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ON THE NUMBER OF BRANCHES OF AN 1-DIMENSIONAL SEMIANALYTIC SET

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1. Introduction.

Let $F=(F_1, \cdots, F_{n-1}): (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^{n-1}, 0)$ be a germ of an analytic map, and let $\tilde{F}: (B, 0)\to(\boldsymbol{R}^{n-1}, 0)$ be a representative mapping of F, where B is a small ball centered at the origin in $\boldsymbol{R}^n$. Let us donote $X=\tilde{F}^{-1}(0)\cap B$. Assume that $0\in\boldsymbol{R}^n$ is an isolated singular point in X (i.e. $0\in\boldsymbol{R}^n$ is an isolated point in $\{x\in X \mid \mathrm{rank}[D\tilde{F}(x)]<n-1\}$). If B is small enough, the set $X-\{0\}$ is void or a finite disjoint union of analytic curves.

Let $G: (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$ be an analytic germ. We may suppose that a representative $\tilde{G}$ of G is defined in B.

DEFINITION 1.1. We shall say that a pair (G, F) has property $\mathscr{A}$ if $0\in\boldsymbol{R}^n$ is isolated in $\{x\in X \mid \tilde{G}(x)=0\}$.

Assume that a pair (G, F) has property $\mathscr{A}$. There is a well-known fact that if B is small enough then the function $\tilde{G}$ has a constant sign on each connected component of $X-\{0\}$. Let

$b(F)$=the number of branches of $X-\{0\}$,

$b_+(G, F)$=the number of branches of $X-\{0\}$ on which $\tilde{G}$ is positive,

$b_-(G, F)$=the number of branches of $X-\{0\}$ on which $\tilde{G}$ is negative.

Of course, $b_+(G, F)+b_-(G, F)=b(F)$.

Let $(x_1, \cdots, x_n)$ be a coordinate system in $\boldsymbol{R}^n$. Let $\Delta=\dfrac{\partial(\tilde{G}, \tilde{F}_1, \cdots, \tilde{F}_{n-1})}{\partial(x_1, \cdots, x_n)}$ be the Jacobian of a map $(\tilde{G}, \tilde{F}_1, \cdots, \tilde{F}_{n-1}): B\to\boldsymbol{R}^n$, and let $H=(\Delta, \tilde{F}_1, \cdots, \tilde{F}_{n-1}): (B, 0)\to(\boldsymbol{R}^n, 0)$. In this paper we show (Theorem 3.1) that

$$b_+(G, F)-b_-(G, F)=2\deg(H),$$

where $\deg(H)$ is the topological degree of the map-germ $H: (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^n, 0)$ at the origin.

Let $\omega=x_1^2+\cdots+x_n^2$. Clearly, a pair (ω, F) has property $\mathscr{A}$ and $b_+(\omega, F)=b(F)$, $b_-(\omega, F)=0$. Thus, as a consequence of the above fact, we get a formula for the number $b(F)$. This formula was proved by Kenji Aoki, Takuo Fukuda, Wei-Zhi Sun and Takashi Nishimura (in case $n=2$ [1], in general case [2]).

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Let $\Theta=x_n$, and let us assume that a pair (Θ, F) has property $\mathscr{A}$. Thus there are $b_+(\Theta, F)$ branches of $X-\{0\}$ contained in the half region $\{x_n>0\}$ and $b_-(\Theta, F)$ branches contained in the half region $\{x_n<0\}$. In this case we get a formula for a number $b_+(\Theta, F)-b_-(\Theta, F)$. This formula was proved by K. Aoki, T. Fukuda and T. Nishimura [3].

A proof presented here differs from that which are presented in [1, 2, 3]. It seems to be more geometrical.

Our result may be used in a more general case. Let $G_1, \cdots, G_s: (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}, 0)$ be germs of analytic functions. Assume that each pair (G_i, F), $1\leqq i\leqq s$, has property $\mathscr{A}$. Let $\beta=(\beta_1, \cdots, \beta_s)\in\{0, 1\}^s$. If B is small enough then a semianalytic set

$$X_\beta=\{x\in X-\{0\}\,|\,(-1)^{\beta_1}\tilde{G}_1(x)>0, \cdots, (-1)^{\beta_s}\tilde{G}_s(x)>0\}$$

is void or a finite union of curves. We shall show how to compute the number of branches of X_β in terms of topological degrees of some finite family of map-germs $H_\alpha: (\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^n, 0)$, $\alpha\in\{0, 1\}^s$ (see Theorem 3.4).

There is possible a different aproach to the same problem in case $n=2$. In [4] is described another algorithm of calculating of the number of branches of X_β in terms of Puiseux series of F and $G_1, \cdots, G_s$.

2. Preliminaries.

The following lemma is the most essential for the further part of this paper.

LEMMA 2.1. *Let* $F=(F_1, \cdots, F_{n-1}): U\to\boldsymbol{R}^{n-1}$, $G: U\to\boldsymbol{R}$, *be* C^2-*functions defined in an open set* $U\subset\boldsymbol{R}^n$. *Assume that* $\operatorname{rank}[DF(x_0)]=n-1$, *where* $x_0\in U$. *From the implicit function theorem* $W=\{x\in U\,|\,F(x)=F(x_0)\}$ *is an* 1-*dimensional* C^2-*manifold in some neighbourhood of* x_0.

Let $\Delta=\dfrac{\partial(G, F_1, \cdots, F_{n-1})}{\partial(x_1, \cdots, x_n)}$ *be the Jacobian of a map* $(G, F_1, \cdots, F_{n-1}): U\to\boldsymbol{R}^n$, *let* $H=(\Delta, F_1, \cdots, F_{n-1}): U\to\boldsymbol{R}^n$, *and let* $\Delta_1=\dfrac{\partial(\Delta, F_1, \cdots, F_{n-1})}{\partial(x_1, \cdots, x_n)}=\det[DH]$. *Then*

(i) $G|W$ *has a critical point at* x_0 *if and only if* $\Delta(x_0)=0$,

(ii) $G|W$ *has a non-degenerate critical point at* x_0 *if and only if* $\Delta(x_0)=0$ *and* $\Delta_1(x_0)\neq0$,

(iii) *if* $\Delta(x_0)=0$ *and* $\Delta_1(x_0)>0$ *then* $G|W$ *has a minimum at* x_0,

(iv) *if* $\Delta(x_0)=0$ *and* $\Delta_1(x_0)<0$ *then* $G|W$ *has a maximum at* x_0.

Proof. We may assume that $x_0=0\in\boldsymbol{R}^n$. Clearly, $G|W$ has a critical point at $0\in\boldsymbol{R}^n$ if and only if a vector $\operatorname{grad}G(0)$ belongs to the linear space spaned by vectors $\operatorname{grad}F_1(0), \cdots, \operatorname{grad}F_{n-1}(0)$. Thus $G|W$ has a critical point at the origin if and only if $\Delta(0)=0$.

Assume that $\Delta(0)=0$. After an ortogonal change of coordinates we can find a new well-oriented coordinate system $(y_1, \cdots, y_n)$ such that

(1) $$D_1F_1(0)=\cdots=D_1F_{n-1}(0)=0,$$

where $D_i f$ is the i-th partial derivative of f. Hence the tangent space T_0W is spaned by a vector $(1, 0, \cdots, 0)$ and there are C^2-functions $\phi_2, \cdots, \phi_n: (\boldsymbol{R}, 0)\to(\boldsymbol{R}, 0)$ such that $W=\{(y_1, \phi_2(y_1), \cdots, \phi_n(y_1)) \mid y_1\in\boldsymbol{R}\}$ in some neighbourhood of the origin. Clearly

(2) $$D_1\phi_2(0)=\cdots=D_1\phi_n(0)=0.$$

Let $g(y_1)=G(y_1, \phi_2(y_1), \cdots, \phi_n(y_1))$. The function $G|W$ has a critical point at the origin, and then from (2) we have

(3) $$D_1g(0)=D_1G(0)=0,$$

(4) $$D_1^2g(0)=D_1^2G(0)+\sum_{i=2}^{n} D_iG(0)D_1^2\phi_i(0).$$

Since $F_j(y_1, \phi_2(y_1), \cdots, \phi_n(y_1))\equiv$constant, then from (2) we have

$$D_1^2F_j(0)+\sum_{i=2}^{n} D_iF_j(0)D_1^2\phi_i(0)=0.$$

Let $M(x)=\det[D_iF_j(x)]$, where $2\leqq i\leqq n$, $1\leqq j\leqq n-1$, and let

$$N_i(x)=\det\begin{bmatrix} D_2F_1(x) & \cdots & D_1^2F_1(x) & \cdots & D_nF_1(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ D_2F_{n-1}(x) & \cdots & D_1^2F_{n-1}(x) & \cdots & D_nF_{n-1}(x)\end{bmatrix},$$

where $2\leqq i\leqq n$, and the column $(D_1^2F_1(x), \cdots, D_1^2F_{n-1}(x))$ is situated at the $(i-1)$-th place. By (1) we have $M(0)\neq 0$, and then from Cramer's rule

$$D_1^2\phi_i(0)=-N_i(0)/M(0).$$

From (4) we have

(5) $$\begin{aligned}\operatorname{sign}(D_1^2g(0))&=\operatorname{sign}\Big(\Big(D_1^2G(0)M(0)-\sum_{i=2}^{n} D_iG(0)N_i(0)\Big)\Big/M(0)\Big)\\&=\operatorname{sign}\Big(M(0)\Big(D_1^2G(0)M(0)-\sum_{i=2}^{n} D_iG(0)N_i(0)\Big)\Big).\end{aligned}$$

Let $M_i(x)=\det\begin{bmatrix} D_1F_1(x) & \cdots & \widehat{D_iF_1(x)} & \cdots & D_nF_1(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ D_1F_{n-1}(x) & \cdots & \widehat{D_iF_{n-1}(x)} & \cdots & D_nF_{n-1}(x)\end{bmatrix}$, where $2\leqq i\leqq n$. From (1) we have

(6) $$M_2(0)=\cdots=M_n(0)=0.$$

The change of coordinates was ortogonal and then

$$\Delta(x)=D_1G(x)M(x)-D_2G(x)M_2(x)+\cdots\pm D_nG(x)M_n(x),$$

for any $x\in U$. By (3) and (6) we have

$$D_1\Delta(0)=D_1^2G(0)M(0)-D_2G(0)D_1M_2(0)+\cdots\pm D_nG(0)D_1M_n(0).$$

From (1) we have

$$D_1M_i(0)=\det\begin{bmatrix} D_1^2F_1(0) & \cdots & \widehat{D_iF_1(0)} & \cdots & D_nF_1(0) \\ \hline D_1^2F_{n-1}(0) & \cdots & \widehat{D_iF_{n-1}(0)} & \cdots & D_nF_{n-1}(0)\end{bmatrix}$$

$$=(-1)^iN_i(0).$$

Hence $D_1\Delta(0)=D_1^2G(0)M(0)-\sum_{i=2}^{n}D_iG(0)N_i(0)$. From (1) and (5) we have $\Delta_1(0)=D_1\Delta(0)M(0)$ and $\operatorname{sign}(\Delta_1(0))=\operatorname{sign}(D_1^2g(0))$, and the lemma is proved. □

Let $F=(F_1,\cdots,F_{n-1})\colon(\boldsymbol{R}^n,0)\to(\boldsymbol{R}^{n-1},0)$ and $G\colon(\boldsymbol{R}^n,0)\to(\boldsymbol{R},0)$ be germs of analytic maps. We may suppose that representatives of F and G are defined in an open neighbourhood U of the origin. Assume that $0\in\boldsymbol{R}^n$ is an isolated singular point in $X=F^{-1}(0)\cap U$. Let $B_r=\{x\in\boldsymbol{R}^n\mid\|x\|<r\}$, $S_r=\{x\in\boldsymbol{R}^n\mid\|x\|=r\}$. Using well-known facts from the theory of semianalytic sets we get

Remark 2.2. If a pair (G,F) has property $\mathscr{A}$ then there is $r>0$ such that $(X-\{0\})\cap B_r$ is a finite disjoint union of 1-dimensional connected analytic manifolds $Y_1,\cdots,Y_k$, $k\geqq 0$ (if $k=0$ then $(X-\{0\})\cap B_r$ is vide). For any $r'\in(0,r)$ the sphere $S_{r'}$ is transverse to each Y_i and $S_{r'}\cap Y_i$ has exactly one point. Moreover, a restricted function $G|Y_i$ has a constant sign for each $i\in\{1,\cdots,k\}$. Thus numbers $b(F)=k$, $b_+(G,F)=\#\{x\in X\cap S_{r'}\mid G(x)>0\}$, $b_-(G,F)=\#\{x\in X\cap S_{r'}\mid G(x)<0\}$ are well-defined. Of course $b(F)=b_+(G,F)+b_-(G,F)$.

Let $\Delta=\dfrac{\partial(G,F_1,\cdots,F_{n-1}}{\partial(x_1,\cdots,x_n)}$ be the Jacobian of the map $\boldsymbol{R}^n\ni x\mapsto(G(x),F(x))\in\boldsymbol{R}^n$, and let $H=(\Delta,F)\colon(\boldsymbol{R}^n,0)\to(\boldsymbol{R}^n,0)$.

LEMMA 2.3. *If the pair (G,F) has property $\mathscr{A}$ then $0\in\boldsymbol{R}^n$ is isolated in $H^{-1}(0)$.*

Proof. From Remark 2.2 there are 1-dimensional analytic manifolds $Y_1,\cdots,Y_k$ such that $(X-\{0\})\cap B_r=Y_1\cup\cdots\cup Y_k$. If r is sufficiently small then from the Curve Selection Lemma there are analytic maps $p_i\colon[0,\varepsilon)\to Y_i\cup\{0\}$ such that $p_i^{-1}(0)=\{0\}$ and $p_i\colon(0,\varepsilon)\to Y_i$ is an analytic diffeomorphism. The function G is analytic, $G(0)=0$, and from Remark 2.2, $G^{-1}(0)\cap Y_i=\emptyset$. Thus if r and ε are small enough then $G\circ p_i$ is a monotonic function, and then $G|Y_i$ has no critical points. Hence, from Lemma 2.1,

$$\Delta(x)=\frac{\partial(G,F_1,\cdots,F_{n-1})}{\partial(x_1,\cdots,x_n)}(x)\neq 0$$

for every $x\in Y_i$. Clearly $H^{-1}(0)\cap B_r\subset F^{-1}(0)\cap B_r=Y_1\cup\cdots\cup Y_k\cup\{0\}$. Then $0\in\boldsymbol{R}^n$ is isolated in $H^{-1}(0)$. □

Let M be a compact 1-dimensional manifold with a boundary ∂M. Clearly, ∂M is a finite set. Let $G: M\rightarrow \boldsymbol{R}$ be a C^2-function. Assume that a set C of critical points of G is a finite subset of $M-\partial M$ and that each critical point of G is non-degenerate. Let

$$m_1=\#\{x\in C \mid G \text{ has a minimum at } x\},$$

$$m_2=\#\{x\in C \mid G \text{ has a maximum at } x\}.$$

LEMMA 2.4. *Let the notation be as above. Suppose that*
(i) *if* $x\in\partial M$ *then* $G(x)\neq 0$,
(ii) *if* $x\in\partial M$ *and* $G(x)<0$ *then* G *has a minimum at* x,
(iii) *If* $x\in\partial M$ *and* $G(x)>0$ *then* G *has a maximum at* x.

Then

$$\#\{x\in\partial M \mid G(x)>0\}-\#\{x\in\partial M \mid G(x)<0\}=2(m_1-m_2).$$

The proof is straightforward. □

3. Main theorem.

Let the notation be as above. Let $\deg(H)$ be the topological degree of the mapping $x\mapsto H(x)/\|H(x)\|$ from a small sphere S_r centered at the origin to the unit sphere in $\boldsymbol{R}^n$.

THEOREM 3.1. *Assume that a pair* (G, F) *has property* $\mathscr{A}$. *Then*

$$b_+(G, F)-b_-(G, F)=2\deg(H).$$

Proof. Let $y\in\boldsymbol{R}^{n-1}$ be a regular value of F, and let $S_r\subset\boldsymbol{R}^n$ be a small sphere centered at the origin. From Remark 2.2, $X=F^{-1}(0)$ is transverse to S_r. Hence, if y is sufficiently close to the origin then $F^{-1}(y)$ is transverse to S_r too. Moreover, we may assume that

$$\begin{aligned} b_+(G, F)&=\#\{x\in X\cap S_r \mid G(x)>0\}=\#\{x\in F^{-1}(y)\cap S_r \mid G(x)>0\},\\ b_-(G, F)&=\#\{x\in X\cap S_r \mid G(x)<0\}=\#\{x\in F^{-1}(y)\cap S_r \mid G(x)<0\}. \end{aligned} \tag{1}$$

In the proof of Lemma 2.3 we have shown that $G|(X-\{0\})$ has no critical points in some neighbourhood of the origin. Since $G^{-1}(0)\cap X=\{0\}$ then if $x\in X\cap S_r\cap\{G>0\}$ then $G|B_r\cap X$ has a local maximum at x, if $x\in X\cap S_r\cap\{G<0\}$ then $G|B_r\cap X$ has a local minimum at x. Moreover, if y is close to the origin then critical points of $G|F^{-1}(y)\cap B_r$ belong to $F^{-1}(y)\cap B_{r/4}$. There is a function $\tilde{G}$ such that the first and second derivatives of $\tilde{G}$ uniformly approximate those of G, $\tilde{G}|F^{-1}(y)\cap B_r$ is a Morse function and the set $\tilde{C}$ of critical points of $\tilde{G}|F^{-1}(y)\cap B_r$ is contained in $F^{-1}(y)\cap B_{r/2}$. We can also assume that

(i) if $x\in F^{-1}(y)\cap S_r$ then $\tilde{G}(x)\neq 0$,

(ii) if $x\in F^{-1}(y)\cap S_r$ and $\tilde{G}(x)<0$ then $\tilde{G}|F^{-1}(y)\cap B_r$ has a local minimum at x,

(iii) if $x\in F^{-1}(y)\cap S_r$ and $\tilde{G}(x)>0$ then $\tilde{G}|F^{-1}(y)\cap B_r$ has a local maximum at x.

Let $\tilde{\Delta}=\dfrac{\partial(G, F_1, \cdots, F_{n-1})}{\partial(x_1, \cdots, x_n)}$. Of course, $x\in F^{-1}(y)$ is a critical point of $\tilde{G}|F^{-1}(y)$ if and only if $\tilde{\Delta}(x)=0$. Thus $\tilde{C}=\tilde{H}^{-1}(0, y)$, where $\tilde{H}=(\tilde{\Delta}, F_1, \cdots, F_{n-1})$. From Lemma 2.1 we have

$$
\begin{aligned}
m_1&=\#\{x\in\tilde{C}\mid \tilde{G}|F^{-1}(y) \text{ has a minimum at } x\}\\
&=\#\{x\in\tilde{H}^{-1}(0, y)\cap B_r\mid \det[D\tilde{H}(x)]>0\},\\
m_2&=\#\{x\in\tilde{C}\mid \tilde{G}|F^{-1}(y) \text{ has a maximum at } x\}\\
&=\#\{x\in\tilde{H}^{-1}(0, y)\cap B_r\mid \det[D\tilde{H}(x)]<0\}.
\end{aligned}
\tag{2}
$$

The function $\tilde{G}|F^{-1}(y)\cap B_r$ has only non-degenerate critical points and then, from Lemma 2.1,

$$\{x\in\tilde{H}^{-1}(0, y)\cap B_r\mid \det[D\tilde{H}(x)]=0\}=\emptyset.$$

Hence the point $(0, y)$ is a regular value of $\tilde{H}|B_r$.

Let d be the degree of the mapping

$$S_r\ni x\longmapsto \tilde{H}(x)/\|\tilde{H}(x)\|\in S^{n-1}.$$

From (2), $m_1-m_2=d$. Clearly, if y is sufficiently close to the origin and $\tilde{G}$ is sufficiently close to G then $d=\deg(H)$, and then $m_1-m_2=\deg(H)$.

The function $\tilde{G}|F^{-1}(y)\cap B_r$ satysfies all assumptions of Lemma 2.4. Thus

$$
\begin{aligned}
&\#\{x\in F^{-1}(y)\cap S_r\mid \tilde{G}(x)>0\}-\#\{x\in F^{-1}(y)\cap S_r\mid \tilde{G}(x)<0\}\\
&=2(m_1-m_2).
\end{aligned}
$$

Then from (1) we have

$$b_+(G, F)-b_-(G, F)=2\deg(H). \qquad \square$$

Let $\omega=x_1^2+\cdots+x_n^2$. Clearly a pair (ω, F) has property $\mathscr{A}$. Of course, $b_+(\omega, F)=b(F)$, $b_-(\omega, F)=0$. As a consequence of Theorem 3.1 we get a theorem which was proved by K. Aoki, T. Fukuda, W. Z. Sun and T. Nishimura [1, 2].

THEOREM 3.2. *Let* $\Delta=\dfrac{\partial(\omega, F_1, \cdots, F_{n-1})}{\partial(x_1, \cdots, x_n)}$, *and let* $H=(\Delta, F_1, \cdots, F_{n-1})$: $(\boldsymbol{R}^n, 0)\to(\boldsymbol{R}^n, 0)$. *Then* $0\in\boldsymbol{R}^n$ *is isolated in* $H^{-1}(0)$ *and*

$$b(F)=2\deg(H). \qquad \square$$

Let $\theta=x_1$. Then a pair (θ, F) has property $\mathscr{A}$ if and only if $0\in\boldsymbol{R}^n$ is isolated in $X\cap\{x_1=0\}$. In this case

$b_+(\theta, F)=$the number of branches of $X-\{0\}$ which are contained in the half region $\{x_1>0\}$,

$b_-(\theta, F)=$the number of branches of $X-\{0\}$ which are contained in the half region $\{x_1<0\}$.

Let

$$\Delta=\frac{\partial(\theta, F_1, \cdots, F_{n-1})}{\partial(x_1, \cdots, x_n)}=\frac{\partial(F_1, \cdots, F_{n-1})}{\partial(x_2, \cdots, x_n)},$$

and let

$$H=\left(\frac{\partial(F_1, \cdots, F_{n-1})}{\partial(x_2, \cdots, x_n)}, F_1, \cdots, F_{n-1}\right):(\boldsymbol{R}^n, 0)\longrightarrow(\boldsymbol{R}^n, 0).$$

As a consequence of Theorem 3.1 we get a following theorem which was proved in [3].

THEOREM 3.3. *Assume that a pair (θ, F) has property $\mathscr{A}$. Then $0\in\boldsymbol{R}^n$ is isolated in $H^{-1}(0)$ and*

$$b_+(\theta, F)-b_-(\theta, F)=2\deg(H). \qquad \square$$

Let $G_1, \cdots, G_s:(\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}, 0)$ be analytic functions. For any $\alpha=(\alpha_1, \cdots, \alpha_s)\in\{0, 1\}^s$ let us define a germ $G_\alpha:(\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}, 0)$ by

$$G=\begin{cases}\omega, & \text{if } \alpha=(0, \cdots, 0)\\ \prod_{i=1}^{s} G_i^{\alpha_i}, & \text{if } \alpha\neq(0, \cdots, 0).\end{cases}$$

Assume that each pair (G_i, F) has property $\mathscr{A}$. Then for each $\alpha\in\{0, 1\}^s$ a pair (G_α, F) has property $\mathscr{A}$ too. According to Lemma 2.3 and Theorem 3.1 there is a map $H_\alpha:(\boldsymbol{R}^n, 0)\rightarrow(\boldsymbol{R}^n, 0)$ defined in terms of G_α and F such that $b_+(G_\alpha, F)-b_-(G_\alpha, F)=2\deg(H_\alpha)$. From Remark 2.2 there is a small constant $r>0$ such that each function G_i has a constant sign on each branch of $(X-\{0\})\cap B_r$. For any $\beta=(\beta_1, \cdots, \beta_s)$ let

$$b_\beta=\#\{x\in X\cap S_r\,|\,(-1)^{\beta_1}G_1(x)>0, \cdots, (-1)^{\beta_s}G_s(x)>0\}.$$

Thus b_β is the number of branches of $(X-\{0\})\cap B_r$ on which G_i has a sign $(-1)^{\beta_i}$, for every $i\in\{1, \cdots, s\}$.

THEOREM 3.4. *The numbers b_β, $\beta\in\{0, 1\}^s$, are determined* ***by*** *numbers* $\deg(H_\alpha)$, $\alpha\in\{0, 1\}^s$.

Proof. If $s=1$ then the theorem is a consequence of Theorems 3.1 and 3.2. We shall prove the theorem in case $s=2$.

We have a non-singular system of linear equations:

$$\begin{cases} b_{(0,0)}+b_{(0,1)}+b_{(1,0)}+b_{(1,1)}=b(F) \\ b_{(0,0)}+b_{(0,1)}-b_{(1,0)}-b_{(1,1)}=b_+(G_1, F)-b_-(G_1, F) \\ b_{(0,0)}-b_{(0,1)}+b_{(1,0)}-b_{(1,1)}=b_+(G_2, F)-b_-(G_2, F) \\ b_{(0,0)}-b_{(0,1)}-b_{(1,0)}+b_{(1,1)}=b_+(G_1G_2, F)-b_-(G_1G_2, F) \end{cases}$$

By Theorem 3.1, numbers b_β, $\beta\in\{0, 1\}^2$, are determined by numbers $\deg(H_\alpha)$, $\alpha\in\{0, 1\}^2$.

The case $s>2$ is left to the reader. □

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