*

$$P(X=k)=\sum_{[C_0]}\frac{N!}{\prod_{i\in E_0}\alpha_i!}\prod_{i\in E_0}p_i^{\alpha_i}. \tag{3.1}$$

Proof. See Kawamura [1].

THEOREM 2. *Let X be a binomial distribution $B(N, p)$ then we have for any $k\geqq 0$*

$$\lim_{\substack{N\to\infty\\ Np_i\to\lambda_i,\, i\in E}} P(X=k)=\sum_{[C]}\prod_{i\in E}p(\alpha_i;\lambda_i) \tag{3.2}$$

where $p(\alpha;\lambda)$ is an univariate Poisson probability density.

Proof. See Kawamura [1].

In this paper we will show the fact that the limiting condition

$$\text{“}Np_i\longrightarrow\lambda_i \quad \text{as } N\longrightarrow\infty \qquad \text{for } i\in E\text{”}$$

is essential to the approximation.

THEOREM 3. *For given independent Bernoulli sequence $X_1, X_2, \cdots, X_N$ each having a distribution $B(1, p)$ if we assume that the sum $X=\sum X_j$ has the property of Poisson approximation:*

$$\lim_{N\to\infty}P(X=k)=\sum_{[C]}\prod_{i\in E}p(\alpha_i;\lambda_i) \qquad (k\geqq 0) \tag{3.3}$$

then we can derive the condition

$$\text{“}Np_i\longrightarrow\lambda_i \quad as\ N\longrightarrow\infty \qquad for\ i\in E\text{”} \tag{3.4}$$

Proof. Step 1) Defining $\#v$ is a number of positive components in the vector v (usually components of v are nonnegative integers).

If $\# k=0$ then $k=0=(0, 0, \cdots, 0)$. Put $k=0$ then in the left side of limiting equation (3.3) becomes

$$P(X=0)=\sum_{[C_0]} \frac{N!}{\prod_{i\in E_0} \alpha_i !} \prod_{i\in E_0} p_i^{\alpha_i}=p_o^N .$$

Solution of $[C_0]$ with $k=0$ is simply expressed

$$\alpha_i=\begin{cases} N, & i=0 , \\ 0, & i\neq 0 \end{cases}$$

for $i\in E_0$, then we get

$$P(X=O)=p_o^N ,$$

and left side of (3.3) becomes

$$\lim_{N\to\infty} P(X=O)=\lim_{N\to\infty} p_o^N .$$

Solution of $[C]$ with $k=0$ is also expressed

$$\alpha_i=0 , \qquad \text{for } i\in E$$

then right side of (3.3) becomes

$$\sum_{[C]} \prod_{i\in E} p(\alpha_i ; \lambda_i)=\prod_{i\in E} p(0 ; \lambda_i)=\exp\{-\sum_{i\in E} \lambda_i\}.$$

Therefore, we get

$$\lim_{N\to\infty} p_o^N=\exp\{-\sum_{i\in E} \lambda_i\}, \tag{3.5}$$

where

$$p_o=1-\sum_{i\in E} p_i$$

$$\lim_{N\to\infty} \left\{1-\frac{N\sum p_i}{N}\right\}^N=\exp\{-\sum_{i\in E} \lambda_i\}.$$

Then we can conclude

$$\lim_{N\to\infty} N\sum_{i\in E} p_i=\lim \sum_{i\in E} Np_i=\sum_{i\in E} \lambda_i .$$

LEMMA. *Under the condition of theorem* 3, *we have*

$$\lim_{N\to\infty} N\sum_{i\in E} p_i=\sum_{i\in E} \lambda_i . \tag{3.6}$$

Step 2) In the case of $k=(100\cdots 0)$ the solution of $[C_0]$ with $k=(100\cdots 0)$ is

$$\alpha_o=N-1 ,$$

$$\alpha_k=1 ,$$

$$\alpha_i=0\,,\quad i\neq 0,\ k \qquad \text{for } i\in E_0\,.$$

The solution of $[C]$ is $\alpha_k=1$ and $\alpha_i=0$ for $i\neq k$, $i\in E$ then from (3.3)

$$\lim_{N\to\infty} N\,p_k\,p_o^{N-1}=p(1\,;\lambda_k)\prod_{i\in E,\,i\neq k} p(0\,;\lambda_i)$$

then

$$\lim_{N\to\infty} N\,p_k \lim_{N\to\infty} p_o^{N-1}=\lambda_k \prod_{i\in E}\exp\{-\lambda_i\}$$

and us'ng (3.6) we can get

$$\lim_{N\to\infty} N\,p_k=\lambda_k\,. \tag{3.7}$$

In the same way, we get (3.7) for any k satisfying $\#k=1$; that is, under the condition of the theorem if $k\in E$ and $\#k=1$ then we have

$$\lim_{N\to\infty} N\,p_k=\lambda_k\,.$$

Step 3) Let us proceed to prove the conclusion (3.4) of theorem 3 that for any $k\in E$ we have

$$\lim_{N\to\infty} N\,p_k=\lambda_k\,,$$

by the induction of the number of positive components of $k\in E:\#k=r\ (1\leqq r\leqq n)$. In step 2 we have proved that the conclusion of the theorem is valid for $r=1$.

HYPOTHESIS OF THE INDUCTION. If we assume the conclusion also valid for every $r:r\leqq r_0$ where $1\leqq r_0\leqq n$, then we can prove the relation of (3.7) for $r=r_0+1$, as follows and finish the induction.

Put $k\in E$, $\#k=r_0+1$ then if we have to decompose the vector k as following

$$k=j_1+j_2+\cdots+j_s \tag{3.8}$$

where $j_1, j_2, \cdots, j_s$ are n-dimensional vectors and having nonnegative integral components, so $j_1, j_2, \cdots, j_s\geqq 0$ and $j_1, j_2, \cdots, j_s\neq 0$ are satisfied.

Let us define the relation of vectors V and $O: V>O \Leftrightarrow V\geqq O$ and $V\neq O$, then the decomposed vectors must satisfy $j_1, j_2, \cdots, j_s>0$. If the vector k in the left side of (3.3) is the sum of N independent Bernoulli vectors having a distribution $B(1, p)$ then we express a sequence of N observations:

$$k\geqq j_1, j_2, \cdots, j_s, \underbrace{O, O, \cdots, O}_{N-s}\geqq O\,,\qquad (\#k=r_0+1)\,.$$

That is a decomposition of k is given by

$$k=j_1+j_2+\cdots+j_s+\underbrace{O+O+\cdots+O}_{N-s}\,. \tag{3.9}$$

The probability of an occurrence of the decomposition is

$$p_{j_1}\, p_{j_2} \cdots p_{j_s} \underbrace{p_o \cdots p_o}_{N-s}.$$

The combination for the decomposition satisfying (3.9) is

$$\frac{N!}{1!\,1!\cdots 1!(N-s)!}=N(N-1)\cdots(N-s+1).$$

Therefore the probability of the decomposition having $j_1, j_2, \cdots, j_s > O$ becomes

$$N(N-1)\cdots(N-s+1)p_{j_1}\, p_{j_2} \cdots p_{j_s}\, p_o^{N-s}.$$

To calculate $P(X=k)$ of (3.3) we have to summarize such probabilities for all decompositions satisfying (3.9).

$$\sum_{s=1}^{r+1} \sum_{\substack{j_1+j_2+\cdots+j_s=k,\\ 0<j_1, j_2, \cdots, j_s \leqq 1}} N(N-1)\cdots(N-s+1)p_{j_1}\, p_{j_2} \cdots p_{j_s}\, p_o^{N-s} \tag{3.10}$$

where $O=(0, 0, \cdots, 0)$ and $1=(1, 1, \cdots, 1)$ are n-dimensional vectors. And from the condition (3.3) of the theorem, the fact (3.6) in the lemma is already proved:

$$\lim_{N\to\infty} p_o^N=\exp\{-\sum_{i\in E}\lambda_i\}$$

and the hypothesis of the induction we have

$$\begin{aligned}\lim_{N\to\infty} P(X=k)&=\sum_{s=1}^{r+1}\sum \lim N\, p_{j_1}(N-1)p_{j_2}\cdots(N-s+1)p_{j_s}\lim p_o^{N-s}\\ &=\lim N\, p_k \lim p_o^{N-1}+\sum_{s=2}^{r+1}\sum\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_s}\exp\{-\sum_{i\in E}\lambda_i\} \qquad (3.11)\\ &=[\lim N\, p_k+\sum_{s=2}^{r+1}\sum\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_s}]\exp\{-\sum_{i\in E}\lambda_i\}.\end{aligned}$$

On the other hand, in the right side of (3.3) the solution of $[C]$ with $\#k=r_0+1$ is expressed

$$\alpha_{j_1}=\alpha_{j_2}=\cdots=\alpha_{j_s}=1 \quad\text{and}\quad \alpha_i=0 \qquad \text{for } i\neq j_1, j_2, \cdots, j_s$$

where

$$j_1+j_2+\cdots+j_s=k \quad\text{and}\quad 0<j_1, j_2, \cdots, j_s \leqq 1.$$

And $\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_s}$ are consist with the numbers of the decomposed vectors. $j_1, j_2, \cdots, j_s$ in (3.8). In the right side of (3.3)

$$\sum_{[C]}\prod_{i\in E} p(\alpha_i\,;\lambda_i),$$

the solution of $[C]$ is depending on the decomposition and we have $\alpha_i=0$ or 1 from $k\in E$. We have $\alpha_i=1$ if $i\in\{j_1, j_2, \cdots, j_s\}$ of the decomposition and otherwise $\alpha_i=0$ for $i\in E$. And we can check from (3.8) and $k\in E$ the decomposed vectors $j_1, j_2, \cdots, j_s$ are mutually different vectors.

$$P(X=k)=\sum_{[C]}\prod_{i=1}^{s} p(\alpha_{j_i}\,;\lambda_{j_i})\prod_{i\in E,\, i\neq j_1, j_2, \cdots, j_s} p(0\,;\lambda_i)$$

$$=\sum_{s=1}^{\#k}\sum_{\mathrm{Dec}}\prod_{i=1}^{s}\lambda_{j_i}\exp\{-\lambda_{j_i}\}\prod_{i\in E,\, i\neq j_1, j_2, \cdots, j_s}\exp\{-\lambda_i\}$$

where

$$\mathrm{Dec}=\{j_1, j_2, \cdots, j_s : j_1+j_2+\cdots+j_s=k,\ 0<j_i\leqq 1 \text{ and } 0<j_i\leqq k\}$$

$$=\sum_{s=1}^{\#k}\sum_{\mathrm{Dec}}\prod_{i=1}^{s}\lambda_{j_i}\prod_{i\in E}\exp\{-\lambda_i\}$$

$$=\lambda_k\exp\{-\sum_{i\in E}\lambda_i\}+\sum_{s=2}^{\#k}\sum_{\mathrm{Dec}}\prod_{i=1}^{s}\lambda_{j_i}\exp\{-\sum_{i\in E}\lambda_i\} \tag{3.12}$$

Finally we can conclude by (3.11) and (3.12)

$$\lim_{N\to\infty} Np_k=\lambda_k \qquad \text{for } \#k=r+1\,.$$

Now, we have finished the induction: the proof of the validity of (3.7) in the theorem for every k satisfying $\#k=r+1$. So we can conclude for any $k: k\in E$ and $\#k=1, 2, \cdots, n$

$$\lim_{N\to\infty} Np_k=\lambda_k\,.$$

This is the conclusion of the theorem. ■

These theorems have a variation theorem rather a mathematical one, that is, summarizing Theorem 2 and 3 we conclude next theorem.

THEOREM 4. *Necessary and sufficient condition for the convergence of p.g.f. of $B(N, p)$ to p.g.f. of $P(\lambda)$:*

$$\lim_{N\to\infty}(\sum_{i\in E_0} p_i s^i)^N=\prod_{i\in E}\exp\{-\lambda_i+\lambda_i s^i\} \tag{3.13}$$

is

$$\lim_{N\to\infty} Np_i=\lambda_i \qquad \textit{for } i\in E\,. \tag{3.7}$$

That is the condition "$\lim_{N\to\infty} Np_i=\lambda_i$ for $i\in E$" (3.4) is essential for the convergence of distribution.

§ 4. Bivariate case.

Let $X_1, X_2, \cdots, X_N$ be a sequence of independent Bernoulli distribution $B(1, p)$ where $p=(p_{00}\ p_{10}\ p_{01}\ p_{11})$, $p_{ij}\geqq 0$ and $\sum p_{ij}=1$ $(\sum_{i\in E_0} p_i=1)$

$$P(X_j=i)=p_i\,,\quad i\in E_0\,,\qquad \text{for } j=1, 2, \cdots, N\,.$$

Then from theorem 1 we have

$$P\Big(\sum_{j=1}^{N} X_j = k\Big) = \sum_{\substack{\beta+\delta=k_1 \\ \gamma+\delta=k_2 \\ \alpha+\beta+\gamma+\delta=N \\ \alpha,\beta,\gamma,\delta\geq 0 \text{ integer}}} \frac{N!}{\alpha!\beta!\gamma!\delta!} p_{00}{}^{\alpha} p_{10}{}^{\beta} p_{01}{}^{\gamma} p_{11}{}^{\delta} \tag{4.1}$$

where $k=(k_1, k_2)\geqq O:\ k_1, k_2\geqq 0$.

THEOREM 5. *Let $X_1, X_2, \cdots, X_N$ be an independent Bernoulli sequence then we have*

$$\lim_{\substack{Np_{10}\to\lambda_{10} \\ Np_{01}\to\lambda_{01} \\ Np_{11}\to\lambda_{11} \\ N\to\infty}} P\Big(\sum_{j=1}^{N} X_j = k\Big) = \sum_{\substack{\beta+\delta=k_1 \\ \gamma+\delta=k_2 \\ \beta,\gamma,\delta\geq 0 \text{ integer}}} \frac{\lambda_{10}{}^{\beta}\lambda_{01}{}^{\gamma}\lambda_{11}{}^{\delta}}{\beta!\gamma!\delta!} \exp\{-\lambda_{10}-\lambda_{01}-\lambda_{11}\} \tag{4.2}$$

for every $k=(k_1, k_2)\geqq 0$.

Proof. See theorem 2.

THEOREM 6. *Let $X_1, X_2, \cdots, X_N$ be an independent bivariate Bernoulli sequence and we assume*

$$\lim_{N\to\infty} P\Big(\sum_{j=1}^{N} X_j = k\Big) = \sum_{\substack{\beta+\delta=k_1 \\ \gamma+\delta=k_2 \\ \beta,\gamma,\delta\geq 0 \text{ integer}}} \frac{\lambda_{10}{}^{\beta}\lambda_{01}{}^{\gamma}\lambda_{11}{}^{\delta}}{\beta!\gamma!\delta!} \exp\{-\lambda_{10}-\lambda_{01}-\lambda_{11}\} \tag{4.3}$$

then we have $Np_{10}\to\lambda_{10}$, $Np_{01}\to\lambda_{01}$ and $Np_{11}\to\lambda_{11}$ as $N\to\infty$.

Proof. See theorem 3.

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