S. DESHMUKH
KODAI MATH. J.
11 (1988), 155—158

REDUCTION IN CODIMENSION OF MIXED FOLIATE *CR*-SUBMANIFOLDS OF A KAEHLER MANIFOLD

BY SHARIEF DESHMUKH

Abstract

Starting with a proper mixed foliate CR-submanifold M of dimension $2p+q$ in a hyperbolic complex space form $\bar{M}(-4)$ of dimension $m(m \geqq 2p+2q)$, it has been proved that under a suitable condition on second fundamental form there exists a $(2p+2q)$-dimensional totally geodesic submanifold M' of $\bar{M}$ such that M is a mixed foliate CR-submanifold of M'.

1. Let $\bar{M}$ be an m-dimensional hyperbolic complex space form, that is, a Kaehler manifold of constant holomorphic sectional curvature -4. The curvature tensor $\bar{R}$ or $\bar{M}$ is given by

$$\bar{R}(X, Y)Z = -\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\}, \tag{1.1}$$

where J is the almost complex structure of $\bar{M}$ and g is the hermitian metric.

A $(2p+q)$-dimensional submanifold M of $\bar{M}$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions D and $D^{\perp}$ such that $JD=D$ and $JD^{\perp} \subset \nu$, where ν is the normal bundle of M and $\dim D=2p$, $\dim D^{\perp}=q$, [1]. A CR-submanifold is said to be proper if neither $D=\{0\}$ nor $D^{\perp}=\{0\}$. We shall denote by $\bar{\nabla}$, ∇, $\nabla^{\perp}$ the Riemannian connections on $\bar{M}$, M, and the normal bundle respectively. They are related by the Gauss and Weingarten formulae

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad N \in \nu \tag{1.2}$$

where $h(X, Y)$ and $A_N X$ are second fundamental forms and $g(h(X, Y), N) = g(A_N X, Y)$.

A CR-submanifold M is said to be mixed foliate if (i) D is integrable, and (ii) $h(X, Y)=0$, $X \in D$, $Y \in D^{\perp}$. Mixed foliate CR-submanifolds have been studied by A. Bejancu [1] and B. Y. Chen [2]. It is known that if M is a mixed foliate CR-submanifold of a complex space form $\bar{M}(c)$, then $c<0$, that is why we consider mixed foliate CR-submanifolds of $\bar{M}(-4)$. In a mixed foliate CR-submanifold of a Kaehler manifold the following hold good [2]

Received May 2, 1987

(1.3) $$h(X, JY)=h(JX, Y), \quad X, Y\in D$$

(1.4) $$A_N X\in D, \quad X\in D \quad \text{and} \quad A_N X\in D^\perp, \quad X\in D^\perp$$

(1.5) $$A_N JX=-JA_N X, \quad X\in D$$

(1.6) $$A_{JX}Y=A_{JY}X, \quad X, Y\in D^\perp.$$

The normal bundle splits as $\nu=JD^\perp\oplus\mu$, where μ is a J-invariant sub-bundle of ν.

For a submanifold M, the first normal space N_p^1 and the first osculating space O_p^1 at $p\in M$ are defined by

$$N_p^1=\{h_p(X_p, Y_p): X_p, Y_p\in T_pM\} \quad \text{and} \quad O_p^1=T_pM\oplus N_p^1,$$

where T_pM is the tangent space of M at p. A subspace V of $T_p\overline{M}$ is said to define a Lie-triple system if $\bar{R}_p(X_p, Y_p)Z_p\in V$ for $X_p, Y_p, Z_p\in V$. For a Lie-triple system V in a symmetric space $\overline{M}$, there exists a unique complete totally geodesic submanifold M' of $\overline{M}$ such that $T_pM'=V$, (cf. [6]).

The equations of Gauss, Codazzi and Ricci are

(1.7) $$R(X, Y; Z, W)=\bar{R}(X, Y; Z, W)+g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W))$$

(1.8) $$[\bar{R}(X, Y)Z]^\perp=(\bar{\nabla}_X h)(Y, Z)-(\bar{\nabla}_Y h)(X, Z)$$

(1.9) $$\bar{R}(X, Y; N_1, N_2)=R^\perp(X, Y; N_1, N_2)-g([A_{N_1}, A_{N_2}](X), Y),$$

where $[\ ]^\perp$ denotes the normal component, $R^\perp$ is the curvature tensor of $\nabla^\perp$, and $(\bar{\nabla}_X h)(Y, Z)=\nabla_X^\perp h(Y, Z)-h(\nabla_X Y, Z)-h(Y, \nabla_X Z)$.

2. Let M be a mixed foliate CR-submanifold of $\overline{M}(-4)$. Then it follows that (cf. [2]).

(2.1) $$\nabla_X Y D^\perp, \quad X, Y\in D^1; \ \nabla_X Y\in D, \quad X\in D^\perp, \quad Y\in D \quad \text{and}$$
$$\nabla_X^1 JY\in JD^1, \quad X\in D, \quad Y\in D^\perp.$$

PROPOSITION 2.1. *Let M be a proper mixed foliate CR-submanifold of hyperbolic complex space form $\overline{M}(-4)$. Then $h(X, Y)\in JD^1$, $X, Y\in D$.*

Proof. For $X, Y\in D$ and $Z\in D^\perp$, the equation (1.1) gives

(2.2) $$[\bar{R}(X, Y)Z]^\perp=-2g(X, JY)JZ.$$

Using this in equation (1.8) and noting that $h([X, Y], Z)=0$, we get

(2.3) $$-2g(X, JY)JZ=h(X, \nabla_Y Z)-h(Y, \nabla_X Z).$$

Taking the inner product with $JW\in JD^\perp$ and replacing X by JX we get

$$2g(X, Y)g(Z, W)=g(A_{JW}X, A_{JZ}Y)+g(A_{JZ}X, A_{JW}Y), \tag{2.4}$$

where we have used $A_{JW}X$, $JA_{JW}X\in D$ and the equations (1.2), (1.5).

Also for $X, Y\in D$, $Z\in D^{\perp}$ and $N\in\mu$, using (1.1) and (1.9), we get

$$g([A_{JZ}, A_N](X), Y)=0, \tag{2.5}$$

where we use the fact $R^{\perp}(X, Y)JZ\in JD^{\perp}$ by virtue of (2.1). Taking inner product with $N\in\mu$ in (2.3), we get with similar techniques as in (2.4) that

$$g(A_N X, A_{JZ}Y)+g(A_{JZ}X, A_N Y)=0. \tag{2.6}$$

Combination of (2.5) and (2.6) gives $A_{JZ}(A_N X)=0$, $X\in D$. Since, for every $X\in D$, $A_N X\in D$, replacing X by $A_N X$ in (2.4) and using $A_{JZ}(A_N X)=A_{JW}(A_N X)=0$, we get

$$2g(A_N X, Y)g(Z, W)=0$$

and M being a proper CR-submanifold we get $g(h(X, Y), N)=0$.

COROLLARY 2.1. *Let M be a proper mixed foliate CR-submanifold of a hyperbolic complex space form $\bar{M}(-4)$. Then*

(a) $A_{JZ}^2X=X$, $X\in D$ *and* Z *a unit vector in* D^1

(b) $A_{JZ}A_{JW}X=-A_{JZ}A_{JZ}X$, $X\in D$ *and* $Z\perp W$, $Z, W\in D^{\perp}$.

The proof follows from equation (2.4).

LEMMA 2.1. *Let M be a mixed foliate CR-submanifold of $\bar{M}(-4)$. Then for $X, Y, Z\in X(M)$, the Lie-algebra of vector fields on M*

$$\bar{R}_p(X_p, Y_p)Z_p,\ \bar{R}_p(JX_p, Y_p)Z_p,\ \bar{R}(JX_p, JY_p)JZ_p\in T_pM\oplus JD_p^1$$

for $p\in M$.

The proof is trivial and follows at once from (1.1).

LEMMA 2.2. *Let M be a proper mixed foliate CR-submanifold of $\bar{M}(-4)$ satisfying $h(X, Y)\in JD^{\perp}$ for $X, Y\in D^{\perp}$. Then $T_pM\oplus JD_p^{\perp}$ is the first osculating space at $p\in M$.*

Proof. It suffices to prove

$$JD^1=\{h(X, Y):\ X, Y\in *(M)\}.$$

From proposition 2.1 it follows that $\{h(X, Y): X, Y\in X(M)\}\subset JD^{\perp}$. Now suppose there exists a unit vector $JZ\in JD^{\perp}$ such that $g(h(X, Y), JZ)=0$ for all $X, Y\in X(M)$. This gives $g(A_{JZ}X, Y)=0$ for all Y. In particular we have for $X\in D$ and $Y=A_{JZ}X$, $g(A_{JZ}X, A_{JZ}X)=0$. Which is impossible by Corollary 1.1 (a), unless M is totally real (that is $D=\{0\}$). Hence we get the quality.

Now we prove the main result.

THEOREM. *Let M be a $(2p+q)$-dimensional proper mixed foliate CR-submanifold of the hyperbolic complex space form $\overline{M}(-4)$ of dimension m $(m \geqq 2p+2q)$ satisfying $h(X, Y) \in JD^{\perp}$ for $X, Y \in D^{\perp}$. Then there exists a complete totally geodesic complex submanifold M' of dimension $2p+2q$ of $\overline{M}$ such that M is a mixed foliate CR-submanifold of M'.*

Proof. By Lemmas 2.1 and 2.2, it follows that the osculating space O_p^1 is a Lie-triple system at each point $p \in M$. Hence there exists a complete totally geodesic submanifold M' of $\overline{M}(-4)$ of dimension$=\dim O_p^1 = 2p+2q$. By a result of Chen and Ogiue [4], submanifolds of complex space forms with parallel second fundamental form are either complex or totally real. Since M' is totally geodesic it satisfies the hypothesis of this Theorem, and therefore M' is either a complex or totally real submanifold. But owing to the presence of the non-trivial J-invariant D, it cannot be totally real. In fact M' is a hyperbolic complex space form, and M is a mixed foliate CR-submanifold of $M'(-4)$.

REFERENCES

[1] A. BEJANCU, *CR*-submanifold of a Kaehler manifold, I, II, Proc. Amer. Math. Soc. **69** (1978), 135-142, Trans. Amer. Math. Soc. **250** (1979), 333-345.

[2] B.Y. CHEN, *CR*-submanifolds of a Kaehler manifold II, J. Diff. Geom. **16** (1981), 493-509.

[3] B.Y. CHEN, Geometry of Submanifolds and its Applications, Science Univ. Tokyo, (1981).

[4] B.Y. CHEN AND K. OGIUE, On totally real submanifolds, Trans. Amer. Math. Soc. **193** (1975), 257-266.

[5] J. ERBACHER, Reduction of codimension of an isometric immersion, J. Diff. Geom. **5** (1971), 333-340.

[6] S. HELGASON, Differential Geometry and Symmetric spaces, Academic Press, New York (1962).

DEPARTMENT OF MATHEMATICS
KING SAUD UNIVERSITY
P.O. BOX 2455, RIYADH 11451
SAUDI ARABIA