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KODAI MATH. J.
11 (1988), 233—243

ON THE GROWTH OF MEROMORPHIC FUNCTIONS OF ORDER LESS THAN 1/2; IV

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Introduction. This note is a continuation of [10]. The notational conventions of [10] are adopted without modifications and strictly adhered to. We supplement Theorems 1, 2 and 3 of [10] by the information contained in the theorems of the present note.

In everything that follows

(i) ρ and δ are numbers such that $0<\rho<1/2$ and $1-\cos\pi\rho<\delta\leqq 1$;

(ii) $\alpha(f)=\limsup_{r\to\infty} T(r, f)/r^{\rho}$, $\beta(f)=\liminf_{r\to\infty} T(r, f)/r^{\rho}$, where $f(z)$ is a meromorphic function of order ρ.

We first prove in §1

THEOREM 6. *Let $f(z)\in\mathcal{M}_{\rho,\delta}$ be of minimal type. Then there is an $h(r)\in S_2$ such that*

$$\log m^*(r, f)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1+h(r))T(r, f) \tag{1}$$

for certain arbitrarily large values of r.

Our second result, which is proved in §2, is the following

THEOREM 7. *Let $h(r)\in S_1$ be given. If $f(z)\in\mathcal{M}_{\rho,\delta}$ satisfies $\beta(f)=0$, then the estimate* (1) *holds for a sequence of $r\to\infty$.*

Remarks. (i) Theorem 3 of [10] is contained in the above Theorem 7. (ii) Modifying a part of the proof of Theorem 7, we are able to show the following

THEOREM 8. *Let $k=k(\rho)$ and $K_1=K_1(\rho)$ be positive constants which appear in Lemma* 13 *and* (2.14), *respectively. If $f(z)\in\mathcal{M}_{\rho,\delta}$ satisfies $0<\beta(f)<(k/K_1)\alpha(f)\leqq+\infty$, then the estimate* (1) *holds with any $h(r)\in S_1$ on an unbounded sequence of r.*

In §3, we use our results stated above and in [10] to refine the estimate

Received November 25, 1987

$$(2)\qquad \log m^*(r, f) > \frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho - 1 + \delta)(1-\varepsilon)T(r, f) \quad (\varepsilon > 0,\ r = r_n \to \infty)$$

for all $f(z) \in \mathcal{M}_{\rho,\delta}$ whose characteristics vary regularly with order ρ. It was this refinement that provided the impetus for the previous and the present works. We say, according to Baernstein [2], that the function $\phi(r)$ *varies regularly with order* ρ if $\phi(r) \sim r^\rho L(r)$ $(r \to \infty)$ for some slowly varying function $L(r)$.

1. We start by showing the following

LEMMA 12. *Given $G(r)$ positive and continuous for $r \geqq r_0$, $G(r) \to \infty$ $(r \to \infty)$, there exists a function $h(r) \in S_2$ such that*

$$(1.1)\qquad \int_1^r \frac{h(t)}{t}\,dt \leqq G(r) + C \qquad (r \geqq 1),$$

where C is a positive constant depending only on $G(r)$.

Proof. By assumptions on $G(r)$, we find a positive integer n_0 and an increasing unbounded sequence $\{r_n\}_{n_0}^\infty$ with the property that $G(r) > G(r_n) = n$ $(r > r_n)$. Choose $\{R_n\}_{n_0}^\infty$ such that

$$R_{n_0} = r_{n_0}, \quad R_{n_0+1} = r_{n_0+1},$$
$$R_n \geqq r_n \quad (n \geqq n_0 + 2),$$
$$R_{n+2}/R_{n+1} \geqq (R_{n+1}/R_n)^2 \quad (n \geqq n_0).$$

Define a function $h_1(r)$ $(r \geqq r_{n_0})$ by

$$h_1(r) = \{\log(R_{n+1}/R_n)\}^{-1} \quad (R_n \leqq r < R_{n+1},\ n \geqq n_0).$$

Then $h_1(r)$ is positive, decreasing, and tends to 0 as $r \to \infty$. We define

$$H_1(r) = n_0 - 1 + \int_{r_{n_0}}^r h_1(t) t^{-1}\,dt.$$

Then if $R_n \leqq r < R_{n+1}$ $(n \geqq n_0)$,

$$H_1(r) \leqq H_1(R_{n+1}) = n_0 - 1 + (n - n_0 + 1) = n = G(r_n) \leqq G(r).$$

Now, define $h(r)$ $(r \geqq 0)$ by

$$h(r) = \{\log(R_{n_0+1}/R_{n_0})\}^{-1} \quad (0 \leqq r \leqq \sqrt{R_{n_0} R_{n_0+1}}),$$
$$h(r) = \{\log(R_{n+1}/R_n)\}^{-1} \quad (R_n \leqq r \leqq \sqrt{R_n R_{n+1}},\ n \geqq n_0 + 1),$$

and by linear interpolation otherwise. Clearly $h(r) \in S_2$, and if we put

$$H(r) = n_0 - 1 + \int_{r_{n_0}}^r h(t) t^{-1}\,dt,$$

then $H(r) \leqq H_1(r) \leqq G(r) (r \geqq r_{n_0})$. Thus, with a suitable positive constant C $\left(\geqq \int_1^{r_{n_0}} h(t) t^{-1} dt - n_0 + 1\right)$, we obtain (1.1).

The proof of Theorem 6 is a combination of Lemma 12 and Theorem 2 in [10].

Proof of Theorem 6. Let $f(z) \in \mathcal{M}_{\rho,\delta}$ be of minimal type, and set

$$G(r) = \log(r^{\rho}/T(r, f)) \quad (r > 0). \tag{1.2}$$

Then $G(r)$ satisfies the assumptions of Lemma 12, so we find a function $h(r) \in S_2$ satisfying (1.1) with a suitable positive constant C. Now, choose a positive number $K < C(\rho, \delta)$ arbitrarily, where $C(\rho, \delta)$ is defined by (5) in [10], and put $h_1(r) = K h(r) \in S_2$. Then in view of (1.2)

$$T(r, f) = r^{\rho} \exp\{-G(r)\} \leqq e^{C} r^{\rho} \exp\left\{-K^{-1} \int_1^r h_1(t) t^{-1} dt\right\}.$$

Hence from Theorem 2 we deduce (1) with $h(r)$ replaced by $h_1(r)$ for certain arbitrarily large values of r.

2. Let $f(z) \in \mathcal{M}_{\rho,\delta}$ be given, and let a be a complex number satisfying $f(0) \neq a$ and

$$N(r, \infty, f) < (1-\delta) N(r, a, f) + O(1) \quad (r \to \infty). \tag{2.1}$$

We set

$$F(z) = f(z) - a = c z^{-p} \frac{\Pi(1 - z/a_n)}{\Pi(1 - z/b_n)} = c z^{-p} \frac{P(z)}{Q(z)} = c z^{-p} F_1(z), \tag{2.2}$$

where c is a nonzero constant and p is a nonnegative integer. It is convenient to introduce the notation

$$\hat{P}(z) = \Pi(1 + z/|a_n|), \quad \hat{Q}(z) = \Pi(1 - z/|b_n|), \quad \hat{F}_1(z) = \hat{P}(z)/\hat{Q}(z). \tag{2.3}$$

Our proofs of Theorems 7 and 8 make use of the following

LEMMA 13. (See [1, Lemma 1].) *Let $F_1(z)$ be defined by* (2.2). *Then there exist constants $K = K(\rho)$, $k = k(\rho)$ depending only on ρ satisfying $0 < k < K < 4\pi + 2\pi^2/\log 2$, such that for any $r_2 > r_1 > 0$,*

$$\int_{r_1}^{r_2} \{\pi\rho N(t, \infty, \hat{F}_1) + \sin\pi\rho \log m^*(t, \hat{F}_1) - \pi\rho\cos\pi\rho N(t, 0, \hat{F}_1)\} t^{-1-\rho} dt$$
$$> k T(r_1, \hat{F}_1) r_1^{-\rho} - K T(2r_2, \hat{F}_1) r_2^{-\rho}.$$

Now choose R sufficiently large so that $F_1(z)$ has N zeros and M poles in $|z| < R$, where $\max(M, N) > 0$. Let

$$f_2(z)=\frac{\prod_{n=1}^{N}(1-z/a_n)}{\prod_{m=1}^{M}(1-z/b_m)},\quad \hat{f}_2(z)=\frac{\prod_{n=1}^{N}(1+z/|a_n|)}{\prod_{m=1}^{M}(1-z/|b_m|)}=\frac{\hat{P}_2(z)}{\hat{Q}_2(z)},$$

and define $f_3(z)$ by $F_1(z)=f_2(z)f_3(z)$. Using a result of Edrei [4, Lemma A] we have for $r<R/2$

$$T(r, F_1)\leqq T(r, f_2)+T(r, f_3)\leqq T(r, \hat{f}_2)+\frac{14r}{R}T(2R, F_1). \tag{2.4}$$

Here we apply Lemma 13 to $\hat{f}_2(z)$ to obtain for any r_1, r_2, $0<r_1<r_2<R$

$$\begin{aligned}&\int_{r_1}^{r_2}\{\pi\rho N(t, \infty, \hat{F}_1)+\sin\pi\rho\log m^*(t, \hat{f}_2)-\pi\rho\cos\pi\rho N(t, 0, \hat{F}_1)\}t^{-1-\rho}dt\\ &\qquad>kT(r_1, \hat{f}_2)r_1^{-\rho}-KT(2r_2, \hat{f}_2)r_2^{-\rho}.\end{aligned} \tag{2.5}$$

Proof of Theorem 7. Suppose that $f(z)\in\mathcal{M}_{\rho,\delta}$ satisfies $0=\beta(f)\leqq\alpha(f)\leqq+\infty$ and

$$\begin{aligned}&\pi\rho N(r, \infty, F)+\sin\pi\rho\log m^*(r, F)-\pi\rho\cos\pi\rho N(r, 0, F)\\ &\leqq\pi\rho(\cos\pi\rho-1+\delta)h(r)T(r, F)+K_2\log r\qquad(r\geqq r_0=r_0(K_2)),\end{aligned} \tag{2.6}$$

where $F(z)$ is defined by (2.2) and K_2 is any fixed positive number. By (2.2) and (2.3) we have

$$\begin{cases}N(r, \infty, F)=N(r, \infty, \hat{F}_1)+p\log r,\\ \log m^*(r, F)=\log|c|-p\log r+\log m^*(r, F_1),\\ N(r, 0, F)=N(r, 0, \hat{F}_1).\end{cases} \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain

$$\begin{aligned}&\pi\rho N(r, \infty, \hat{F}_1)+\sin\pi\rho\log m^*(r, F_1)-\pi\rho\cos\pi\rho N(r, 0, \hat{F}_1)\\ &\leqq\pi\rho(\cos\pi\rho-1+\delta)h(r)T(r, F)+\{K_2-p(\pi\rho-\sin\pi\rho)\}\log r\\ &\qquad\qquad-\sin\pi\rho\log|c|\quad(r\geqq r_0).\end{aligned} \tag{2.8}$$

Hence from (2.5) and (2.8) it follows that for any r_1, r_2, $r_0<r_1<r_2<R$

$$\begin{aligned}&\pi\rho(\cos\pi\rho-1+\delta)\int_{r_1}^{r_2}h(t)T(t, F)t^{-1-\rho}dt+K_3\int_{r_1}^{r_2}(\log t)t^{-1-\rho}dt\\ &\quad+\sin\pi\rho\int_{r_1}^{r_2}\{\log m^*(t, \hat{f}_2)-\log m^*(t, F_1)\}t^{-1-\rho}dt\\ &\quad>kT(r_1, \hat{f}_2)r_1^{-\rho}-KT(2r_2, \hat{f}_2)r_2^{-\rho},\end{aligned} \tag{2.9}$$

where $K_3(\geqq K_2)$ is a suitable constant. Using a result of Edrei [4, Lemma A] again, we have for $0<t<R/2$

$$\log m^*(t, F_1) \geqq \log m^*(t, f_2) + \log m^*(t, f_3)$$

(2.10)

$$\geqq \log m^*(t, \hat{f}_2) - 14T(2R, F_1)t/R.$$

By (2.4)

$$T(r_1, \hat{f}_2) r_1^{-\rho} \geqq T(r_1, F_1) r_1^{-\rho} - 14 \cdot 2^{\rho} (r_1/R)^{1-\rho} T(2R, F_1)(2R)^{-\rho}. \tag{2.11}$$

Also, if we choose $r_2 = R/2$, we have

$$T(2r_2, \hat{f}_2) = T(R, \hat{f}_2) \leqq N(R, 0, \hat{P}) + N(R, 0, \hat{Q}) + \log \hat{P}_2(R) + \log \hat{Q}_2(-R)$$

$$\leqq 2T(R, F_1) + n(R, 0, \hat{P}_2) \log 2 + N(R, 0, \hat{P}_2) + n(R, 0, \hat{Q}_2) \log 2 + N(R, 0, \hat{Q}_2)$$

$$\leqq 2T(R, F_1) + 2(T(2R, F_1) + T(R, F_1)) \leqq 6T(2R, F_1),$$

so that

$$T(2r_2, \hat{f}_2) r_2^{-\rho} \leqq 6 \cdot 4^{\rho} T(2R, F_1)(2R)^{-\rho}. \tag{2.12}$$

Further, with $r_2 = R/2$ (>1) we have

$$\int_{r_1}^{r_2} (\log t) t^{-1-\rho} dt = -\rho^{-1} (\log r_2) r_2^{-\rho} + \rho^{-1} (\log r_1) r_1^{-\rho}$$

(2.13)

$$-\rho^{-2} r_2^{-\rho} + \rho^{-2} r_1^{-\rho} < \rho^{-2} (\rho \log r_1 + 1) r_1^{-\rho}.$$

Incorporating (2.10)-(2.13) into (2.9), it follows that for $r_0 < r_1 < R/2$

$$\pi\rho(\cos \pi\rho - 1 + \delta) \int_{r_1}^{R/2} h(t) T(t, F) t^{-1-\rho} dt + K_3 \rho^{-1} (\log r_1) r_1^{-\rho} + K_3 \rho^{-2} r_1^{-\rho}$$

$$+ 14(1-\rho)^{-1} 2^{2\rho-1} \sin \pi\rho T(2R, F_1)(2R)^{-\rho} > kT(r_1, F_1) r_1^{-\rho}$$

$$-7 \cdot 4^{\rho} kT(2R, F_1)(2R)^{-\rho} - 6 \cdot 4^{\rho} KT(2R, F_1)(2R)^{-\rho}, \quad \text{i.e.,}$$

$$\pi\rho(\cos \pi\rho - 1 + \delta) \int_{r_1}^{R/2} h(t) T(t, F) t^{-1-\rho} dt + K_3 \rho^{-1} (\log r_1) r_1^{-\rho} + K_3 \rho^{-2} r_1^{-\rho}$$

(2.14)

$$> kT(r_1, F_1) r_1^{-\rho} - K_1 T(2R, F_1)(2R)^{-\rho}$$

with a suitable positive constant $K_1 = K_1(\rho)$.

Case 1. Assume first that $\alpha(f) = 0$. Let $R \to \infty$ in (2.14) to get

$$\pi\rho(\cos \pi\rho - 1 + \delta) \int_{r_1}^{\infty} h(t) T(t, F) t^{-1-\rho} dt + K_3 \rho^{-1} (\log r_1) r_1^{-\rho}$$

(2.15)

$$+ K_3 \rho^{-2} r_1^{-\rho} \geqq kT(r_1, F_1) r_1^{-\rho}.$$

Choose a sequence $\{(r_1)_n\} \to \infty$ such that

$$T(t, F) t^{-\rho} < T((r_1)_n, F)(r_1)_n^{-\rho} \qquad (t > (r_1)_n).$$

Then we deduce from (2.15) and (2.2) that for $n \geqq n_0$

$$\pi\rho(\cos\pi\rho-1+\delta)\int_{(r_1)_n}^{\infty}h(t)t^{-1}dt+K_3\rho^{-1}\log(r_1)_n/T((r_1)_n, F)$$
(2.16)
$$+K_3\rho^{-2}/T((r_1)_n, F)\geqq kT((r_1)_n, F_1)/T((r_1)_n, F)>k/2.$$

Since $h(r)\in S_1$, the left hand side of (2.16)$\rightarrow 0$ $(n\rightarrow\infty)$. This is a contradiction.

Case 2. Next we consider the case $\alpha=\alpha(f)\in(0, +\infty)$. Given $\varepsilon>0$, there is a number $R_0(\geqq r_0)$ such that $t\geqq R_0$ implies $T(t, F)t^{-\rho}<\alpha+\varepsilon$. Hence by (2.14) we have for $R_0<r_1<R/2$

$$\pi\rho(\cos\pi\rho-1+\delta)(\alpha+\varepsilon)\int_{r_1}^{R/2}h(t)t^{-1}dt+K_3\rho^{-1}(\log r_1)r_1^{-\rho}+K_3\rho^{-2}r_1^{-\rho}$$
(2.17)
$$>kT(r_1, F_1)r_1^{-\rho}-K_1T(2R, F_1)(2R)^{-\rho}.$$

Choose $\{(r_1)_n\}\rightarrow\infty$, $\{2R_n\}\rightarrow\infty$ such that $R_0<(r_1)_n<R_n/2$ $(n=1, 2, \cdots)$ and $T((r_1)_n, F_1)(r_1)_n^{-\rho}\rightarrow\alpha$, $T(2R_n, F_1)(2R_n)^{-\rho}\rightarrow 0$ $(n\rightarrow\infty)$. Then from (2.17) it follows that for $n\geqq n_0=n_0(\varepsilon)$

$$\pi\rho(\cos\pi\rho-1+\delta)(\alpha+\varepsilon)\int_{(r_1)_n}^{R_n/2}h(t)t^{-1}dt+\varepsilon>(\alpha-\varepsilon)k-\varepsilon K_1.$$

Now, let $n\rightarrow\infty$ to get $\varepsilon\geqq(\alpha-\varepsilon)k-\varepsilon K_1$. Since $\varepsilon(>0)$ was arbitrary, this implies $k\leqq 0$, a contradiction.

Case 3. It remains to consider the case $\alpha(f)=+\infty$. First, choose $\{2R_n\}\rightarrow\infty$ such that $R_1>2$, and

(2.18) $$T(2R_n, F_1)(2R_n)^{-\rho}\rightarrow 0 \qquad (n\rightarrow\infty).$$

Next, define $\{(r_1)_n\}$ $(1\leqq(r_1)_n\leqq R_n/2)$ by

(2.19) $$\max_{1\leqq t\leqq R_n/2}T(t, F)t^{-\rho}=T((r_1)_n, F)(r_1)_n^{-\rho}.$$

Then the fact that $\alpha(f)=+\infty$ and (2.19) give

(2.20) $$T((r_1)_n, F)(r_1)_n^{-\rho}\rightarrow\infty \qquad (n\rightarrow\infty),$$

which, in particular, implies $\{(r_1)_n\}\rightarrow\infty$. Further, in view of (2.18) and (2.20) we see that $(r_1)_n<R_n/2$ $(n\geqq n_0)$. Now, we use (2.14) with $r_1=(r_1)_n$ and $R=R_n$ $(n\geqq n_0)$. Taking (2.19) into consideration, we have

$$\pi\rho(\cos\pi\rho-1+\delta)(T((r_1)_n, F)(r_1)_n^{-\rho}\int_{(r_1)_n}^{R_n/2}h(t)t^{-1}dt+K_3\rho^{-1}(\log(r_1)_n)(r_1)_n^{-\rho}$$
(2.21)
$$+K_3\rho^{-2}(r_1)_n^{-\rho}>kT((r_1)_n, F_1)(r_1)_n^{-\rho}-K_1T(2R_n, F_1)(2R_n)^{-\rho}.$$

Since $h(r)\in S_1$, we deduce from (2.21), (2.18) and (2.2) that

$$T((r_1)_n, F)(r_1)_n^{-\rho}\rightarrow 0 \qquad (n\rightarrow\infty),$$

which contradicts (2.20).

Thus we see that (2.6) is not valid. Hence there is a sequence $\{r_n\}\to\infty$ such that

$$\begin{aligned}(2.22)\qquad &\pi\rho N(r,\infty,F)+\sin\pi\rho\log m^*(r,F)-\pi\rho\cos\pi\rho N(r,0,F)\\ &>\pi\rho(\cos\pi\rho-1+\delta)h(r)T(r,F)+K_2\log r\quad(r=r_n),\end{aligned}$$

where K_2 is any fixed positive number. As in the proof of Theorem 1 of [9], we deduce from (2.22) that

$$\sin\pi\rho\log m^*(r,F)>\pi\rho(\cos\pi\rho-1+\delta)(1+h(r))T(r,F)+K_2\log r-O(1)\ (r=r_n).$$

From this and (2.2) it follows that

$$\begin{aligned}\sin\pi\rho\log m^*(r,f)&>\pi\rho(\cos\pi\rho-1+\delta)(1+h(r))T(r,f)+K_2\log r-O(1)\\ &>\pi\rho(\cos\pi\rho-1+\delta)(1+h(r))T(r,f)\quad(r=r_n).\end{aligned}$$

3. Edrei proved the following Theorem A in [5].

THEOREM A. *Assume that* $f(z)\in\mathcal{M}_{\rho,\delta}$ *satisfies the relation*

$$\limsup_{r\to\infty}\frac{\log m^*(r,f)}{T(r,f)}=\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta).$$

Then there exist three positive sequences $\{r_n\}\to\infty$, $\{r'_n\}\to\infty$, $\{r''_n\}\to\infty$ *having all the following properties.*

(i) $r'_n<r_n<r''_n<r'_{n+1}$ $(n=1,2,3,\cdots)$.

(ii) $r_n/r'_n\to\infty$, $r''_n/r_n\to\infty$ *as* $n\to\infty$.

(iii) $\displaystyle\lim_{n\to\infty}\frac{\log m^*(r_n,f)}{T(r_n,f)}=\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)$.

(iv) *Put* $L(r)=T(r,f)/r^{\rho}$ $(r>0)$, *and let* $\Lambda=\bigcup_{n=1}^{\infty}(r'_n,r''_n)$. *Then*

$$\lim_{\substack{r\to\infty\\ r,\sigma r\in\Lambda}}\frac{L(\sigma r)}{L(r)}=1\qquad(\sigma>0)$$

and

$$\lim_{\substack{r\to\infty\\ r\in\Lambda}}\frac{N(r,\infty,f)}{N(r,a,f)}=1-\delta$$

hold, where $a\in\boldsymbol{C}$ *is any number satisfying* $f(0)\neq a$ *and* (1) *of* [10].

(v) *Let* $s>0$ *and* $\varepsilon>0$ *be given. Consider the annuli* $A_n(s)=\{z=re^{i\theta};\ e^{-s}<r/r_n<e^{s}\}$, *the sectors* $S_n(s;\varphi-\varepsilon,\varphi+\varepsilon)=\{z=re^{i\theta}\in A_n(s);\ \varphi-\varepsilon<\theta<\varphi+\varepsilon\}$, *and let* $\{\omega_n\}$ *be any real sequence defined by the conditions* $m^*(r_n,f-a)=|f(r_ne^{i\omega_n})-a|$ $(k=1,2,3,\cdots)$. *Let* $\nu_n(a)$ *be the number of zeros of* $f(z)-a$ *in the sector* $A_n(s)$

$-S_n(s\,;\,\omega_n-\varepsilon,\,\omega_n+\varepsilon)$, and $\nu_n(\infty)$ the number of poles of $f(z)$ in $A_n(s)-S_n(s\,;\,\omega_n+\pi-\varepsilon,\,\omega_n+\pi+\varepsilon)$. Then

$$\lim_{n\to\infty}\frac{\nu_n(a)+\nu_n(\infty)}{T(r_n,\,f)}=0\,.$$

The above Edrei's result implies that the extremal functions $f(z)$ for the estimate (2) satisfy the relation $T(r,\,f)\sim r^\rho L(r)$ (with slowly varying functions $L(r)$) at least locally as $r\to\infty$.

In this section we first prove the following

THEOREM 9. *Let $f(z)$ be a meromorphic function of the form*

$$f(z)=\frac{\Pi(1+z/a_n)}{\Pi(1-z/b_n)}\equiv\frac{P(z)}{Q(z)}\qquad(0<a_n\leqq a_{n+1},\ 0<b_n\leqq b_{n+1}),$$

and let $L(r)$ be a slowly varying function. Then

$$T(r,\,f)\sim r^\rho L(r)\qquad(r\to\infty,\ 0<\rho<1/2) \tag{3.1}$$

and

$$N(r,\,\infty,\,f)\sim(1-\delta)N(r,\,0,\,f)\qquad(r\to\infty,\ 1-\cos\pi\rho<\delta<1) \tag{3.2}$$

or

$$N(r,\,\infty,\,f)=0\qquad(r\geqq0,\ \delta=1) \tag{3.2$'$}$$

imply that for $\varepsilon>0$

$$\log m^*(r,\,f)<\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1+\varepsilon)T(r,\,f)\quad(r\geqq r_0(\varepsilon)). \tag{3.3}$$

Proof. Let $\{r_n\}$ be any positive, increasing, unbounded sequence. Then the hypothesis (3.1) implies that $\{r_n\}$ is a sequence of Pólya peaks of order ρ for $T(r,\,f)$. (See [2, p 94].) Using the assumption (3.2) or (3.2)$'$, we easily deduce that

$$\delta(\infty,\,f)\geqq\delta>1-\cos\pi\rho\,. \tag{3.4}$$

Now, put $J(r)=\{\theta\in(-\pi,\,+\pi]\,;\ |f(re^{i\theta})|\geqq1\}$. Then the spread relation (See [3].) and (3.4) yield

$$\liminf_{n\to\infty}\ \text{meas}\ J(r_n)\geqq\min\left\{\frac{4}{\rho}\sin^{-1}\sqrt{\frac{\delta(\infty,\,f)}{2}},\ 2\pi\right\}=2\pi,\ \text{i.e.}$$

$$\lim_{n\to\infty}\ \text{meas}\ J(r_n)=2\pi\,. \tag{3.5}$$

From the first fundamental theorem and the Edrei-Fuchs Lemma (See [6, p 322]), it follows that

$$T(r_n, f)-N(r_n, 0, f)=m(r_n, 0, f)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\log^+\left|\frac{1}{f(r_n e^{i\theta})}\right|d\theta$$

$$(3.6)\qquad =\frac{1}{2\pi}\int_{J(r_n)^c}\log\left|\frac{1}{f(r_n e^{i\theta})}\right|d\theta$$

$$\leqq 11T(2r_n, f)\text{ meas } J(r_n)^c\left\{1+\log^+\left(\frac{1}{\text{meas } J(r_n)^c}\right)\right\}.$$

In view of (3.1) we have

$$(3.7)\qquad T(2r_n, f)\sim 2^\rho T(r_n, f)\quad (n\to\infty).$$

Substituting (3.5) and (3.7) into (3.6), we deduce that

$$(3.8)\qquad T(r_n, f)-N(r_n, 0, f)=o(T(r_n, f))\quad (n\to\infty).$$

Since the sequence $\{r_n\}$ was arbitrary, (3.8) gives

$$(3.9)\qquad N(r, 0, f)\sim T(r, f)\quad (r\to\infty),$$

and so by (3.1) and (3.2)

$$(3.10)\qquad N(r, 0, f)\sim r^\rho L(r)\quad (r\to\infty),$$

$$(3.11)\qquad N(r, \infty, f)\sim(1-\delta)r^\rho L(r)\quad (r\to\infty,\ 1-\cos\pi\rho<\delta<1).$$

Then an abelian argument (See, for example, [7, Theorem 2].) may be used to prove

$$(3.12)\qquad \log|P(re^{i\theta})|=\frac{\pi\rho}{\sin\pi\rho}\{\cos\theta\rho+o(1)\}r^\rho L(r)\quad (r\to\infty,\ |\theta|<\pi),$$

and

$$(3.13)\qquad \log|Q(re^{i\theta})|=\frac{\pi\rho}{\sin\pi\rho}(1-\delta)\{\cos(\pi-\theta)\rho+o(1)\}r^\rho L(r)\quad (r\to\infty,\ 0<\theta<2\pi).$$

Given $\varepsilon>0$, choose $\eta>0$ with the property that $\cos(\pi-\eta)\rho-1+\delta<(\cos\pi\rho-1+\delta)(1+\varepsilon/2)$. Then (3.12), (3.13) and (3.1) give

$$\log m^*(r, f)=\log|P(-r)|-\log Q(-r)<\log|P(re^{i(\pi-\eta)})|-\log Q(-r)$$

$$<\frac{\pi\rho}{\sin\pi\rho}\{\cos(\pi-\eta)\rho-(1-\delta)+o(1)\}r^\rho L(r)$$

$$<\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1+\varepsilon)T(r, f)\quad (r\geqq r_0(\varepsilon)).$$

This completes the proof of Theorem 9.

We conclude from Theorems A and 9 that the simplest and the most typical growth of the characteristic functions of $f(z)\in\mathcal{M}_{\rho,\delta}$ satisfying (3.3) is regular variation of order ρ.

Now, we refine the estimate (2) for all $f(z)\in\mathcal{M}_{\rho,\delta}$ whose characteristics vary regularly with order ρ.

Case 1. $\alpha(f)=0$. Choose $h(r)\in S_2$ arbitrarily satisfying

$$T(r, f)=O\left(r^{\rho}\exp\left\{-\frac{1}{(1-\varepsilon)C(\rho, \delta)}\int_1^r\frac{h(t)}{t}dt\right\}\right)\quad (r\to\infty)$$

with some $\varepsilon>0$. Such an $h(r)\in S_2$ certainly exists. (See Lemma 12.) Then the estimate (1) holds on an unbounded sequence of r. (See Theorem 2.)

Case 2. $\beta(f)=0$ or $0<\beta(f)<\dfrac{k}{K_1}\alpha(f)\leqq+\infty$. In these cases, for any $h(r)\in S_1$, we have the estimate (1) for certain arbitrarily large values of r. (For the proof, see Theorems 7 and 8.)

Case 3. $0<\beta(f)\leqq\alpha(f)\leqq\dfrac{K_1}{k}\beta(f)<+\infty$. Let $h(r)\in S_2$ be given. Then the estimate

$$\log m^*(r, f)>\frac{\pi\rho}{\sin\pi\rho}(\cos\pi\rho-1+\delta)(1-h(r))T(r, f) \tag{3.14}$$

holds for a sequence of $r\to\infty$. (See Corollary 1 of [10].)

Case 4. $\beta(f)=+\infty$. Choose $h(r)\in S_2$ arbitrarily such that

$$T(r, f)=O\left(r^{\rho}\exp\left\{\frac{1}{(1+\varepsilon)C(\rho, \delta)}\int_1^r\frac{h(t)}{t}dt\right\}\right)\quad (r\to\infty).$$

with some $\varepsilon>0$. To see such a $h(r)\in S_2$ exists, we may note that any slowly varying function can be written as

$$L(r)=c(r)\exp\left(\int_1^r\varepsilon(t)t^{-1}dt\right),$$

where $\lim_{r\to\infty}c(r)=c>0$ and $\lim_{t\to\infty}\varepsilon(t)=0$. (See [8, p 45].) Then the estimate (3.14) holds for a sequence of $r\to\infty$. (See [10, Theorem 1].)

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