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ON PSEUDO-PRIMALITY OF THE $2n$-TH POWER OF PRIME ENTIRE FUNCTIONS

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1. Introduction

In [8] Guo Dong Song and Jue Huang proved the following theorem:

THEOREM A. *Let $g_0(z)$ be a pseudo-prime entire function, and $n(\geqq 3)$ be an odd number. Then $F(z)=g_0^n(z)$ is also pseudo-prime.*

They used a prime entire function $\sin z e^{\cos z}$ to show that there exists a prime entire function $g_0(z)$ such that $g_0^{2n}(z)(n\geqq 1)$ is not pseudo-prime. Because the order of $\sin z e^{\cos z}$ is infinite, in [8] the authors naturally proposed the following question (1): Does there exist an entire function $g_0(z)$ which is prime and of finite order such that $g_0^2(z)$ is not pseudo-prime?

In this paper we shall give an affirmative answer to above question (1). That is, there exists an entire function $g_0(z)$ which is prime and of finite order such that $g_0^2(z)$ is not pseudo-prime. Further we shall prove that the prime entire functions of which the $2n$-th power are not pseudo-prime are only some special periodic functions.

We assume that the reader is familiar with the fundamental concepts of Nevanlinna's theory and adopt with their usual meaning, classical symbols such as $m(r, a, f)$, $n(r, a, f)$, $N(r, a, f)$, $T(r, f)$, $M(r, f)$ etc, ……(see [4]).

2. Main results

THEOREM 1. *Let $H(w)$ be an odd transcendental entire function. Suppose that the order of $H(\sin z)$ is finite. Put $H_a(z)=(\cos z)\cdot(H(\sin z)+2a)$. Then the set*

$$E=\{a\in \boldsymbol{C}\,;\ H_a(z) \text{ is not prime}\}$$

is at most countable set.

Remark 1. It is easy to choose a transcendental entire function $h(w)$ such that

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$$0<\overline{\lim_{r\to\infty}}\frac{\log\log M(r, h)}{\log\log r}\leqq\mu<\infty .$$

In Theorem 1, we choose $H(w)=wh(w^2)$. Obviously $H(u)$ is an odd transcendental entire function and

$$\overline{\lim_{r\to\infty}}\frac{\log\log M(r, H)}{\log\log r}=\overline{\lim_{r\to\infty}}\frac{\log\log M(r, h)}{\log\log r}\leqq\mu<\infty .$$

From Theorem 2 of paper [5] we know that the order of $H(\sin z)$ can not be larger than μ. Therefore, by Theorem 1 $H_a(z)$ is a prime entire function of finite order for any $a\notin E$. But

$$H_a^2(z)=[(1-w^2)(H(w)+2a)^2]\circ\sin z$$

is not pseudo-prime. This is an affirmative answer to question (1).

THEOREM 2. *Let $F(z)$ be a right-prime entire function and $F^{2n}(z)$ is not pseudo-prime for some natural number n. Then there must exist a transcendental entire function $h(w)$ such that*

$$F(z)=\cos(az+b)h(\sin(az+b)),\ (a \text{ and } b \text{ are constants}).$$

Remark 2. From above Theorem 2 we can easily know that for a right-prime entire function $F(z)$, $F^{2n}(z)$ is pseudo-prime for some natural number n if and only if $F^2(z)$ is pseudo-prime.

3. Some lemmas

To prove Theorem 1 and 2 we need some lemmas. At first, we prove the following Lemma 1 which is similar to Lemma 3 of paper [6].

LEMMA 1. *Let $h(w)$ be a single-valued regular function in $0<|w|<\infty$. Then there is a countable set $E\subset \boldsymbol{C}$ such that any two common roots w_1, w_2 of the simultaneous equations*

$$\begin{cases} h(w)+a\left(w+\dfrac{1}{w}\right)=t \\ h'(w)+a\left(1-\dfrac{1}{w^2}\right)=0 \end{cases} \tag{1}$$

satisfy

$$w_1+\frac{1}{w_1}=w_2+\frac{1}{w_2} \tag{2}$$

for any constant $t(\in \boldsymbol{C})$ provided that $a\notin E$.

Proof. Put

$$k(w)=-h'(w)\Big/\left(1-\frac{1}{w^2}\right);\ A=\boldsymbol{C}-(\{0,\ 1,\ -1\}\cup\{p\in\boldsymbol{C};\ k'(p)=0\}).$$

We choose open sets $\{C_i\}_{i=1}^{\infty}$ of A such that

(I) $\bigcup_{i=1}^{\infty} C_i=A$;

(II) $k(w)$ is univalent in $C_i (i=1, 2, \cdots)$;

(III) $\{k(w);\ w\in C_i\}=D_i$ is a disk $(i=1, 2, \cdots)$.

Put

$$K(w)=h(w)+\left(w+\frac{1}{w}\right)k(w);$$

$$q_i(x)=(k\,|\,C_i)^{-1}(x) \qquad x\in D_i; \tag{3}$$

$$r_i(x)=K(q_i(x)) \qquad x\in D_i;$$

$$I=\{(i,\ j)\in N\times N;\ D_i\cap D_j\neq\emptyset \ \text{ and } \ r_i(x)\not\equiv r_j(x)(x\in D_i\cap D_j)\};$$

$$S_{ij}=\{x\in D_i\cap D_j;\ r_i(x)=r_j(x)\},\ (i,\ j)\in I;$$

$$E_0=\bigcup_{i=1}^{\infty} D_i-\bigcup_{(i,j)\in I} S_{ij}.$$

Put $E=\boldsymbol{C}-E_0$, E is obviously a countable set and

$$r_i(x)=h(q_i(x))+\left(q_i(x)+\frac{1}{q_i(x)}\right)\cdot x,\ (x\in D_i) \tag{4}$$

We choose any two common roots w_1, w_2 of simultaneous equations (1). By the same method as in the proof of Lemma 3 of paper [6] we have

1) There is a $(i,\ j)\in I$ such that $a\in D_i\cap D_j$ and

$$w_1=q_i(a), \qquad w_2=q_j(a) \tag{5}$$

2) $r_i'(a)=r_j'(a)$ (6)

By (3) and (4) we see

$$\begin{aligned} r_i'(x)&=q_i(x)+\frac{1}{q_i(x)}+\left(1-\frac{1}{q_i^2(x)}\right)q_i'(x)(x-k(q_i(x))) \\ &=q_i(x)+\frac{1}{q_i(x)}. \end{aligned} \tag{7}$$

By (6) and (7) we easily obtain

$$q_i(a)+\frac{1}{q_i(a)}=q_j(a)+\frac{1}{q_j(a)}.$$

By (5) Lemma 1 is proved.

LEMMA 2[3]. *Let $f(z)$ be an entire function of exponential type σ and periodic on the real axis with period 2π. Then $f(z)$ is of the form*

$$f(z)=\sum_{k=-n}^{n} a_k e^{ikz} \qquad (n\leqq\sigma).$$

LEMMA 3[3]. *Let $f(w)$ be an entire function of order $\rho<1/2$ and $g(z)$ an entire function. Then $f(g(z))$ is periodic if and only if $g(z)$ is periodic.*

LEMMA 4[3]. *Let $f(w)$ be an entire function and $g(z)$ a polynomial of degree $n\geqq2$. If $f(g(z))$ is periodic then $g(z)$ must be a quadratic polynomial.*

LEMMA 5[4]. *Let $f(z)$ be an entire function. Then*

$$\sum_{a\neq\infty}\left(1-\frac{1}{\nu(a)}\right)\leqq1$$

where $\nu(a)$ stands for the least order of almost all a-point of $f(z)$.

LEMMA 6[2]. *All entire solutions of functional equation*

$$f^2(z)+g^2(z)=1$$

are of the form

$$f(z)=\cos\theta(z) \quad \textit{and} \quad g(z)=\sin\theta(z)$$

where $\theta(z)$ is any entire function.

4. Proof of theorems

Proof of Theorem 1. Put

$$h(w)=\frac{w^2+1}{2w}H\left(\frac{w^2-1}{2iw}\right).$$

Then $h(w)$ is a single-valued regular function in $0<|w|<\infty$, and

$$H_a(z)=\left(h(w)+a\left(w+\frac{1}{w}\right)\right)\circ e^{iz}. \tag{8}$$

Because $w=0, \infty$ are essential singularities of $h(w)$, by Picard's theorem there is a constant b such that $h(w)=b$ has infinitely many roots $\{b_k\}_{k=1}^{\infty}$, hence we have

$$\begin{aligned} T(r, h(e^{iz}))&\geqq N(r, b, h(e^{iz}))+O(1)\\ &\geqq\sum_{k=1}^{n} N(r, b_k, e^{iz})+O(1)\\ &\geqq(n-2)T(r, e^{iz})+S(r, e^{iz}). \end{aligned}\tag{9}$$

Taking n sufficiently large, by (9) we have

$$T(r, e^{iz})=o(T(r, h(e^{iz}))) \qquad (r\to\infty). \tag{10}$$

Because $h(e^{iz})$ is an entire function of finite order, by the theorem of paper

[1] and (10) it is easily seen that there is not two distinct values a such that

$$N(r, -a(e^{iz}+e^{-iz})+t, h(e^{iz}))<\alpha m(r, h(e^{iz})) \tag{11}$$

for sufficiently large value of r, any $t\in \boldsymbol{C}$ and any $\alpha\in(0, 1/2)$. Therefore, there is a countable set $E_1\subset \boldsymbol{C}$ such that the conclusion of Lemma 1 holds for $a\notin E_1$, and further there is a sequence $r_n=r_n(a)(r_n\to\infty)$ such that the following inequality holds

$$N(r_n, -a(e^{iz}+e^{-iz})+t, h(e^{iz}))\geqq\alpha m(r_n, h(e^{iz})) \tag{12}$$

for any $t\in \boldsymbol{C}$.

Hence we have

$$N(r_n, t, H_a)\geqq\alpha m(r_n, h(e^{iz})). \tag{13}$$

Put $E_2=E_1\cup\{0\}$. We shall prove $E\subset E_2$.

Assume $a\notin E_2$, let $H_a(z)=f(g(z))$. We discuss the following five cases.

a) Suppose that f and g are transcendental entire functions. Since the order of $H_a(z)=f(g(z))$ is finite, by Pólya's theorem the order of $f(w)$ is zero. Hence $f'(w)$ has infinitely many zeros $\{w_n\}_{n=1}^{\infty}$. Since $H_a'(z)=f'(g(z))\cdot g'(z)$, by (8) any root of $g(z)=w_n$ is also a common root of the simultaneous equations

$$\begin{cases} \left(h(w)+a\left(w+\dfrac{1}{w}\right)\right)\circ e^{iz}=f(w_n) \\ \left(h'(w)+a\left(1-\dfrac{1}{w^2}\right)\right)\circ e^{iz}=0. \end{cases} \tag{14}$$

By Lemma 1 any two roots z_1, z_2 of (14) satisfy

$$e^{iz_1}+e^{-iz_1}=e^{iz_2}+e^{-iz_2}. \tag{15}$$

Hence $z_2-z_1=2k\pi$ or $z_2+z_1=2k\pi$ (k is an integer). This implies

$$n(r, w_n, g)\leqq(1+o(1))\frac{2}{\pi}r \qquad (r\to\infty).$$

By the second fundamental theorem, $g(z)$ is an entire function of exponential $\sigma\leqq4/\pi$. Since $H_a(z)$ is periodic, by Lemma 3 $g(z)$ is periodic. It is easily seen from (15) that the period of $g(z)$ is $2N\pi$ with an integer N. By Lemma 2

$$g(z)=a_{-1}e^{-i(z/N)}+a_0+a_1e^{i(z/N)}. \tag{16}$$

We discuss three subcases.

a$_1$) $a_{-1}=0$. Then

$$H_a(z)=\left(h(w^N)+a\left(w^N+\frac{1}{w^N}\right)\right)\circ e^{i(z/N)},\ H_a(z)=[f(a_0+a_1w)]\circ e^{i(z/N)}.$$

Hence we have

$$h(w^N)+a\left(w^N+\frac{1}{w^N}\right)=f(a_0+a_1w).$$

Then its right hand side is regular at $w=0$, while the left hand side is not regular. This is a contradiction.

a₂) $a_1=0$. Discussing $H_a(-z)$ and treating it by the same method as in case a₁) we can obtain a contradiction.

a₃) $a_1a_{-1}\neq 0$. By (1), (14) and (16) the two roots w_n', w_n'' of $a_{-1}w^{-(1/N)}+a_0+a_1w^{1/N}=w_n$ satisfy

$$w_n'+\frac{1}{w_n'}=w_n''+\frac{1}{w_n''}. \tag{17}$$

Obviously we see $w_n'w_n''=(a_{-1}/a_1)^N$, thus we have

$$w_n'+\frac{1}{w_n'}=\left(\frac{a_1}{a_{-1}}\right)^N w_n''+\left(\frac{a_{-1}}{a_1}\right)^N\frac{1}{w_n''}. \tag{18}$$

By (17) and (18) we have

$$\left[1-\left(\frac{a_1}{a_{-1}}\right)^N\right]w_n''+\left[1-\left(\frac{a_{-1}}{a_1}\right)^N\right]\frac{1}{w_n''}=0 \quad (n=1, 2, \cdots).$$

This implies $a_{-1}=a_1e^{i(2k\pi/N)}$ with an integer k. Thus

$$g(z)=2a_1e^{i(k\pi/N)}\cos\frac{z-k\pi}{N}+a_0.$$

Therefore

$$H_a(z+k\pi)=f\left(2a_1e^{i(k\pi/N)}\cos\frac{z}{N}+a_0\right)$$

is an even function. But

$$H_a(z+k\pi)=(-1)^k(\cos z)(H((-1)^k\sin z)+2a).$$

Thus the fact that $H(w)$ is an odd function implies $H(w)=0$. This is a contradiction.

b) Suppose that f is a transcendental entire function and g is a polynomial with degree $n\geqq 2$. Since $H_a(z)$ is periodic, by Lemma 4 $g(z)$ is a quadratic polynomial. Put

$$g(z)=b(z-c)^2+d \quad (b,\ c \text{ and } d \text{ are constants}).$$

Then $H_a(z+c)=f(bz^2+d)$ is an even function. That is

$$(\cos(z+c))(H(\sin(z+c))+2a)=(\cos(-z+c))(H(\sin(-z+c))+2a).$$

Put $z=\pi/2$. Then we have

$$-\sin c(H(\cos z)+2a)=\sin c(H(-\cos c)+2a).$$

The fact that $H(w)$ is an odd function and $a \notin E_2 (a \neq 0)$ implies

$$C = k\pi \qquad (k \text{ is an integer}).$$

By the same reason as in the case a_3), $H_a(z+c)$ is not even function. This is a contradiction.

c) Suppose that f is a polynomial with degree $n \geqq 2$ and g is a transcendental entire function. Since $H_a(z)$ is periodic, by Lemma 3, $g(z)$ is periodic. We discuss two subcases:

c_1) Suppose that f' has only one zero w_1. Put $f' = b(w-w_1)^{n-1}$ with a constant b and a natural number $n > 1$. Thus we see

$$f = \frac{b}{n}(w-w_1)^n + t, \ H_a(z) = \frac{b}{n}(g(z)-w_1)^n + t, \ (t \in \boldsymbol{C}).$$

Therefore we have

$$N(r, t, H_a) = nN(r, w_1, g). \tag{19}$$

By the discussion in case a)

$$N(r, w_1, g) = O(r) \qquad (r \to \infty). \tag{20}$$

Then (20) (19) and (10) imply

$$N(r, t, H_a) = o(m(r, h(e^{iz}))) \qquad (r \to \infty).$$

This contradicts (13).

c_2) Suppose that f' has at least two zeros w_1, w_2. By the same discussion as in the case a), $g(z)$ must be of the form (16). From the discussion of case a_1), a_2) and a_3) we obtain a contradiction.

d) Suppose that f is a meromorphic (not an entire) function and g is an transcendental entire function. Let w_0 be a pole of f. Since $H_a(z) = f(g(z))$ is an entire function, $g(z)$ does not assume w_0. By Picard's theorem f has only one pole. Put

$$f(w) = f_1(w)/(w-w_0)^p \quad \text{and} \quad g(z) = w_0 + e^{Q(z)},$$

where f_1 is a transcendental entire function, $f_1(w_0) \neq 0$, p is a natural number and $Q(z)$ is an entire function. Since the order of $H_a(z) = f(g(z))$ is finite, by Pólya's theorem the order of f_1 is zero. Now

$$f'(w) = \frac{(w-w_0)f_1'(w) - pf_1(w)}{(w-w_0)^{p+1}}.$$

It is easily seen that $(w-w_0)f_1'(w) - pf_1(w)$ is not a polynomial. Thus $f'(w)$ has infinitely many zeros $\{w_n\}_0^\infty$. By the discussion in case a), $g(z)$ must be an entire function of exponential type. Thus we have

$$g(z) = w_0 + e^{az+b} \quad (a \text{ and } b \text{ are constants}).$$

By (15) it is easily seen that $a=i/N$ with an integer N. Since

$$f(g)=[f_2(w)/w^p]\circ e^{i(z/N)}, \qquad f_2=f_1(w_0+e^b w)/e^{pb}$$

and

$$H_a(z)=\left(h(w^N)+a\left(w^N+\frac{1}{w^N}\right)\right)\circ e^{i(z/N)},$$

thus we have

$$h(w^N)+a\left(w^N+\frac{1}{w^N}\right)=f_2(w)/w^p.$$

Then $w=0$ is a essential singularity of the left hand side, while $w=0$ is a pole of the right hand side. This is a contradiction.

e) Suppose that f is a rational function (not a polynomial) and g is a transcendental meromorphic (not entire) function. Let w_0 be the pole of f. Put

$$g_1(z)=\frac{1}{g(z)-w_0} \quad \text{and} \quad f_1(w)=f\left(\frac{1}{w}+w_0\right).$$

Then $f(g)=f_1(g_1)$. This case can be reduced to the case c) or d).

From a) to e) we know that if $a\notin E_2$ then $H_a(z)$ is prime Hence $E\subset E_2$. Theorem 1 is thus proved.

Proof of Theorem 2. Since $F^{2n}(z)$ is not pseudo-prime, there exist a transcendental meromorphic function f and transcendental entire function g such that $F^{2n}(z)=f(g(z))$. By the same method as in the proof of Theorem 1 of paper [8], it is easily seen that f can not be meromorphic. It is obviously that f must have zeros. Because if not, $f^{1/2n}$ is a transcendental entire function and $F(z)=\mu f^{1/2n}\circ g(z)(\mu^{2n}=1)$. This contradicts that $F(z)$ is right-prime. For the same reason as in the proof of Theorem 2 of paper [7], the following three cases may occur.

a) Let $f(w)=k^{2n}(w)$ with some transcendental entire function $k(w)$. Then $F(z)=\mu k(g(z))(\mu^{2n}=1)$, which is a contradiction.

b) Let $f(w)=(w-w_1)^p k^{2n}(w)$ with $w_1\in \boldsymbol{C}$, a natural number $p<2n$ and a transcendental entire function $k(w)$. Then obviously $g(z)=w_1+S^q(z)$ with a transcendental entire function $S(z)$, a natural number q and $2n|pq$. Thus we obtain

$$F(z)=\mu S^{pq/2n}(z)\cdot k(w_1+S^q(z)) \qquad (\mu^{2n}=1).$$

This is a contradiction.

c) Let $f(w)=(w-w_1)^{p_1}(w-w_2)^{p_2}k^{2n}(w)$ with $w_1, w_2\in \boldsymbol{C}$, a transcendental entire function $k(w)$ and two natural number $p_1, p_2<2n$. At first

$$\frac{p_1}{2}<n \quad \text{and} \quad \frac{p_2}{2}<n. \tag{21}$$

Obviously $g(z)=w_1+S_1^{q_1}(z)$ and $g(z)=w_2+S_2^{q_2}(z)$, where $S_1(z)$ and $S_2(z)$ are two transcendental entire functions, q_1 and q_2 are two natural numbers satisfing

$$2n \mid p_1q_1 \quad \text{and} \quad 2n \mid p_2q_2 \tag{22}$$

By Lemma 5, $\nu(w_1)=\nu(w_2)=2$. Thus $q_1=q_2=2$. By (21), (22) we obtain $p_1=p_2=n$. It is easily seen that

$$\frac{S_1^2(z)}{w_2-w_1}+\frac{S_2^2(z)}{w_1-w_2}=1.$$

By Lemma 6, there exists an entire function $\theta(z)$ such that

$$S_1^2(z)=(w_2-w_1)\cos^2\theta(z) \quad \text{and} \quad S_2^2(z)=(w_1-w_2)\sin^2\theta(z).$$

Thus we obtain

$$F(z)=\mu i(w_2-w_1)\cos\theta(z)\cdot\sin\theta(z)\cdot k(w_2+(w_1-w_2)\sin^2\theta(z)) \quad (\mu^{2n}=1).$$

Put $h(w)=\mu i(w_2-w_1)wk(w_2+(w_1-w_2)w^2)$. Then

$$F(z)=\cos\theta(z)\cdot h(\sin\theta(z)).$$

Since $F(z)$ is right-prime, we have $\theta(z)=az+b$ with constants a and b. Theorem 2 is thus proved.

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