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FINITENESS OF SOME FAMILIES OF MEROMORPHIC MAPS

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1. Introduction.

In [3], H. Cartan proved that there exist at most two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ which have the same inverse images with multiplicities counted for three distinct values. Relating to this the author showed in his paper [5] that, for given $N+2$ hyperplanes $H_1, \cdots, H_{N+2}$ in $P^N(\boldsymbol{C})$ located in general position and effective divisors $E_1, \cdots, E_{N+2}$ on $\boldsymbol{C}^n$, the set of all linearly nondegenerate meromorphic maps f of $\boldsymbol{C}^n$ into $P^N(\boldsymbol{C})$ such that $f^*H_i = E_i$ $(1 \leqq i \leqq N+2)$ as divisors is finite. The purpose of this paper is to give a generalization of this result to the case of meromorphic maps of a compact complex manifold minus a thin analytic set into a projective algebraic manifold.

Let Y be a projective algebraic manifold. For a complex holomorphic line bundle $L \rightarrow Y$ we denote the set of all holomorphic sections of L by $H^0(Y, \mathcal{O}(L))$ and the set of all divisors D_φ associated with zeros of nonzero holomorphic sections φ of L by $|L|$.

DEFINITION 1.1. A meromorphic map f of a complex manifold X into Y is said to be *algebraically nondegenerate* with respect to L if $f(X) \not\subset \mathrm{Supp}(D_\varphi)$ for any $\varphi \in H^0(Y, \mathcal{O}(L^d)) - \{0\}$, where d is a positive integer.

The main result is stated as follows.

MAIN THEOREM. *Let Y be an N-dimensional projective algebraic manifold, $L \rightarrow Y$ a positive holomorphic line bundle and let X be an n-dimensional compact complex manifold minus a thin analytic subset. Take effective divisors $E_1, \cdots, E_{N+2}$ on X and $D_1, \cdots, D_{N+2} \in |L|$ such that*

$$\bigcap_{1 \leqq j \leqq N+2,\, j \neq i} \mathrm{Supp}(D_j) = \emptyset \tag{1.2}$$

for each $i = 1, 2, \cdots, N+2$. Then the set $\mathcal{E}$ of all meromorphic maps of X into Y which are algebraically nondegenerate with respect to L and satisfy the condition $f^*(D_i) = E_i$ $(1 \leqq i \leqq N+2)$ is finite.

In the previous papers ([6], [7]) the author stated that, for the particular case where $X = \boldsymbol{C}^n$ or X is a compact normal complex space minus an irreducible analytic set, the same conclusion holds under the weaker assumption that

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$D_1, \cdots, D_{i-1}, D_{i+1}, \cdots, D_{N+2}$ are algebraically independent with respect to L for each $i=1, 2, \cdots, N+2$. However, he found a gap in the proof of Lemma 4.3 in [6]. It is an open problem whether the assumption (1.2) of Main Theorem can be replaced by this weaker one or not.

In Main Theorem, we can take $E_1=\cdots=E_{N+2}=0$. Then we have

COROLLARY 1.3. *Under the same assumption as in Main Theorem, the set of all meromorphic maps of X into $Y-\bigcup_{1\leq i\leq N+2}\mathrm{Supp}\, D_i$ which are algebraically non-degenerate with respect to L is finite.*

This is closely related to the result of Langmann [10].

2. Preliminaries.

Let X, Y be (σ-compact connected) complex manifolds and $f: X\rightarrow Y$ be a meromorphic map, namely, a many-valued map of X into Y such that (i) the graph $G^f=\{(x, y); y\in f(x)\}$ is an analytic subset of $X\times Y$, (ii) the projection $\pi_X|G^f: G^f\rightarrow X$ is proper and (iii) f is single-valued on a nonempty open set U in X. We denote by I_f the set of all $x\in X$ such that $f(x)$ contains at least two points. Then, I_f is an analytic set in X with $\operatorname{codim} I_f\geqq 2$ and f may be considered a single-valued map on $X-I_f$.

We consider particularly meromorphic maps into $P^N(\boldsymbol{C})$. Taking homogeneous coordinates $(w_1:\cdots:w_{N+1})$ on $P^N(\boldsymbol{C})$, we set $H_{N+1}=\{w_{N+1}=0\}$. By identifying a point $(z_1, \cdots, z_N)$ in $\boldsymbol{C}^N$ with $(z_1:\cdots:z_N:1)$ in $P^N(\boldsymbol{C})$, we may regard as $P^N(\boldsymbol{C})=\boldsymbol{C}^N\cup H_{N+1}$. We can show easily the following:

(2.1) *Every meromorphic map $f: X\rightarrow P^N(\boldsymbol{C})$ with $f(X)\not\subset H_{N+1}$ can be written as*

$$f(x)=(\varphi_1(x):\cdots:\varphi_N(x):1) \tag{$*$}$$

outside a thin analytic set with meromorphic functions $\varphi_1, \cdots, \varphi_N$ on X. Conversely, each system of meromorphic functions $\varphi_1, \cdots, \varphi_N$ on X gives a meromorphic map $f: X\rightarrow P^N(\boldsymbol{C})$ satisfying the identity (*).

We now consider the set $\mathscr{V}(X)$ of all one-codimensional irreducible analytic subsets of X.

DEFINITION 2.2. We define a *divisor* D on X to be a map $D: \mathscr{V}(X)\rightarrow\boldsymbol{Z}$ which satisfies the condition that each $x\in X$ has a neighborhood U such that

$$\#\{V\in\mathscr{V}(X); U\cap V\neq\emptyset, D(V)\neq 0\}<+\infty,$$

where $\boldsymbol{Z}$ denotes the ring of all integers and $\#A$ means the number of elements in a set A.

For a divisor D on X we set $\mathscr{V}_D=\{V; D(V)\neq 0\}$. The *support* of D is defined by $\operatorname{Supp} D=\bigcup_{V\in\mathscr{V}_D} V$. The set $\mathscr{V}_D$ is at most countable. By notation $D=\sum_i m_i V_i$ we mean that $\mathscr{V}_D\subset\{V_i;i=1,2,\cdots\}$ and $m_i=D(V_i)$, and we write $D=0$ if $\mathscr{V}_D=\emptyset$. A divisor D is called *effective* if $D(V_i)\geqq 0$ for each i. For a divisor $D=\sum_i m_i V_i$ and an open subset U of X let each $V_i\cap U$ have the irreducible decomposition $V_i\cap U=\bigcup_j V_{ij}$. Then we define the *restriction* of D to U by $D|U=\sum_{i,j} m_i V_{ij}$.

Let φ be a nonzero holomorphic function on a connected open subset U of X. For each $x\in U$, taking holomorphic local coordinates z with $x=(0)$, we expand φ as

$$\varphi(z)=\sum_{m=0}^{\infty} P_m(z)$$

around x, where $P_m(z)$ is a homogeneous polynomial of degree m or vanishes identically. We set

$$\nu_\varphi(x):=\min\{m;P_m\neq 0\},$$

which does not depend on the choice of holomorphic local coordinates z. Set $Z=\{x\in U;\varphi(x)=0\}$ and consider the irreducible decomposition $Z=\bigcup_i Z_i$. Then, $\nu_\varphi(x)$ is equal to a constant m_i on each $R(Z)\cap Z_i$, where $R(Z)$ denotes the set of all regularities of Z. We define the zero divisor of φ by $D_\varphi:=\sum_i m_i Z_i$. Let f be a nonzero meromorphic function on X. For each $x\in X$, taking nonzero holomorphic functions φ and ψ on a neighborhood of x with $f=\varphi/\psi$, we define the *order* of f at x by $\nu_f:=\nu_\varphi-\nu_\psi$. It is easily seen that there exists exactly one divisor $D_f=\sum_i m_i V_i$ on X such that $\nu_f(x)=0$ on $X-\operatorname{Supp} D_f$ and $\nu_f(x)=m_i$ on $V_i\cap R(\operatorname{Supp} D_f)$. We call $\operatorname{ord}_V(f):=D_f(V)$ the order of f along V for each $V\in\mathscr{V}(X)$. The *zero divisor* Z_f and the *pole divisor* P_f of f are defined by $Z_f:=\sum_{m_i>0} m_i V_i$ and $P_f=\sum_{m_i<0}(-m_i)V_i$ respectively.

PROPOSITION 2.3. *For two nonzero meromorphic functions f_1 and f_2 on X the following three conditions are mutually equivalent;*

(i) *there is a nowhere zero holomorphic function h with $f_2=hf_1$,*

(ii) $D_{f_1}=D_{f_2}$,

(iii) *there exists an analytic set A of pure codimension one such that $A\supset\operatorname{Supp} D_{f_1}\cup\operatorname{Supp} D_{f_2}$ and each irreducible component of A contains at least one point $x\in R(A)$ with $\nu_{f_1}(x)=\nu_{f_2}(x)$.*

Particularly, if X is compact, the condition (i) *can be replaced by*

(i)′ *there exists a nonzero constant c with $f_2=cf_1$.*

Proof. It is obvious that (i) implies (ii) and (ii) implies (iii). Suppose that f_1 and f_2 satisfy the condition (iii), and set $h:=f_1/f_2$. Then,

$$\operatorname{Supp} D_h\subset\operatorname{Supp} D_{f_1}\cup\operatorname{Supp} D_{f_2}\subset A.$$

We can write $D_h=\sum_i m_i A_i$, where m_i are integers and A_i are irreducible components of A. By the assumption, for each i there exists one point $x_i\in R(A)\cap A_i$ such that $\nu_{f_1}(x_i)=\nu_{f_2}(x_i)$. This implies that

$$m_\iota=\nu_h(x_\iota)=\nu_{f_1}(x_\iota)-\nu_{f_2}(x_\iota)=0$$

for each i. Therefore, $D_h=0$. This means that h is a nowhere zero holomorphic function on X and so f_ι $(i=1, 2)$ satisfy the condition (i). Here, h is constant if X is compact.

Let $f: X\to Y$ be a meromorphic map and D be a divisor on Y such that $f(X)\not\subset \operatorname{Supp} D$. For each $x\in X-I_f$ we can take a neighborhood U of x in X and a neighborhood V of $f(x)$ such that $f(U)\subset V$ and $D|V=D_\varphi$ for a nonzero meromorphic function φ on V. Obviously, $\varphi\circ f|U$ is a nonzero meromorphic function on U and the divisor $D_{\varphi\circ f}$ does not depend on the choice of the above φ. Then, there exists exactly one divisor D^* on $X-I_f$ such that $D^*|U=D_{\varphi\circ f}$ for each $\varphi\circ f$ with the above property. Let $D^*=\sum_i n_i V_\iota$ on $X-I_f$. Since I_f is of codimension $\geqq 2$, $\overline{V}_i\in\mathscr{V}(X)$ and $\{\overline{V}_\iota\}$ is locally finite. We call the divisor $f^*(D):=\sum_i n_\iota \overline{V}_\iota$ the *pull-back* of D by f.

3. Langmann's finiteness theorem for nowhere zero holomorphic functions.

For a complex manifold X we denote the field of all meromorphic functions on X by $M(X)$ and the multiplicative group of all nowhere zero holomorphic functions on X by $H^*(X)$.

Let $\tilde{X}$ be a complex manifold and X an open subset of $\tilde{X}$ such that $A:=\tilde{X}-X$ is a thin analytic set in $\tilde{X}$. Regarding $M(\tilde{X})$ and $H^*(X)$ as subsets of $M(X)$ naturally, we set $H^*_{\tilde{X}}(X):=H^*(X)\cap M(\tilde{X})$. The multiplicative group $\boldsymbol{C}^*:=\boldsymbol{C}-\{0\}$ may be considered as a subgroup of the group $H^*_{\tilde{X}}(X)$. We consider the factor group $G:=H^*_{\tilde{X}}(X)/\boldsymbol{C}^*$. For each h in $H^*_{\tilde{X}}(X)$ we denote by $[h]$ the class in G which contains h.

PROPOSITION 3.1 (cf., [9], Satz 3.4). *In the above situation, if $\tilde{X}$ is compact and A has s irreducible components, then* $\operatorname{rank}_{\boldsymbol{Z}} G\leqq s-1$.

Proof. We may assume that each irreducible component A_t $(1\leqq t\leqq s)$ of A is of codimension one because every h in $H^*_{\tilde{X}}(X)$ has no zero on $A_t-(\bigcup_{u\neq t}A_u)$ whenever A_t is of codimension$\geqq 2$. We first consider the case $s=1$. For each h in $H^*_{\tilde{X}}(X)$ h is holomorphic on $\tilde{X}$ if $\operatorname{ord}_A h\geqq 0$, and $1/h$ is holomorphic on $\tilde{X}$ if $\operatorname{ord}_A h<0$. In either case, h is necessarily a constant by the maximum principle. This shows that $\operatorname{rank}_{\boldsymbol{Z}} G=0$. Suppose that $s\geqq 2$. We define a $\boldsymbol{Z}$-homomorphism of G into $\boldsymbol{Z}^{s-1}$ by

$$\Phi(h)=(\operatorname{ord}_{A_1}h, \cdots, \operatorname{ord}_{A_{s-1}}h)\in\boldsymbol{Z}^{s-1} \quad (h\in H^*_{\tilde{X}}(X)).$$

For h_1 and h_2 in $H^*_{\tilde{X}}(X)$, if $\Phi(h_1)=\Phi(h_2)$, the meromorphic function $\varphi:=h_1/h_2$ has neither zero nor pole on $\tilde{X}-A_s$. By the above argument, φ is a constant and so $[h_1]=[h_2]$. Therefore, Φ is injective. The group G may be considered as a subgroup of $\boldsymbol{Z}^{s-1}$. We then have $\mathrm{rank}_{\boldsymbol{Z}} G \leqq s-1$.

We now give the following finiteness theorem.

THEOREM 3.2. *Let $\tilde{X}$ be a compact complex manifold and X be an open subset of $\tilde{X}$ such that $A:=\tilde{X}-X$ is a thin analytic set in $\tilde{X}$. For nonzero meromorphic functions α_i $(1\leqq i\leqq p)$, consider the set $\mathscr{F}$ of all elements $([h_1], \cdots, [h_p]) \in G^p$ with $h_i \in H^*_{\tilde{X}}(X)$ which satisfy the conditions*

$$\sum_{i=1}^{p} \alpha_i h_i = 1$$

and $\sum_{i\in I} \alpha_i h_i \neq 0$ for any $I\subset\{1, 2, \cdots, p\}$. Then, $\#\mathscr{F}$ is bounded by a constant $R(p, s)$ depending only on p and the number s of irreducible components of A.

This is a special case of Langmann [10], Lemma 1.2. We shall give here a function-theoretic direct proof, which provides a better estimate than his, particularly, in the case where $\alpha_i h_i$ $(1\leqq i\leqq p)$ are linearly independent over $\boldsymbol{C}$. For our purpose, we need some lemmas.

Let $U^n:=\{(z_1, \cdots, z_n); |z_i|<1\}$ and $A=\{z_1=0\}\cap U^n$.

LEMMA 3.3. *If V is a d-dimensional $\boldsymbol{C}$-vector space of $M(U^n)$, then*

$$\#\{\mathrm{ord}_A \varphi\,;\, \varphi\in V-\{0\}\}\leqq d\,.$$

Proof. Take a vector subspace W of V with $\dim W=d-1$. It suffices to show that

$$\#\{\mathrm{ord}_A\varphi\,;\,\varphi\in V-\{0\}\}\leqq\#\{\mathrm{ord}_A\varphi\,;\,\varphi\in W-\{0\}\}+1\,,$$

which gives Lemma 3.3 by induction on d. Assume that there exists some φ_0 in $V-\{0\}$ such that $\mathrm{ord}_A\varphi_0 \notin \{\mathrm{ord}_A\varphi\,;\,\varphi\in W-\{0\}\}$. Take any $\varphi\in V-\{0\}$ with $\mathrm{ord}_A\varphi\neq\mathrm{ord}_A\varphi_0$. Then, we can see $\varphi=c\varphi_0+\psi$ for some c in $\boldsymbol{C}$ and ψ in $W-\{0\}$ and we easily see $\mathrm{ord}_A\varphi=\mathrm{ord}_A\psi\in\{\mathrm{ord}_A\chi\,;\,\chi\in W-\{0\}\}$. This completes the proof.

LEMMA 3.4. *Let $\alpha_1, \cdots, \alpha_p \in M(U^n)^*:=M(U^n)-\{0\}$ and P a subset of $M(U^n)^*$ such that $[P]=\{[h]\,;\,h\in P\}$ is a finitely generated subgroup of the factor group $M(U^n)^*/\boldsymbol{C}^*$. Consider the set $\mathscr{G}_p$ of all elements $(\mathrm{ord}_A h_1, \cdots, \mathrm{ord}_A h_p)\in \boldsymbol{Z}^p$ with h_i in P which satisfy the conditions*

$$\sum_{i=1}^{p}\alpha_i h_i=1\,,\qquad \sum_{i\in I}\alpha_i h_i\neq 0 \tag{3.5}$$

for any $I\subset\{1, \cdots, p\}$. Then $\#\mathscr{G}_p$ is bounded by a constant depending only on p and $r=\mathrm{rank}_{\boldsymbol{Z}}[P]$.

Proof. Since $[P]$ is countable, we can find a point $a'=(a_2, \cdots, a_n)$ with

$|a_\iota|<1$ such that, setting $\alpha_i^*(z):=\alpha_i(z, a')$ and $h_i^*(z)=h_i(z, a')$ for $h_1, \cdots, h_p \in P$ satisfying the condition in Lemma 3.4, we have $\sum_{i\in I}\alpha_i^* h_i^* \neq 0$ for any $I \subset \{1, \cdots, p\}$ and $\mathrm{ord}_0\alpha_i^*=\mathrm{ord}_A\alpha_\iota$, $\mathrm{ord}_0 h_i^*=\mathrm{ord}_A h_\iota$. Therefore, we may consider α_i^* and h_i^* instead of α_ι and h_ι. By this reason, we assume $n=1$.

Let $h_1, \cdots, h_p$ satisfy the condition (3.5) and set $f_\iota=\alpha_i h_\iota$ $(1\leqq i\leqq p)$. We first consider systems $(f_1, \cdots, f_p)$ satisfying the additional condition that $f_1, \cdots, f_p$ are linearly independent over $\boldsymbol{C}$. By the assumption that

$$f_1+f_2+\cdots+f_p=1$$

we have

$$\frac{f_1^{(l)}}{f_1}f_1+\frac{f_2^{(l)}}{f_2}f_2+\cdots+\frac{f_p^{(l)}}{f_p}f_p=0 \qquad (1\leqq l\leqq p-1),$$

where $f_i^{(l)}$ denotes the l-th derivatives of f_ι. Therefore,

$$f_\iota=(-1)^{\iota-1}\frac{\det\left(\dfrac{f_1^{(l)}}{f_1}, \cdots, \dfrac{f_{\iota-1}^{(l)}}{f_{\iota-1}}, \dfrac{f_{\iota+1}^{(l)}}{f_{\iota+1}}, \cdots, \dfrac{f_p^{(l)}}{f_p};\ 1\leqq l\leqq p-1\right)}{\det\left(\dfrac{f_1^{(l)}}{f_1}, \cdots, \dfrac{f_p^{(l)}}{f_p};\ 0\leqq l\leqq p-1\right)}. \tag{3.6}$$

We now take $g_1, \cdots, g_r \in M(U^1)$ which give a system of generators of $[P]$, where $r=\mathrm{rank}_Z[P]$. Each h_ι can be written as

$$h_\iota=c_i g_1^{m_{i1}}\cdots g_r^{m_{\iota r}}$$

with some $c_i \in \boldsymbol{C}^*$ and $m_{\iota j} \in \boldsymbol{Z}$. Then,

$$\left(\frac{f_\iota'}{f_\iota}\right)^{(l)}=\left(\frac{\alpha_\iota'}{\alpha_\iota}\right)^{(l)}+m_{i1}\left(\frac{g_1'}{g_1}\right)^{(l)}+\cdots+m_{\iota r}\left(\frac{g_r'}{g_r}\right)^{(l)}$$

for each l. On the other hand, for each l there exists a polynomial $P_l(u_1, \cdots, u_l)$ such that

$$\frac{f_i^{(l)}}{f_\iota}=P_l\left(\frac{f_\iota'}{f_\iota}, \left(\frac{f_\iota'}{f_\iota}\right)', \cdots, \left(\frac{f_\iota'}{f_\iota}\right)^{(l-1)}\right)$$

and P_l is isobaric of weight l if we associate weight k with each variable u_k, namely, if $P_l(u, u^2, \cdots, u^l)$ is homogeneous of degree l as a polynomial in u. From these facts, we can conclude that both of the denominator W_1 and the numerator W_2 of the right hand side of (3.6) are written as polynomials of $(\alpha_i'/\alpha_\iota)^{(l)}$ and $(g_j'/g_j)^{(l)}$ $(1\leqq i\leqq p,\ 1\leqq j\leqq r,\ 0\leqq l\leqq p-2)$ which are isobaric of weight $p(p-1)/2$ if we associate weight l with each $(\alpha_i'/\alpha_\iota)^{(l-1)}$ and $(g_j'/g_j)^{(l-1)}$. Let V be the set of all polynomials of $(\alpha_i'/\alpha_\iota)^{(l)}$ and $(g_j'/g_j)^{(l)}$ $(l=0, 1, \cdots, p-2)$ which are isobaric of weight $p(p-1)/2$. Then, V is a $\boldsymbol{C}$-vector subspace of $M(U^1)$ with $\dim V\leqq d(r+p, p-1, p(p-1)/2)$, where $d(u, v, w)$ denotes the dimension of the $\boldsymbol{C}$-vector space of all polynomials of $u\times v$ variables $x_{\iota j}(1\leqq i\leqq u$, $1\leqq j\leqq v)$ which are isobaric of weight w if we associate weight j with each $x_{\iota j}$. In view of Lemma 3.3, we have

$$\#\{\mathrm{ord}_A\varphi\,;\ \varphi\in V-\{0\}\}\leqq d(r+p, p-1, p(p-1)/2).$$

This shows that the number of possible values of $\mathrm{ord}_A W_1$ and of $\mathrm{ord}_A W_2$ are both at most $d(r+p,\ p-1,\ p(p-1)/2)$. Therefore, the number of possible values of each $\mathrm{ord}_A f_\iota$ is at most $d(r+p,\ p-1,\ p(p-1)/2)^2$. Since $\mathrm{ord}_A h_\iota = \mathrm{ord}_A f_i - \mathrm{ord}_A \alpha_\iota$, we conclude that

$$\#\{(\mathrm{ord}_A h_1, \cdots, \mathrm{ord}_A h_p) \in \mathcal{G}_p\,;\ \alpha_i h_\iota\ (1 \leqq \iota \leqq p) \text{ are linearly independent}\}$$
$$\leqq d(r+p,\ p-1,\ p(p-1)/2)^{2p}.$$

We now start to prove Lemma 3.4 by induction on p. The case $p=1$ is trivial. Assume that Lemma 3.4 is true for the case $\leqq p-1$. Set $\mathcal{F} := \{(f_1, \cdots, f_p)\,; f_1 := \alpha_1 h_1, \cdots, f_p := \alpha_p h_p$ satisfy the condition (3.5)$\}$. For each subset I of $\{1, \cdots, p\}$ we consider the set $\mathcal{F}_I$ of all elements $(f_1, \cdots, f_p)$ in $\mathcal{F}$ such that f_ι $(i \in I)$ are linearly independent over $\boldsymbol{C}$ and they satisfy the identity

$$\sum_{\iota \in I} c_\iota f_\iota = 1 \tag{3.7}$$

for some $c_i \in \boldsymbol{C}^*$ $(i \in I)$. Then, as is easily seen, $\mathcal{F} = \bigcup_I \mathcal{F}_I$. So, it suffices to show that

$$\#\{(\mathrm{ord}_A f_1, \cdots, \mathrm{ord}_A f_p)\,; (f_1, \cdots, f_p) \in \mathcal{F}_I\}$$

is finite for an arbitrarily fixed I. Changing indices, we assume $I = \{1, 2, \cdots, q\}$ $(1 \leqq q < p)$. We next consider a set $\mathcal{J} = (J_{q+1}, \cdots, J_p)$ of proper subsets of $\{1, 2, \cdots, p\}$ such that $l \in J_l$, $J_l \cap \{1, 2, \cdots, q\} \neq \emptyset$, and define the set $\mathcal{F}_I^{\mathcal{J}} := \bigcap_{l=1}^{p} \mathcal{F}_{I, J_l}$, where $\mathcal{F}_{I, J_l}$ is the set of all $(f_\iota) \in \mathcal{F}_I$ satisfying the condition that there exist some $d_i \in \boldsymbol{C}^*$ such that $\sum_{i \in J_l} d_\iota f_\iota = 0$ and $\sum_{i \in I'} d_\iota f_i \neq 0$ for any $I' \subsetneqq J_l$. For an element $(f_1, \cdots, f_p) \in \mathcal{F}_I$ satisfying the identity (3.7), we have

$$(1-c_1) f_1 + \cdots + (1-c_q) f_q + \sum_{l=q+1}^{p} f_l = 0$$

for some c_ι in $\boldsymbol{C}^*$. For each $l = q+1, \cdots, p$, if we take a minimal subset J_l such that $l \in J_l$ and $\sum_{i \in J_l} d_\iota f_\iota = 0$, then J_l intersects with $\{1, 2, \cdots, q\}$ by the condition (3.5), where $d_i = 1 - c_\iota$ for $1 \leqq i \leqq q$ and $d_j = 1$ for $q+1 \leqq j \leqq p$. This shows that (f_ι) is contained in $\mathcal{F}_I^{\mathcal{J}}$ for $\mathcal{J} = (J_l)$. Therefore, $\mathcal{F}_I = \bigcup_{\mathcal{J}} \mathcal{F}_I^{\mathcal{J}}$. On the other hand, by the above shown facts we have

$$\#\{(\mathrm{ord}_A f_1, \cdots, \mathrm{ord}_A f_q)\,; (f_1, \cdots, f_q) \in \mathcal{F}_I\} < \infty.$$

Moreover, for $J_l = \{j_0, j_1, \cdots, j_s\}$ with $1 \leqq j_0 \leqq q$, by applying the induction hypothesis to the functions $f_{j_1}/f_{j_0}, \cdots, f_{j_s}/f_{j_0}$ we see

$$\#\{(\mathrm{ord}_A f_{j_1} - \mathrm{ord}_A f_{j_0}, \cdots, \mathrm{ord}_A f_{j_s} - \mathrm{ord}_A f_{j_0})\,; (f_\iota) \in \mathcal{F}_{I, J_l}\} < \infty.$$

It then follows that

$$\#\{(\mathrm{ord}_A f_{j_1}, \cdots, \mathrm{ord}_A f_{j_q})\,; (f_\iota) \in \mathcal{F}_{I, J_l}\} < \infty.$$

Since $l \in J_l$ for any l $(q+1 \leqq l \leqq p)$, we conclude

$$\sharp\{(\mathrm{ord}_A f_1, \cdots, \mathrm{ord}_A f_p); (f_i) \in \mathscr{F}_I^{\prime}\} < \infty$$

and so $\sharp\{(\mathrm{ord}_A f_i); (f_i) \in \mathscr{F}_I\}$ is finite. As is seen by the above arguments, $\sharp\mathscr{F}$ is bounded by a constant depending only on p and $\mathrm{rank}_{\boldsymbol{Z}}[P]$. This completes the proof of Lemma 3.4.

Proof of Theorem 3.2. Let $A = \bigcup_{t=1}^{s} A_t$ be the irreducible decomposition of A. We may assume $\mathrm{codim}\, A_t \geqq 1$ for each t. For each A_t we take a point $x_t \in R(A_t)$ and choose holomorphic local coordinates $(z_1^{(t)}, \cdots, z_n^{(t)})$ on a neighborhood U_t of x_t with $x_t = (0)$ such that $U_t = \{|z_i^{(t)}| < 1\}$ and $A \cap U_t = \{z_1^{(t)} = 0\} \cap U_t$. Set $P := H_{\tilde{X}}^*(X)$, which may be considered as a subgroup of $M(U_t)^*$ because the restriction map of $M(\tilde{X})$ into $M(U_t)$ is injective. We may also regard $[P] := \{[h]; h \in P\}$ as a subset of $H_{U_t}^*(U_t - A \cap U_t)/\boldsymbol{C}^*$. On the other hand, $[P]$ is of rank$\leqq s-1$ by Proposition 3.1. Therefore, Lemma 3.4 implies that the number of possible cases of $(\mathrm{ord}_{A_t} h_1, \cdots, \mathrm{ord}_{A_t} h_p)$ is bounded by a constant depending only on p and s. On the other hand, two members h, h' in $H_{\tilde{X}}^*(X)$ satisfy the condition $[h] = [h']$ if and only if $\mathrm{ord}_{A_t} h = \mathrm{ord}_{A_t} h'$ for each t. From these facts, we conclude Theorem 3.2.

4. A finiteness theorem of meromorphic maps into $P^N(\boldsymbol{C})$.

Let f be a meromorphic map of a complex space X into $P^N(\boldsymbol{C})$.

DEFINITION 4.1. We say f to be *linearly nondegenerate* if $f(X)$ is not included in any hyperplane in $P^N(\boldsymbol{C})$.

The purpose of this section is to prove the following

THEOREM 4.2. *Let X be a complex space such that $X = \tilde{X} - A$ for a compact complex space $\tilde{X}$ and a thin analytic subset A of $\tilde{X}$. For hyperplanes $H_1, \cdots, H_{N+2}$ on $P^N(\boldsymbol{C})$ located in general position and effective divisors $E_1, \cdots, E_{N+2}$ on X, consider the set $\mathscr{F}$ of all linearly nondegenerate meromorphic maps of X into $P^N(\boldsymbol{C})$ such that $f^*H_i = E_i$ $(1 \leqq i \leqq N+2)$. Then, $\sharp\mathscr{F}$ is bounded by a constant depending only on N and the number of irreducible components of A.*

For the proof, we need some preparations. We first recall the following generalization of the classical Picard-Borel theorem, which was proved by the author in [4] and by M. L. Green in [8] independently.

PROPOSITION 4.3. *Let $U^n = \{(z_1, \cdots, z_n); |z_i| < 1\}$, $A := U^n \cap \{z_1 = 0\}$ and let $f_1, \cdots, f_p$ be nowhere zero holomorphic functions on $U^n - A$. If each f_i/f_j $(i \neq j)$ has essential singularities along A, then $f_1, \cdots, f_p$ are linearly independent over the field $M(U^n)$ of all meromorphic functions on U^n.*

For the proof, see [4], p. 280.

We obtain from this the following:

PROPOSITION 4.4. *Let* $\alpha^1, \cdots, \alpha^p \in M(\tilde{X})^*$ *and* $f_1, \cdots, f_p \in H^*(X)$ *satisfying the condition*

$$\alpha^1 f_1 + \cdots + \alpha^p f_p = 0.$$

Consider a partition of indices

$$\{1, 2, \cdots, p\} = J_1 \cup J_2 \cup \cdots \cup J_k$$

such that i *and* j *are contained in the same class* J_l *if and only if* f_i/f_j *has a meromorphic extension to* $\tilde{X}$. *Then,* $\sum_{i \in J_l} \alpha^i f_i = 0$ *for each* $l = 1, 2, \cdots, k$.

Proof. This is shown by induction on k. We have nothing to prove for the case $k=1$. Assume that $k \geqq 2$ and Proposition 4.3 holds for the case $\leqq k-1$. Then some f_{i_0}/f_{j_0} $(i_0 \neq j_0)$ has an essential singularity and so essential singularities at all points of an irreducible component A_t of A. Take a point $x_0 \in R(A_t)$ and choose holomorphic local coordinates $z_1, \cdots, z_n$ on a neighborhood U^n of x_0 in $\tilde{X}$ such that $x_0 = (0)$, $U^n = \{|z_i| < 1\}$ and $U^n \cap A = U^n \cap \{z_1 = 0\}$. Let

$$\{1, \cdots, p\} = J'_1 \cup \cdots \cup J'_{k'}$$

be a partition such that i and j are in the same class J'_m if and only if f_i/f_j has a meromorphic extension to U^n. Then, we see $k' \geqq 2$ and each J_l is included in some J'_m. Changing indices, we may assume $m \in J'_m$ for $1 \leqq m \leqq k'$. Set

$$\beta^m := \sum_{i \in J'_m} \alpha_i (f_i/f_m) \qquad (\in M(U^n))$$

for each m. Apply Proposition 4.3 to the identity

$$\sum_{1 \leqq m \leqq k'} \beta^m f_m = \sum_{1 \leqq i \leqq p} \alpha_i f_i = 0$$

to show $\beta^m = 0$ on U^n for each m. This concludes

$$\sum_{I_l \subset J'_m} \Big(\sum_{i \in I_l} \alpha_i f_i \Big) = \sum_{i \in J'_m} \alpha_i f_i = 0$$

on X for each m. Since $\sharp\{l; I_l \subset J'_m\} < k$, we have $\sum_{i \in I_l} \alpha_i f_i = 0$ for each l by the induction hypothesis. This completes the proof.

COROLLARY 4.5. *In the same situation as in Proposition 4.4, functions* $g_1, \cdots, g_r$ *in* $H^*(X)$ *satisfying the condition that* $g_1^{l_1} \cdots g_r^{l_r} \in M(\tilde{X})$ $(l_i \in \boldsymbol{Z})$ *only when* $l_1 = \cdots = l_r = 0$ *are algebraically independent over* $M(U^n)$.

Proof. Set $f_{\boldsymbol{l}} := g_1^{l_1} \cdots g_r^{l_r}$ for $\boldsymbol{l} = (l_1, \cdots, l_r)$. By the assumption, $f_{\boldsymbol{l}}/f_{\boldsymbol{m}} \notin M(\tilde{X})$ for any distinct $\boldsymbol{l}$ and $\boldsymbol{m}$. By proposition 4.4, there is no non-trivial linear relation with coefficients in $M^*(\tilde{X})$ among $\{f_{\boldsymbol{l}}\}$. This shows Corollary 4.5.

We next consider $p\times q$ matrices $(h_{ij};1\leqq i\leqq p, 1\leqq j\leqq q)$ with components h_{ij} in $H^{*}(X)$ for various p and q.

PROPOSITION 4.6. *For each* q_0 $(\geqq 1)$ *there exists some constant* $Q(p, q_0)$ *depending only on* p *and* q_0 *such that, if* $q>Q(p, q_0)$ *and*

$$\det(h_{ij}; i=1, \cdots, p, j=j_1, \cdots, j_p)=0 \tag{4.7}$$

for all j_l *with* $1\leqq j_l\leqq q$, *then there exist* r *functions* $k_1, \cdots, k_r\in H^{*}(X)$ *with* $2\leqq r\leqq p$ *such that, after a suitable change of indices if necessary,* $\gamma_{ij}:=h_{ij}/(h_{1j}k_i)\in M(\tilde{X})$ *for* $1\leqq i\leqq r$, $1\leqq j\leqq q_0$ *and*

$$\det(\gamma_{ij}; i=1, \cdots, r, j=j_1, \cdots, j_r)=0$$

for all j_l *with* $1\leqq j_l\leqq q_0$.

Proof. We consider the factor group $G=H^{*}(X)/H^{*}_{\tilde{X}}(X)$ which is obviously torsion free. Choose $\eta_1, \cdots, \eta_t\in H^{*}(X)$ such that $[\eta_1], \cdots, [\eta_t]$ are multiplicatively independent over $\boldsymbol{Z}$ and each h_{ij} is represented as

$$h_{ij}=\alpha_{ij}\eta_1^{l^1_{ij}}\cdots\eta_t^{l^t_{ij}} \qquad (1\leqq i\leqq p, 1\leqq j\leqq q)$$

for some $\alpha_{ij}\in H^{*}_{\tilde{X}}(X)$. Set $\boldsymbol{l}_{ij}=(l^1_{ij}, \cdots, l^t_{ij})\in\boldsymbol{Z}^t$ and take integers $p_1, \cdots, p_t, q_j$ such that

$$l_{ij}:=l^1_{ij}p_1+\cdots+l^t_{ij}p_t+q_j\geqq 0,$$

and $\boldsymbol{l}_{ij}-\boldsymbol{l}_{i'j}=\boldsymbol{l}_{ij'}-\boldsymbol{l}_{i'j'}$ if and only if $l_{ij}-l_{i'j}=l_{ij'}-l_{i'j'}$ for $1\leqq i, i'\leqq p$, $1\leqq j, j'\leqq q$, and minors

$$A^I_J(\eta_1, \cdots, \eta_t):=\det(\alpha_{ij}\eta_1^{l^1_{ij}}\cdots\eta_t^{l^t_{ij}}; i=i_1, \cdots, i_s, j=j_1, \cdots, j_s)$$

satisfy the condition that $A^I_J(\eta_1, \cdots, \eta_t)\neq 0$ if and only if $A^I_J(u^{p_1}, \cdots, u^{p_t})\neq 0$ for any $I=(i_1, \cdots, i_s)$ and $J=(j_1, \cdots, j_s)$. Set $P_{ij}(u):=\alpha_{ij}u^{l_{ij}}\in M(\tilde{X})[u]$, where $M(\tilde{X})[u]$ denotes the ring of all polynomials in u with coefficients in $M(\tilde{X})$. Then, we have

$$\operatorname{rank}(P_{ij}(u); 1\leqq i\leqq p, 1\leqq j\leqq q)<p. \tag{4.8}$$

In fact, by the assumption, we see

$$\det(\alpha_{ij}\eta_1^{l^1_{ij}}\cdots\eta_t^{l^t_{ij}}\eta_{t+1}^{q_j}; i=1, \cdots, p, j=j_1, \cdots, j_p)=0$$

for all (i_l), where η_{t+1} is an arbitrary function in $H^{*}(X)$. This is an identity of rational functions with coefficients in $M(\tilde{X})$ and indeterminates $\eta_1, \cdots, \eta_{t+1}$ by Corollary 4.5. By substituting $\eta_i=u^{p_i}$ $(1\leqq i\leqq t)$ and $\eta_{t+1}=u$, we get (4.8).

We now apply Main Lemma in the previous paper [6], § 2, p. 531, which remains valid if we replace the coefficient field $\boldsymbol{C}$ by $M(\tilde{X})$. We can conclude that for each q_0 $(\geqq 1)$ there exists some constant $Q(p, q_0)$ $(>q_0)$ depending only on p and q_0 such that, if $q>Q(p, q_0)$, then

$$l_{i1}-l_{i'1}=l_{i2}-l_{i'2}=\cdots l_{iq_0}-l_{i'q_0}$$

for all i, i' with $1\leqq i, i'\leqq r$ and

$$\mathrm{rank}(P_{ij}(u)\,;\,1\leqq i\leqq r,\ 1\leqq j\leqq q_0)<r$$

after a suitable change of indices i and j, where $2\leqq r\leqq p$. Then we have

$$\boldsymbol{l}_{i1}-\boldsymbol{l}_{i'1}=\cdots=\boldsymbol{l}_{iq_0}-\boldsymbol{l}_{i'q_0}.$$

Set $(m_{1i},\cdots,m_{ti}):=\boldsymbol{l}_{i1}-\boldsymbol{l}_{11}$ and define $k_i=\eta_i^{m_{1i}}\cdots\eta_t^{m_{ti}}$, which satisfy the desired condition. This completes the proof of Proposition 4.6.

Next, we study functions $\lambda_1,\cdots,\lambda_p$ in $M(\tilde{X})^*$ and $p\times q$ matrices $(\gamma_{ij}\,;\,1\leqq i\leqq p,\ 1\leqq j\leqq q)$ with components in $H_{\tilde{X}}^*(X)$ such that

$$\lambda_1\gamma_{1j}+\cdots+\lambda_p\gamma_{pj}=0 \qquad (1\leqq j\leqq q)$$

for various p and q.

LEMMA 4.9. *For each q_0 $(\geqq 1)$ there exists a constant $Q'(p, q_0)$ such that, if $q>Q'(p, q_0)$, then there is some s_0 with $2\leqq s_0\leqq p$ such that, after a suitable change of indices i and j*

$$\lambda_1\gamma_{1j}+\cdots+\lambda_{s_0}\gamma_{s_0j}=0$$

and $\sum_{i\in I}\lambda_i\gamma_{ij}\neq 0$ for any $I\subsetneqq\{1,\cdots,s_0\}$ and $1\leqq j\leqq q_0$.

Proof. Set $q_1^*=0$ and define

$$q_l^*:=\sum_{1\leqq s\leqq l-1}q_s^*\,{}_pC_s+q_0$$

inductively. We shall show that $Q(p, q_0)=q_p^*$ satisfies the desired condition. Suppose that $q>Q'(p, q_0)$. For each $\iota=(i_1,\cdots,i_s)$ with $1\leqq i_1<\cdots<i_s\leqq p$ $(2\leqq s\leqq p)$ we set

$$I_\iota=I_{i_1\cdots i_s}:=\{j\,;\,\lambda_{i_1}\gamma_{i_1j}+\cdots+\lambda_{i_s}\gamma_{i_sj}=0\}.$$

Take the smallest s_0 with $2\leqq s_0\leqq p$ such that $\#I_\iota>q_{s_0}^*$ for some $\iota=(i_1,\cdots,i_{s_0})$. We note here $\#I_{12\cdots p}=q>q_p^*$. Choose some $(i_1,\cdots,i_{s_0})$ with this property. By changing indices, we assume $i_1=1,\cdots,i_{s_0}=s_0$. Then, if $s<s_0$, we have $\#I_\iota\leqq q_s^*$ for any $\iota=(i_1,\cdots,i_s)$ with $1\leqq i_1<\cdots<i_s\leqq s_0$. Therefore,

$$\begin{aligned}&\#(\cup\{I_\iota\,;\,\iota=(i_1,\cdots,i_s),\ 1\leqq i_1<\cdots<i_s\leqq s_0,\ 2\leqq s<s_0\})\\ &\leqq\sum_{1\leqq s\leqq s_0-1}q_s^*\,{}_{s_0}C_s\\ &\leqq\sum_{1\leqq s\leqq s_0-1}q_s^*\,{}_pC_s=q_{s_0}^*-q_0.\end{aligned}$$

This implies that

$$\#(I_{12\cdots s_0}-\cup\{I_{i_1\cdots i_s};\ 1\leqq i_1<\cdots<i_s\leqq s_0,\ 2\leqq s<s_0\})$$
$$>q_{s_0}^*-(q_{s_0}^*-q_0)=q_0.$$

By changing indices, we can assume that $I_{12\cdots s_0}\supset\{1, 2, \cdots, q_0\}$ and $I_{i_1\cdots i_s}\cap\{1, 2, \cdots, q_0\}=\emptyset$ for any $(i_1, \cdots, i_s)$ with $2\leqq s<s_0$. This shows Lemma 4.9.

We now start to prove Theorem 4.2. We may identify $P^N(\boldsymbol{C})$ with the subspace

$$H_0:=\{(w_1:\cdots:w_{N+2});\ w_1+\cdots+w_{N+2}=0\}$$

of $P^{N+1}(\boldsymbol{C})$ and H_i with $H_0\cap\{w_i=0\}$ $(1\leqq i\leqq N+2)$, where $(w_1:\cdots:w_{N+2})$ is a system of homogeneous coordinates on $P^{N+1}(\boldsymbol{C})$. For convenience sake, we set $p=N+2$ in the following.

Assume that $\mathscr{F}$ contains q distinct maps $f_1, \cdots, f_q$. We shall prove that q is not larger than a constant $Q^*(p, s_0)$ depending only on p and the number s_0 of irreducible components of A. Each f_j can be represented as

$$f_j=(\varphi_{1j}:\cdots:\varphi_{pj})$$

with meromorphic functions φ_{ij} on X satisfying the condition

$$\varphi_{1j}+\cdots+\varphi_{pj}=0$$

where we may assume $\varphi_{pj}=1$ by (2.1). By the assumption, φ_{ij} $(1\leqq i\leqq p-1)$ are linearly independent over $\boldsymbol{C}$. Moreover, since $D_{\varphi_{ij}}=f^*H_i-f^*H_p=E_i-E_p$ for every j, we see $h_{ij}:=\varphi_{ij}/\varphi_{i1}\in H^*(X)$. We then have

$$\varphi_{11}h_{1j}+\cdots+\varphi_{p1}h_{pj}=0 \tag{4.10}$$

for $1\leqq j\leqq q$. Therefore, h_{ij} $(1\leqq i\leqq p, 1\leqq j\leqq q)$ satisfy the assumption of Proposition 4.6.

Assume that q_1 mappings among the maps f_j, say $f_1, \cdots, f_{q_1}$, have meromorphic extensions to $\tilde{X}$. Then, for $j=1, \cdots, q_1$, $h_{ij}\in H_{\tilde{X}}^*(X)$, $\sum_{1\leq i\leq p}\varphi_{i1}h_{ij}=0$ and $\sum_{i\in I}\varphi_{i1}h_{ij}\neq 0$ whenever $I\subsetneqq\{1, 2, \cdots, p\}$. Therefore, we can apply Theorem 3.2 to these functions to show that the number of the distinct systems $([h_{1j}], \cdots, [h_{pj}])$ $(1\leqq j\leqq q_1)$ is bounded by a constant $Q^*(p, s_0)$ depending only on p and s_0. On the other hand, if

$$([h_{1j}], \cdots, [h_{pj}])=([h_{1j'}], \cdots, [h_{pj'}])$$

for some j, j', then we can write $\varphi_{ij'}=c_i\varphi_{ij}$ for some $c_i\in\boldsymbol{C}^*$. In this case, we have $c_1\varphi_{1j}+\cdots+c_p\varphi_{pj}=\varphi_{1j'}+\cdots+\varphi_{pj'}=0$. Since $\varphi_{1j}, \cdots, \varphi_{p-1j}$ are linearly independent over $\boldsymbol{C}$, we get $c_1=\cdots=c_p$ and so $j=j'$. This concludes $q_1\leqq Q^*(p, s_0)$.

For our purpose, by the above shown fact we may assume that every f_j $(1\leqq j\leqq q)$ has essential singularities along A. For the case $p=3$, it suffices to take $Q^*(p, s_0)=Q(3, 2)$, where $Q(p, q_0)$ is the quantity given in Proposition 4.6.

In fact, if $q>Q(3, 2)$, then after a suitable change of indices we have

$$\gamma_{ij}:=h_{ij}/(h_{1j}k_i)\in M(\tilde{X})$$

for some $k_1, \cdots, k_r\in H^*(X)$ and $\mathrm{rank}(\gamma_{ij}; i=1, 2, 3, j=1, 2)<r$, where $2\leqq r\leqq 3$. In the case where $\det(\gamma_{ij}; 1\leqq i, j\leqq 2)=0$, there exists some γ in $M(\tilde{X})^*$ with $\gamma_{12}=\gamma\gamma_{11}$, $\gamma_{22}=\gamma\gamma_{21}$. Then, $\varphi_{i2}=h_{i2}\varphi_{i1}=\gamma_{i2}h_{12}k_i\varphi_{i1}=\gamma\gamma_{i1}h_{12}\varphi_{i1}k_i=\gamma h_{12}\varphi_{i1}$ for $i=1, 2$, and $\varphi_{32}=-(\varphi_{12}+\varphi_{22})=\gamma h_{12}\varphi_{31}$. So, $f_1=f_2$. This is a contradiction. In the case where $\det(\gamma_{ij}; 1\leqq i, j\leqq 2)\neq 0$, we have necessarily $r=3$ and the identities

$$\varphi_{11}k_1\gamma_{1j}+\varphi_{21}k_2\gamma_{2j}+\varphi_{31}k_3\gamma_{3j}=0 \qquad (j=1, 2)$$

imply that $(\varphi_{i1}k_i)/(\varphi_{11}k_1)\in M(\tilde{X})$ for $i=2, 3$. This concludes that f_2 has a meromorphic extension to $\tilde{X}$, which contradicts the assumption.

Assume that there exist $Q^*(3, s_0), \cdots, Q^*(p-1, s_0)$ with the desired properties for each s_0, where $Q^*(l-1, s_0)<Q^*(l, s_0)$ for $l=4, \cdots, p-1$. Let $R(p, s_0)$, $Q(p, q_0)$ and $Q'(p, q_0)$ be the quantities given by Theorem 3.2, Proposition 4.6 and Lemma 4.9 respectively, where we may assume $R(p-1, s_0)\leqq R(p, s_0)$. We now define inductively the numbers $Q^{(l)}(p, s_0)$ for $l=1, 2$ and $Q^*(p, s_0)$ by the following conditions;

(4.11) $Q^{(1)}(p, s_0)>R(p, s_0)(Q^*(p-1, s_0)+1)$

(4.12) $Q^{(2)}(p, s_0)>Q'(p, Q^{(1)}(p, s_0)+1)$,

(4.13) $Q^*(p, s_0)\geqq Q(p, Q^{(2)}(p, s_0))$,

(4.14) $Q^{(l)}(p, s_0)\geqq Q^{(l)}(p-1, s_0)$ for each $l=1, 2$ and $Q^*(p, s_0)\geqq Q^*(p-1, s_0)$.

Suppose that $q>Q^*(p, s_0)$. Then, by the use of Proposition 4.6 and (4.13), after a suitable change of indices we can find some $k_1, \cdots, k_r\in H^*(X)$ $(2\leqq r\leqq p)$ such that $\gamma_{ij}=h_{ij}/(h_{1j}k_i)\in M(\tilde{X})$ for $1\leqq i\leqq r$ and $1\leqq j\leqq Q^{(2)}(p, s_0)$ and

$$\mathrm{rank}(\gamma_{ij}; 1\leqq i\leqq r, 1\leqq j\leqq Q^{(2)}(p, s_0))<r.$$

Therefore, there exists some $\lambda_1, \cdots, \lambda_r\in M(\tilde{X})$ with $(\lambda_1, \cdots, \lambda_r)\neq(0, \cdots, 0)$ such that

$$\lambda_1\gamma_{1j}+\cdots+\lambda_r\gamma_{rj}=0 \qquad (1\leqq j\leqq Q^{(2)}(p, s_0)).$$

Changing indices if necessary, we may assume that $\lambda_1\neq 0, \cdots, \lambda_u\neq 0$, $\lambda_{u+1}=\cdots=\lambda_r=0$. Then, by the use of Lemma 4.9, we can assume that

$$\lambda_1\gamma_{1j}+\cdots+\lambda_u\gamma_{uj}=0 \qquad (4.15)$$

for any $j=1, 2, \cdots, Q^{(1)}(p, s_0)+1$ and $\sum_{i\in I}\lambda_i\gamma_{ij}\neq 0$ for $I\subsetneqq\{1, 2, \cdots, u\}$. Apply Theorem 3.2 to the functions $\alpha_1=\lambda_1, \cdots, \alpha_u=\lambda_u$ to show that the number of distinct systems among

$$\{([\gamma_{1j}], \cdots, [\gamma_{uj}])\in\oplus^u(H^*_{\tilde{X}}(X)/\boldsymbol{C}^*); 1\leqq j\leqq Q^{(1)}(p, s_0)\}$$

is at most $R(u, s_0)$ $(\leqq R(p, s_0))$. Among $Q^{(1)}(p, s_0)$ systems $(\gamma_{1j}, \cdots, \gamma_{uj})$ which belongs to the same class $([\gamma_{1j}], \cdots, [\gamma_{uj}])$. Therefore, after changing indices and renewing φ_{ij}, we can write

$$f_j=(c_{1j}k_1^*:\cdots:c_{uj}k_u^*:\varphi_{u+1j}:\cdots:\varphi_{pj})$$

with some $c_{ij}\in C^*$ and $k_1^*, \cdots, k_u^*\in H^*(X)$ for $j=1, 2, \cdots, Q^*(p-1, s_0)$. Then by (4.15) we see

$$\mathrm{rank}(c_{ij};\ 1\leqq i\leqq u,\ 1\leqq j\leqq Q^*(p-1, s_0)+1)<u.$$

We may write

$$c_{1j}=\sum_{2\leqq i\leqq u} c_{ij}d_i$$

for some $d_i\in C$ $(2\leqq i\leqq u)$. Set $k_i^{**}:=k_i^*+d_ik_1^*$ for $2\leqq i\leqq u$ and define the maps

$$\tilde{f}_j=(c_{2j}k_2^{**}:\cdots:c_{uj}k_u^{**}:\varphi_{u+1j}:\cdots:\varphi_{p-1j})$$

of X into $P^{N-1}(C)$ for $j=1, 2, \cdots, Q^*(p-1, s_0)+1$. Then $\tilde{f}_j$ are all nondegenerate. For $k_1^*, \cdots, k_u^*, \varphi_{u+1j}, \cdots, \varphi_{p-1j}$ are linearly independent by the assumption and so $k_2^*+d_2k_1^*, \cdots, k_u^*+d_uk_1^*, \varphi_{u+1j}, \cdots, \varphi_{p-1j}$ are also linearly independent. Moreover, if

$$(c_{2j}k_2^{**}:\cdots:c_{uj}k_u^{**}:\varphi_{u+1j}:\cdots:\varphi_{p-1j})$$
$$=(c_{2j'}k_2^{**}:\cdots:c_{uj'}k_u^{**}:\varphi_{u+1j'}:\cdots:\varphi_{p-1j'}),$$

then $c_{ij}=dc_{ij'}$ $(2\leqq i\leqq u)$ for some $d\in C^*$ and

$$c_{1j}k_1^*=-(c_{2j}k_2^*+\cdots+c_{uj}k_u^*+\varphi_{u+1j}+\cdots+\varphi_{pj})$$
$$=-d(c_{2j'}k_2^*+\cdots+c_{uj'}k_u^*+\varphi_{u+1j'}+\cdots+\varphi_{pj'})$$
$$=dc_{1j'}k_1^*,$$

which implies $f_j=f_{j'}$. Therefore, the set $\mathscr{F}'$ of all meromorphic maps $\tilde{f}$ of X into $P^{N-1}(C)=P^N(C)\cap\{w_1=0\}$ with $\tilde{f}^*H_i=D_{k_i^{**}}$ $(2\leqq i\leqq u)$ and $\tilde{f}^*H_i=E_i$ $(u+1\leqq i\leqq p)$ contains $Q^*(p-1, s_0)+1$ distinct elements. This contradicts the induction hypothesis. The proof of Theorem 4.2 is completed.

5. Proof of Main Theorem.

For the proof of Main Theorem, we need some lemmas.

LEMMA 5.1 ([1]). *Let $L\to Y$ be a very ample line bundle over an N-dimensional smooth projective algebraic manifold Y and $\varphi_1, \cdots, \varphi_{N+1}\in H^0(Y, \mathcal{O}(L))^*$. If*

$$\bigcap_{1\leqq j\leqq N+1}\mathrm{Supp}\,D_{\varphi_j}=\emptyset,$$

then $\varphi_1/\varphi_{N+1}, \cdots, \varphi_N/\varphi_{N+1}$ are algebraically independent over C.

For the proof, see [1], p. 213.

LEMMA 5.2 ([1]). *Let* $L\rightarrow Y$ *be a line bundle as in Lemma* 5.1 *and* $\varphi_1, \cdots, \varphi_{N+2}\in H^0(Y, \mathcal{O}(L))^*$ *satisfy the condition that*

$$\mathrm{Supp}\, D_{\varphi_1}\cap\cdots\cap\mathrm{Supp}\, D_{\varphi_{j-1}}\cap\mathrm{Supp}\, D_{\varphi_{j+1}}\cap\cdots\cap\mathrm{Supp}\, D_{N+2}=\emptyset$$

for each $j=1, \cdots, N+2$. *Take a nonzero irreducible homogeneous polynomial* $R(u_1, \cdots, u_{N+2})$ *such that* $R(\varphi_1, \cdots, \varphi_{N+2})=0$ *on* Y, *and set*

$$R(u)=\sum_{i_1+\cdots+i_{N+2}=k} a_{i_1\cdots i_{N+2}}u_1^{i_1}\cdots u_{N+2}^{i_{N+2}}.$$

Then,

$$a_{k0\cdots0}\neq 0,\ a_{0k0\cdots0}\neq 0,\ \cdots,\ a_{00\cdots0k}\neq 0.$$

For the proof, see [1], pp. 213~216.

LEMMA 5.3. *Let* L *be a line bundle over an* N*-dimensional compact complex manifold* Y *which has at least one system of* $N+1$ *algebraically independent holomorphic sections. Then, there exists a positive constant* k_L *depending only on* L *such that for arbitrary algebraically independent* $\varphi_1, \cdots, \varphi_{N+1}\in H^0(Y, \mathcal{O}(L))$ *the meromorphic map* $\Phi:=(\varphi_1:\cdots:\varphi_{N+1}): Y\rightarrow P^N(\boldsymbol{C})$ *satisfies the condition that* $\#\Phi^{-1}\Phi(w)\leqq k_L$ *for every point* w *in a nonempty Zariski open subset* G *of* Y.

For the proof, see [6], p. 537.

Now, we start to prove Main Theorem. By the assumption, there exists a positive integer d such that L^d is very ample. For our purpose, we may replace L by L^d and so assume that L is very ample from the beginning. Indeed, the set $\mathcal{E}$ is included in the set of all meromorphic maps of X into Y which are algebraically nondegenerate with respect to L and satisfy the condition $f^*(dD_i)=dE_i$. Moreover, the divisors $dD_1, \cdots, dD_{N+2}\in|L^d|$ satisfy the assumption of Main Theorem. Therefore, it suffices to prove Main Theorem for L^d.

Take holomorphic sections $\varphi_1, \cdots, \varphi_{N+2}$ of L with $D_i=D_{\varphi_i}$ $(1\leqq i\leqq N+2)$. Then, $\varphi_1/\varphi_{N+2}, \cdots, \varphi_{N+1}/\varphi_{N+2}$ are algebraically dependent and $\varphi_1/\varphi_{N+1}, \cdots, \varphi_N/\varphi_{N+1}$ are algebraically independent by Lemma 5.1. It follows from these facts that there exists a nonzero homogeneous polynomial $R(u)$ of degree $k\geqq 1$ such that

$$R(\varphi_1, \cdots, \varphi_{N+2})=0.$$

We write

$$R(u)=\sum_{1\leq j\leq s+2} R_j(u),$$

where $R_j(u)$ are nonzero monomials. By virtue of Lemma 5.2, we may assume

$$R_1(u)=c_1u_1^k, \cdots, R_{N+2}(u)=c_{N+2}u_{N+2}^k, \tag{5.4}$$

where $c_i\in\boldsymbol{C}^*$ $(1\leqq i\leqq N+2)$.

We now consider a holomorphic map $\Psi: Y\rightarrow P^s(\boldsymbol{C})$ defined by

$$\Psi(y)=(R_1(\varphi_1(y), \cdots, \varphi_{N+2}(y)) : \cdots : R_{s+1}(\varphi_1(y), \cdots, \varphi_{N+2}(y))).$$

Instead of the set $\mathcal{E}$ we study the set $\tilde{\mathcal{E}}$ of all meromorphic maps $\tilde{f}:=\Psi\cdot f$ of X into $P^s(\boldsymbol{C})$ with $f\in\tilde{\mathcal{E}}$. Each $\tilde{f}\in\tilde{\mathcal{E}}$ is linearly nondegenerate because f is algebraically nondegenerate with respect to L. We set

$$\tilde{H}_j:=\{v_j=0\} \qquad (1\leqq j\leqq s+1)$$

$$\tilde{H}_{s+2}:=\{v_1+\cdots+v_{s+1}=0\},$$

where $(v_1:\cdots:v_{s+1})$ denotes homogeneous coordinates on $P^s(\boldsymbol{C})$. Then, the hyperplanes $\tilde{H}_1, \cdots, \tilde{H}_{s+1}$ are located in general position. Moreover, we set

$$\tilde{E}_j=l_1E_1+\cdots+l_{N+2}E_{N+2}$$

if $R_j(u)=cu_1^{l_1}\cdots u_{N+2}^{l_{N+2}}$ $(c\in\boldsymbol{C}^*)$. We then have

$$f^*(\tilde{H}_j)=f^*(\Psi^*(\tilde{H}_j))=\tilde{E}_j \qquad (1\leqq j\leqq s+2).$$

As a consequence of Theorem 4.2, we obtain $\sharp\tilde{\mathcal{E}}<\infty$. Take an arbitrary map $f_0\in\mathcal{E}$. It suffices to show that

$$\sharp\{f\in\mathcal{E}\,;\Psi\cdot f=\Psi\cdot f_0\}<\infty.$$

To see this, we apply Lemma 5.3 to algebraically independent sections $(\varphi_1)^k, \cdots, (\varphi_{N+1})^k$. By the help of (5.4) we can conclude that there exists a positive constant d_0 such that $\sharp\Psi^{-1}\Psi(w)\leqq d_0$ for every point w in a nonempty Zariski open subset G of Y. Suppose that there are mutually distinct $q+1$ meromorphic maps $f_0, \cdots, f_q\in\mathcal{E}$ such that $\Psi\cdot f_j=\Psi\cdot f_0$. Set

$$G^*:=\{x\in X\,; f_j(x)\in G \text{ for all } j \text{ and } f_j(x)\neq f_{j'}(x) \text{ for } 0\leqq j<j'\leqq q\}.$$

By the assumption of nondegeneracy of f_i, G^* is an open dense subset of X. For a point $x_0\in G^*$ we have $f_0(x_0)\in G$ and

$$\{f_0(x_0), \cdots, f_q(x_0)\}\subset\Psi^{-1}\Psi(x_0),$$

whence $q+1\leqq d_0$. This completes the proof of Main Theorem.

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