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SUBMANIFOLDS WITH PARALLEL RICCI TENSOR

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Let $\tilde{M}^{n+p}(\tilde{c})$ be a Riemannian $(n+p)$-manifold of constant sectional curvature $\tilde{c}$, which is called a *real space form*. If $\tilde{c}=0$, then $\tilde{M}^{n+p}(0)$ denotes the Euclidean $(n+p)$-space R^{n+p}. If $\tilde{c}>0$ (resp. $\tilde{c}<0$), then $\tilde{M}^{n+p}(\tilde{c})$ denotes the Euclidean $(n+p)$-sphere $S^{n+p}(\tilde{c})$ (resp. the hyperbolic $(n+p)$-space $H^{n+p}(\tilde{c})$) in R^{n+p+1}. We consider submanifolds isometrically immersed in a real space form. Ryan [5] showed: Let $M^n(n>2)$ be a hypersurface in $\tilde{M}^{n+1}(\tilde{c})$. If M is not of constant curvature $\tilde{c}$ and if the Ricci tensor of M is parallel, then either M is locally isometric to the product $M_1^k \times M_2^{n-k}$, $0 \leqq k \leqq n$ (if $\tilde{c}=0$, then $k \neq 2$), or $\tilde{c}=0$ and the rank of the second fundamental form A $(=A_1)$ is equal to 2 everywhere. Here, M_1^k is a sphere of some radius contained in some Euclidean space R^{k+1} (resp. M_2^{n-k} is one in some Euclidean space perpendicular to R^{k+1}), except possibly one of M_i $(i=1, 2)$ is a Euclidean space (this can only occur if $\tilde{c} \leqq 0$, and $k=0$ or n if $\tilde{c}<0$) or a hyperbolic space with some negative curvature $\bar{c}$ (this can only occur if $\tilde{c}<0$). In order to prove the above result Ryan made use of the following remarkable result ([5]): let M^n be as above. If the mean curvature is constant, then the second fundamental form of M is parallel.

On the other hand, in [3], [4] the author proved:

THEOREM. *Let M^n be an $n(>2)$-dimensional minimal Einstein submanifold in an $(n+2)$-dimensional space form $\tilde{M}^{n+2}(\tilde{c})$ with constant curvature $\tilde{c}$. Then the second fundamental form of M is parallel and that* (1) *if $\tilde{c} \leqq 0$, then M is totally geodesic and that* (2) *if $\tilde{c}>0$, then either M is totally geodesic or locally isometric to the product $S^m\left(\frac{1}{\sqrt{2\tilde{c}}}\right) \times S^m\left(\frac{1}{\sqrt{2\tilde{c}}}\right)$ $(n=2m)$ of two spheres in totally geodesic $\tilde{M}^{n+1}(\tilde{c})$ in $\tilde{M}^{n+2}(\tilde{c})$ or the product $S^m\left(\frac{1}{\sqrt{3\tilde{c}}}\right) \times S^m\left(\frac{1}{\sqrt{3\tilde{c}}}\right) \times S^m\left(\frac{1}{\sqrt{3\tilde{c}}}\right)$ $(n=3m)$ of three spheres in $\tilde{M}^{n+2}(\tilde{c})$.*

In this paper we would like to prove the following:

THEOREM 1. *Let M^n be an $n(>2)$-dimensional submanifold in $\tilde{M}^{n+p}(\tilde{c})$ with the parallel Ricci tensor. If the mean curvature normal H is parallel and the normal connection of M is trivial, then the second fundamental form of M is parallel and M is locally isometric to the product $M^{n_1} \times \cdots \times M^{n_l}$, $1 \leqq l \leqq p+1$, where each M^{n_i} is an n_i-dimensional sphere of some radius contained in some*

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Euclidean space N^{n_i+1} *of dimension* n_i+1, $N^{n_i+1}\perp N^{n_j+1}$ *for* $i\neq j$, *except possibly one of the* M^{n_i} *is a Euclidean space* N^{n_i} *(this can only occur if* $\tilde{c}\leqq 0$*) or hyperbolic space* $H^{n_i}(\bar{c})$ *with some negative curvature* $\bar{c}$ *(this can only occur if* $\tilde{c}<0$*).*

COROLLARY 2. *Let* M^n *be an Einstein submanifold in* $\tilde{M}^{n+2}(\tilde{c})$. *If the mean curvature normal* H *is parallel, then the second fundamental form of* M *is parallel.*

Remark 1. Let $f: M\to R^3$ be a surface of constant curvature $c\neq 0$, which is not contained in a sphere (See [1], p. 432). Embedd R^3 into R^4 and let u be a unit vector orthogonal to R^3. Then $\tilde{f}: M\times R\to R^4$ which defined by $\tilde{f}(x, t)=f(x)+tu$ gives an example of hypersurfaces in R^4 with parallel Ricci tensor, of which the second fundamental tensor is not parallel. By this example, the assumption on the mean curvature vector in Theorem 1 is necessary. Also, the assumption on the normal connection in Theorem 1 is not necessary if $p=2$ and $H\neq 0$ (See [2], Lemma 7).

Remark 2. In $S^5(1)$ the product $S^1\left(\frac{1}{\sqrt{3}}\right)\times S^1\left(\frac{1}{\sqrt{3}}\right)\times S^1\left(\frac{1}{\sqrt{3}}\right)$ of three spheres is a minimal Einstein submanifold. In $S^6(1)$ the product $S^2\left(\frac{1}{\sqrt{2}}\right)\times S^1\left(\frac{1}{2}\right)\times S^1\left(\frac{1}{2}\right)$ of three spheres is a minimal submanifold with the parallel Ricci tensor. In $S^7(1)$ the product $S^2\left(\frac{\sqrt{2}}{\sqrt{5}}\right)\times S^2\left(\frac{\sqrt{2}}{\sqrt{5}}\right)\times S^1\left(\frac{1}{\sqrt{5}}\right)$ of three spheres and the product $S^1\left(\frac{1}{2}\right)\times S^1\left(\frac{1}{2}\right)\times S^1\left(\frac{1}{2}\right)\times S^1\left(\frac{1}{2}\right)$ of four spheres are minimal submanifolds with the parallel Ricci tensor. And, these normal connections are trivial·········(See [7]).

Finally, we would like to thank the referee for valuable comments.

1. Submanifolds.

Let f be an isometric immersion of a connected Riemannian n-manifold M^n into a real space form $\tilde{M}^{n+p}(\tilde{c})$ of constant curvature $\tilde{c}$. For all local formulas we may consider f as an imbedding and thus identify $x\in M$ with $f(x)\in\tilde{M}$. The tangent space T_xM is identified with a subspace of the tangent space $T_x\tilde{M}$. The normal space $T_x^\perp$ is the subspace of $T_x\tilde{M}$ consisting of all $X\in T_x\tilde{M}$ which are orthogonal to T_xM with respect to the Riemannian metric g. Let ∇ (resp. $\tilde{\nabla}$) denote the covariant differentiation in M (resp. $\tilde{M}$), and D the covariant differentiation in the normal bundle.

With each $\xi\in T_x^\perp$ is associated a linear transformation of T_xM in the following way. Extend ξ to a normal vector field defined in a neighborhood of x and define $-A_\xi X$ to be the tangential component of $\tilde{\nabla}_X\xi$ for $X\in T_xM$. $A_\xi X$ depends only on ξ at x and X. Given an orthonormal basis $\xi_1, \cdots, \xi_p$ of $T_x^\perp$ we write $A_\alpha=A_{\xi_\alpha}$ and call the A_α's the second fundamental forms associated with $\xi_1, \cdots, \xi_p$. If $\xi_1, \cdots, \xi_p$ are now orthonormal normal vector fields in a neighborhood U of x, they determine normal connection forms $s_{\alpha\beta}$ in U by

$$D_x\xi_\alpha=\sum_\beta s_{\alpha\beta}(X)\xi_\beta, \quad s_{\alpha\beta}+s_{\beta\alpha}=0$$

for $X\in T_xM$. Let X and Y be tangent to M and $\xi_1, \cdots, \xi_p$ orthonormal normal vector fields. Then we have the following relationships (in this section Greek indices run from 1 to p) [3]:

$$\tilde{\nabla}_XY=\nabla_XY+\sigma(X, Y), \tag{1.1}$$

$$\sigma(X, Y)=\sum_\alpha g(A_\alpha X, Y)\xi_\alpha, \quad g(A_\alpha X, Y)=g(A_\alpha Y, X),$$

$$(\nabla_XA_\alpha)Y-\sum_\beta s_{\alpha\beta}(X)A_\beta Y=(\nabla_YA_\alpha)X-\sum_\beta s_{\alpha\beta}(Y)A_\beta X \tag{1.2}$$

——Codazzi equation,

$$R^N(X, Y)\xi_\alpha=\sum_\beta g([A_\alpha, A_\beta]X, Y)\xi_\beta, \tag{1.3}$$

$$Ric=(n-1)\tilde{c}I+\sum_\alpha(\text{trace } A_\alpha)A_\alpha-\sum_\alpha A_\alpha^2, \tag{1.4}$$

where σ is also called the second fundamental form of f, and R^N, Ric and I denote the curvature tensor with respect to D, the Ricci tensor for M and the identity transformation of T_xM, respectively.

The *mean curvature normal* H is defined by

$$H=\sum_\alpha(\text{trace } A_\alpha)\xi_\alpha,$$

where the right side is independent of our choice of the orthonormal basis for $T_x^\perp$. An immersion is said to be *minimal* if its mean curvature normal vanishes identically, i.e., if trace $A_\alpha=0$ for all α.

2. Proofs of Theorem 1 and Corollary 2.

Let f be an isometric immersion of M^n into $\tilde{M}^{n+p}(\tilde{c})$ with the assumption of Theorem 1. From the results of [2], [6] and [7] we have only to prove that the second fundamental form of M is parallel.

If $p=1$, then the theorem follows from Proposition 5 of Ryan [5].

We may assume that $p\geqq 2$. If $H\neq 0$ at x, then we can choose an orthonormal normal vector fields $\xi_1, \cdots, \xi_p$ defined in a neiborhood U of x such that $\xi_1=\dfrac{H}{|H|}$. Then on U we have

$$\text{trace } A_1=\text{constant and trace } A_\beta=0,\ 2\leqq\beta\leqq p. \tag{2.1}$$

If M is minimal, then as we of course have for any α

$$\text{trace } A_\alpha=0,$$

we may assume that (2.1) holds on M. Now since the normal connection is trivial, by continuity it is sufficient to prove that $\nabla A_\alpha=0$. In terms of (1.4) we have

$$(1.4)' \qquad Ric=(n-1)\tilde{c}I+\sum_\alpha(\text{trace } A_\alpha)A_\alpha-\sum_\alpha A_\alpha^2 .$$

Then from (2.1) we have

$$(2.2) \qquad \sum_\alpha(\nabla A_\alpha)A_\alpha+\sum_\alpha A_\alpha(\nabla A_\alpha)-(\text{trace } A_1)\nabla A_1=0 .$$

On the other hand, from the triviality of the normal connection, i.e., $A_\beta A_\gamma\equiv A_\gamma A_\beta$ for $1\leqq\beta,\ \gamma\leqq p$ we have

$$(2.3) \qquad (\nabla A_\beta)A_\gamma+A_\beta(\nabla A_\gamma)-(\nabla A_\gamma)A_\beta-A_\gamma(\nabla A_\beta)=0 .$$

Since A_α's are also simultaneously parallelizable, we may consider $\lambda_i: T_x^\perp\to R$ so that

$$A_\alpha=\begin{bmatrix}\lambda_1(\xi_\alpha) & & 0\\ & \ddots & \\ 0 & & \lambda_n(\xi_\alpha)\end{bmatrix}.$$

Moreover we can take an orthonormal frame $\{e_1, \cdots, e_n\}$ of M such that $A_\alpha e_i=\lambda_i(\xi_\alpha)e_i$. Theorem 1 is trivial when M^n is totally umblic. So we may assume that for some β, $A_\beta\neq\rho I$. Therefore we get $i\neq j$ such that $\lambda_i(\xi_\beta)\neq\lambda_j(\xi_\beta)$. For simplicity, put $\lambda=\lambda_i$ and $\mu=\lambda_j$. Now, put $A_\beta X=\lambda(\xi_\beta)X$, $A_\gamma X=\lambda(\xi_\gamma)X$, $A_\beta Y=\mu(\xi_\beta)Y$ and $A_\gamma Y=\mu(\xi_\gamma)Y$, $\beta\neq\gamma$. Then from (2.2) we have

$$(2.4) \qquad \sum_\alpha\lambda(\xi_\alpha)(\nabla_Y A_\alpha)X+\sum_\alpha A_\alpha(\nabla_Y A_\alpha)X-(\text{trace } A_1)(\nabla_Y A_1)X=0 ,$$

$$(2.5) \qquad \sum_\alpha\mu(\xi_\alpha)(\nabla_X A_\alpha)Y+\sum_\alpha A_\alpha(\nabla_X A_\alpha)Y-(\text{trace } A_1)(\nabla_X A_1)Y=0 .$$

Similarly, from (2.3) we have

$$(2.6) \qquad \lambda(\xi_\gamma)(\nabla_Y A_\beta)X+A_\beta(\nabla_Y A_\gamma)X-\lambda(\xi_\beta)(\nabla_Y A_\gamma)X-A_\gamma(\nabla_Y A_\beta)X=0 ,$$

$$(2.7) \qquad \mu(\xi_\gamma)(\nabla_X A_\beta)Y+A_\beta(\nabla_X A_\gamma)Y-\mu(\xi_\beta)(\nabla_X A_\gamma)Y-A_\gamma(\nabla_X A_\beta)Y=0 .$$

Subtracting (2.5) from (2.4), using Codazzi equations (1.2), we have

$$(2.8) \qquad \sum_\alpha(\lambda(\xi_\alpha)-\mu(\xi_\alpha))(\nabla_X A_\alpha)Y=0 .$$

Similarly, from (2.6) and (2.7) we have

$$(\lambda(\xi_\gamma)-\mu(\xi_\gamma))(\nabla_X A_\beta)Y-(\lambda(\xi_\beta)-\mu(\xi_\beta))(\nabla_X A_\gamma)Y=0 .$$

Since $\lambda(\xi_\beta)\neq\mu(\xi_\beta)$ by the assumption, we get

$$(2.9) \qquad (\nabla_X A_\gamma)Y=\frac{\lambda(\xi_\gamma)-\mu(\xi_\gamma)}{\lambda(\xi_\beta)-\mu(\xi_\beta)}(\nabla_X A_\beta)Y$$

for any γ. Substituting this into (2.8), we obtain

$$(\nabla_X A_\beta)Y=0. \tag{2.10}$$

From (2.9), it follows $(\nabla_X A_\gamma)Y=0$ for all γ.

This proves Theorem 1.

Next, we prove Corollary 2. Let M^n be an Einstein submanifold in $\tilde{M}^{n+2}(\tilde{c})$ with the parallel mean curvature normal H.

If $H\neq 0$ at x, then as in the above we choose an orthonormal normal vector fields ξ_1, ξ_2 defined in a neighborhood U of x such that $\xi_1=\dfrac{H}{|H|}$. Now $DH=0$ implies $D\xi_1=0$ and hence $s_{12}=0$ in U. This implies U satisfies the assumption of Theorem 1. Hence the second fundamental form of M is parallel in U. If there exists a neighborhood V which satisfies $H\equiv 0$, then V holds the assumption of Theorem. Hence the second fundamental form of M is parallel in V. By continuity we obtain that the second fundamental form of M is parallel.

This proves Corollary 2.

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