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REMARKS ON A RESULT OF HAYMAN

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1. Introduction.

In this paper, we use the usual notation of Nevanlinna theory[3].

Suppose that $f(z)=\sum_{n=0}^{\infty} a_n z^{\lambda_n}$ is a transcendental entire function, where $a_n \neq 0$ $(n=0, 1, 2, \cdots)$ and $\{\lambda_n\}$ is arranged in increasing order. Also let $g(z)$ be an arbitrary entire function growing slowly compared with the function $f(z)$, i.e., $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$. Following Hayman[4], if $f(z)$ has finite order, we define

$$\delta_S(g(z), f)=1-\underline{\lim}_{r\to\infty}\frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}.$$

If $f(z)$ has infinite order, let E be any set in $(1, \infty)$ having finite length. We define

$$\delta_S(g(z), f)=1-\sup_E \underline{\lim}_{r\to\infty, r\notin E}\frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}=\inf_E \overline{\lim}_{r\to\infty, r\notin E}\frac{m\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}.$$

Obviously,

$$\delta(g(z), f)=1-\overline{\lim}_{r\to\infty}\frac{N\left(r, \frac{1}{f-g(z)}\right)}{T(r, f)}\leqq\delta_S(g(z), f).$$

In particular, when $g(z)\equiv a$ (a is a constant) we get the definition of $\delta_S(a, f)$ defined by Hayman[4].

Under the above definitions, Hayman[4] proved

THEOREM A. *Let d_n be the highest common factor of all the numbers $\lambda_{m+1}-\lambda_m$ for $m\geqq n$ and suppose that*

$$d_n \longrightarrow \infty \quad as \quad n\to\infty.$$

Then $\delta_S(a, f)=0$ for every finite complex number a.

With the hypotheses of Theorem A, we proved in [2] $\Theta_S(g(z), f)\leqq 1/2$ for every function $g(z)$ satisfying $T(r, g)=o\{T(r, f)\}$. Now we further prove

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$\delta_S(g(z), f)=0$ for every entire function $g(z)$ satisfying $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$. That is, we shall prove

THEOREM 1. *Let d_n be the highest common factor of all the numbers $\lambda_{m+1}-\lambda_m$ for $m\geqq n$ and suppose that*

$$d_n \longrightarrow \infty \qquad as \quad n\to\infty .$$

Then $\delta_S(g(z), f)=0$ for every entire function $g(z)$ satisfying $T(r, g)=o\{T(r, f)\}$ as $r\to\infty$.

Clearly, Theorem 1 is an extension of Theorem A.

2. Proof of Theorem 1.

From now on we denote by $S(r, f)$ any term which satisfies $S(r, f)=o\{T(r, f)\}$ as $r\to\infty$ outside some set of finite length. In particular, if $f(z)$ is of finite order, then we assume that $S(r, f)=o\{(T(r, f))\}$ as $r\to\infty$ without exceptional set. Also suppose that a and b are two positive numbers. "$a|b$" implies that there exists a positive integer n such that $b=n\cdot a$.

In order to obtain Theorem 1, we need the following lemma of [1].

LEMMA 1. *If $F(z)$ is an entire function and $g_1(z), g_2(z), \cdots, g_m(z)$ are distinct entire functions satisfying $T(r, g_j)=o\{T(r, F)\}$ as $r\to\infty$ $(j=1, 2, \cdots, m)$, then*

$$\sum_{j=1}^{m} m\left(r, \frac{1}{F-g_j(z)}\right)\leqq T(r, F)+S(r, F).$$

Proof of Theorem 1. Suppose that $g(z)$ is any entire function satisfying

$$(1) \qquad T(r, g)=o\{T(r, f)\} \qquad \text{as} \quad r\to\infty .$$

We then discuss two cases separately.

Case (A). $g(z)=\sum_{v=0}^{t} b_v z^{h_v}$ is a polynomial.

We write

$$f(z)-g(z)=\sum_{v=0}^{\infty} \eta_v z^{\beta_v},$$

where $\eta_v\neq 0$ and $\{\beta_v\}$ is arranged in increasing order. Let d_n^* be the highest common factor of all the numbers $\beta_{m+1}-\beta_m$ for $m\geqq n$. Obviously we have $d_n^*\to\infty$ as $n\to\infty$. Hence by Theorem A we have, in view of (1),

$$\delta_S(g(z), f)=\delta_S(0, f-g)=0 .$$

Case (B). $g(z)=\sum_{v=0}^{\infty} b_v z^{h_v}$ is transcendental, where $b_v\neq 0$ $(v=0, 1, 2, \cdots)$ and

$\{h_v\}$ is arranged in increasing order.

Suppose contrary to hypothesis that $\delta_S(g(z), f)>0$ and choose a positive integer T such that

$$\delta_S(g(z), f)>\frac{1}{T}. \tag{2}$$

We write

$$f(z)-g(z)=\sum_{v=0}^{\infty}\eta_v z^{\beta_v}, \tag{3}$$

where $\eta_v\neq 0$ $(v=0, 1, 2, \cdots)$ and $\{\beta_v\}$ is arranged in increasing order.

Let us choose an integer n so large that

$$n>T+2 \quad\text{and}\quad d_n>T\,!\,T\beta_{T+2}. \tag{4}$$

We assume h_{v_n} as the minimum of numbers h_v $(v=0, 1, 2, \cdots)$ satisfying $h_v\geqq\lambda_n$. Again we consider two subcases separately.

Case (B.1). There exists an integer $v_N>v_n$ such that

$$(h_{v_N}-\lambda_n, d_n)<\frac{d_n}{T}. \tag{5}$$

In this case we write

$$f(z)-g(z)=\sum_{v=0}^{\infty}a_v z^{\lambda_v}-\sum_{v=0}^{\infty}b_v z^{h_v}=\phi_0(z)+\phi_1(z),$$

say, where $\phi_0=\sum_{v=0}^{n-1}a_v z^{\lambda_v}-\sum_{v=0}^{\infty}b_v z^{h_v}$, $\phi_1=\sum_{v=n}^{\infty}a_v z^{\lambda_v}$.

By $d_n|(\lambda_{m+1}-\lambda_m)$ for $m\geqq n$, we have

$$d_n|(\lambda_v-\lambda_n) \qquad \text{for} \quad v\geqq n+1.$$

We set $\omega_n=\exp(2\pi i/d_n)$. Then for each non-negative integer j we have

$$\begin{aligned}
&\phi_1(\omega_n^j z)=\omega_n^{j\lambda_n}\phi_1(z),\\
&\phi_1(\omega_n^j z)+\phi_0(\omega_n^j z)=\omega_n^{j\lambda_n}\{\phi_1(z)+\omega_n^{-j\lambda_n}\phi_0(\omega_n^j z)\}\\
&\qquad\qquad=\omega_n^{j\lambda_n}\{\phi_1(z)-Q_j(z)\},
\end{aligned}$$

say, where $Q_j(z)=-\omega_n^{-j\lambda_n}\phi_0(\omega_n^j z)$. We deduce that

$$m\left(r, \frac{1}{\phi_1(z)-Q_j(z)}\right)=m\left(r, \frac{1}{\phi_1(z)+\phi_0(z)}\right)=m\left(r, \frac{1}{f-g(z)}\right). \tag{6}$$

Also according to (1) we have

$$T(r, Q_j(z))=o\{T(r, \phi_1)\} \qquad \text{as} \quad r\to\infty. \tag{7}$$

Obviously, $h_{v_N}>h_{v_n}\geqq\lambda_n$. Thus the coefficient of $z^{h_{v_N}}$ in $Q_j(z)$ is

$$I_j = b_{v_N}\omega_n^{-j\lambda_n}\omega_n^{jh_{v_N}} = b_{v_N}\omega_n^{j(h_{v_N}-\lambda_n)} = b_{v_N}\exp(2\pi i j(h_{v_N}-\lambda_n)/d_n).$$

By (5), the number of distinct values of I_j is precisely

$$d_n/(h_{v_N}-\lambda_n,\ d_n) > T.$$

Thus more than T of $Q_j(z)$ are distinct from each other. Using (6), (7) and Lemma 1 we deduce that

$$(T+1)m\left(r, \frac{1}{f-g(z)}\right) \leqq T(r,\ \phi_1) + S(r,\ \phi_1)$$
$$\leqq T(r,\ f) + S(r,\ f) \qquad (r\to\infty).$$

Hence

$$\delta_S(g(z),\ f) \leqq \frac{1}{T+1} < \frac{1}{T}.$$

This contradicts (2).

Case (B.2). For all $v > V_n$ we have $(h_v-\lambda_n,\ d_n) \geqq d_n/T$.
In this case we have

$$(h_v-\lambda_n,\ d_n) = \frac{d_n}{c_v}$$

where c_v is an integer and $1 \leqq c_v \leqq T$.

Clearly $d_n/T! \mid d_n/c_v$. Thus

$$\text{(8)} \qquad \frac{d_n}{T!}\Big| d_n \quad \text{and} \quad \frac{d_n}{T!}\Big|(h_v-\lambda_n) \qquad \text{for} \quad v > v_n.$$

By (3)

$$f(z)-g(z) = \sum_{v=0}^{\infty} a_v z^{\lambda_v} - \sum_{v=0}^{\infty} b_v z^{h_v} = \sum_{v=0}^{\infty} \eta_v z^{\beta_v}.$$

Thus we easily deduce, in view of (8) and the definition of d_n, that there exists an integer v_M such that $v_M \geqq n$ and

$$\text{(9)} \qquad \frac{d_n}{T!}\Big|(\beta_v-\beta_{v_M}) \qquad \text{for} \quad v \geqq v_M+1.$$

Obviously we have

$$f(z)-g(z) = \sum_{v=0}^{v_M-1} \eta_v z^{\beta_v} + \sum_{v=v_M}^{\infty} \eta_v z^{\beta_v} = \psi_0(z) + \psi_1(z),$$

say, where $\psi_0 = \sum_{v=0}^{v_M-1} \eta_v z^{\beta_v}$ and $\psi_1 = \sum_{v=v_M}^{\infty} \eta_v z^{\beta_v}$.

We set $\zeta_n = \exp(2\pi i T!/d_n)$. Noticing (9), we deduce that

$$\psi_1(\zeta_n^j z) = \sum_{v=v_M}^{\infty} \eta_v \zeta_n^{j\beta_v} z^{\beta_v} = \zeta_n^{j\beta_{v_M}} \psi_1(z),$$

where j is a non-negative integer.

Hence

$$\phi_1(\zeta_n^j z)+\phi_0(\zeta_n^j z)=\zeta_n^{j\beta_{v_M}}\{\phi_1(z)+\zeta_n^{-j\beta_{v_M}}\phi_0(\zeta_n^j z)\}$$
$$=\zeta_n^{j\beta_{v_M}}\{\phi_1(z)-R_j(z)\},$$

say, where $R_j(z)=-\zeta_n^{-j\beta_{v_M}}\phi_0(\zeta_n^j z)$.

We easily see that

(10) $$T(r, R_j(z))=o\{T(r, \phi_1)\},$$

(11) $$m\left(r, \frac{1}{\phi_1-R_j(z)}\right)=m\left(r, \frac{1}{\phi_0+\phi_1}\right)=m\left(r, \frac{1}{f-g(z)}\right).$$

Now we set

$$\Delta_v=(T!(\beta_{v_M}-\beta_v), d_n) \qquad (v=2, 3, \cdots, T+2).$$

Then there must exist an integer p such that $2\leqq p\leqq T+2$ and

(12) $$\Delta_p=(T!(\beta_{v_M}-\beta_p), d_n)<\frac{d_n}{T}.$$

In fact if $\Delta_v\geqq d_n/T$ for each v $(v=2, 3, \cdots, T+2)$, then $\Delta_v=d_n/q_v$ where q_v is an integer and $1\leqq q_v\leqq T$. Thus there can be at most T different values of q_v and so of Δ_v. But Δ_v $(v=2, 3, \cdots, T+2)$ must all be distinct (If $\Delta_u=\Delta_v=m$ for $2\leqq u<v\leqq T+2$, then $m|T!(\beta_v-\beta_u)$, which is impossible since by (4) we have $0<T!(\beta_v-\beta_u)<T!\beta_{T+2}<d_n/T\leqq m$). Therefore the number of distinct values of Δ_v $(v=2, 3, \cdots, T+2)$ is precisely $T+1$. This is a contradiction, which shows that (12) is valid.

Now let us recall the definition of v_M and p and notice (4). Clearly we have

$$\beta_p\leqq\beta_{T+2}<\beta_n\leqq\beta_{v_M}.$$

Hence the coefficient of z^{β_p} in $R_j(z)$ is

$$L_j=-\zeta_n^{-j\beta_{v_M}}\eta_p\zeta_n^{j\beta_p}$$
$$=-\eta_p\zeta_n^{-j(\beta_{v_M}-\beta_p)}=-\eta_p\exp\left(-\frac{2\pi ij(\beta_{v_M}-\beta_p)T!}{d_n}\right).$$

By (12), the number of distinct values of L_j is precisely

$$\frac{d_n}{(T!(\beta_{v_M}-\beta_p), d_n)}>T.$$

Thus more than T of the functions $R_j(z)$ are distinct from each other. Using (10), (11) and Lemma 1, we obtain that

$$(T+1)m\left(r, \frac{1}{f-g(z)}\right)\leqq T(r, \phi_1)+S(r, \phi_1)$$
$$\leqq T(r, f)+S(r, f) \qquad (r\to\infty).$$

Therefore

$$\delta_S(g(z),\ f) \leqq \frac{1}{T+1} \leqq \frac{1}{T}.$$

This contradicts (2).
According to the above discussion, we deduce that in case (B)

$$\delta_S(g(z),\ f)=0$$

This completes the proof of Theorem 1.

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