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CONVEX CURVES WHOSE POINTS ARE VERTICES OF BILLIARD TRIANGLES

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Abstract

We find out convex curves (other than ellipses) all points of which are vertices of periodic orbits of billiard balls with period three.

0. Introduction.

Let C be a plane convex curve and let D be the reigion inside C. Let a point P move over D with constant speed along a straight line until it hits C where it is reflected so that the angle of reflection with C is equal to the angle of incidence. The motion appears in the geometrical optics and the billiard systems (cf. [1], [4]). We say that C has *constant width* if each point of C has a double normal. There exist C^∞ convex curves with constant width other than circles (cf. [3]). From the viewpoint of the geometrical optics and the billiard problems, the double normal property implies that all points of C are vertices of periodic orbits of billiard balls with period two. Combined with a property of homofocal ellipses, the theorem of Poncelet proves that all points of any ellipse are vertices of *billiard n-gons* for all $n \geqq 3$, i.e., periodic orbits of billiard balls with period n ([2], p. 196). It would be natural to ask whether the converse of this phenomenon is true. In the present note we will see that it is not true if the existence of billiard triangles is assumed alone. Namely, we construct C^∞ convex curves C other than ellipses such that all points of C are vertices of billiard triangles. The example we will show has the following properties: (a) There is a subarc A of C such that all points of A are vertices of billiard equilateral triangles. (b) All billiard triangles of C are isosceles triangles.

In Section 1 we derive the differential equation which the example of convex curves must satisfy. In Section 2 we give a special solution of the differential equation. We prove in Section 3 that if all billiard triangles are equilateral, then the convex curve C is a circle.

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1. The differential equation.

Let $c:[0, L]\to \boldsymbol{E}^2$ be a C^∞ closed convex curve with $|\dot{c}|=1$. Assume that each point $c(s)$, $0\leqq s\leqq L$, is a vertex of a billiard triangle and that the billiard triangles depend differentiably on the parameter $s\in[0, L]$. We denote the vertices of the billiard triangle attached to $c(s)$, $0\leqq s\leqq L$, by $c_1(s)=c(s)$, $c_2(s)$, $c_3(s)$, and the lengths of the sides by $l_1(s)=|c_2(s)-c_1(s)|$, $l_2(s)=|c_3(s)-c_2(s)|$, $l_3(s)=|c_1(s)-c_3(s)|$, and the angles of the sides with the curve c at each vertex by $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$, respectively. If $\{e_1, e_2\}$ is the orthonormal frame field along c with $e_1=\dot{c}$, then we see that

$$c_2(s)=c_1(s)+l_1(s)(\cos\alpha_1(s)e_1(s)+\sin\alpha_1(s)e_2(s)) \tag{1.1}$$

$$c_3(s)=c_1(s)+l_3(s)(-\cos\alpha_1(s)e_1(s)+\sin\alpha_1(s)e_2(s)) \tag{1.2}$$

for any $s\in[0, L]$. By the law of sine we have

$$\frac{l_2(s)}{\sin 2\alpha_1(s)}=\frac{l_3(s)}{\sin 2\alpha_2(s)}=\frac{l_1(s)}{\sin 2\alpha_3(s)} \tag{1.3}$$

$$\alpha_1(s)+\alpha_2(s)+\alpha_3(s)=\pi \tag{1.4}$$

for any $s\in[0, L]$. By differentiating l_1, l_2, l_3, we have

$$\begin{aligned} l_1'(s)&=|c_2(s)|\cos\alpha_2(s)-\cos\alpha_1(s)\\ l_2'(s)&=|c_3(s)|\cos\alpha_3(s)-|c_2(s)|\cos\alpha_2(s)\\ l_3'(s)&=\cos\alpha_1(s)-|c_3(s)|\cos\alpha_3(s)\end{aligned} \tag{1.5}$$

for any $s\in[0, L]$, and, hence,

$$l_1(s)+l_2(s)+l_3(s)=\text{const.}=:R \tag{1.6}$$

for any $s\in[0, L]$. By (1.3) and (1.6) we have

$$\begin{aligned} l_1(s)&=\frac{R\cos a_3(s)}{2\sin\alpha_1(s)\sin\alpha_2(s)}\\ l_2(s)&=\frac{R\cos\alpha_1(s)}{2\sin\alpha_2(s)\sin\alpha_3(s)}\\ l_3(s)&=\frac{R\cos\alpha_2(s)}{2\sin\alpha_3(s)\sin\alpha_1(s)}\end{aligned} \tag{1.7}$$

for any $s\in[0, L]$. By differentiating these we get

$$l_1'(s)=\frac{R}{2(\sin\alpha_1(s)\sin\alpha_2(s))^2}\times(\dot{\alpha}_1(s)\sin\alpha_2(s)\cos\alpha_2(s)+\dot{\alpha}_2(s)\sin\alpha_1(s)\cos\alpha_1(s))$$

$$(1.8)\qquad l_2'(s)=\frac{R}{2(\sin\alpha_2(s)\sin\alpha_3(s))^2}\times(\dot{\alpha}_2(s)\sin\alpha_3(s)\cos\alpha_3(s)+\dot{\alpha}_3(s)\sin\alpha_2(s)\cos\alpha_2(s))$$

$$l_3'(s)=\frac{R}{2(\sin\alpha_3(s)\sin\alpha_1(s))^2}\times(\dot{\alpha}_3(s)\sin\alpha_1(s)\cos\alpha_1(s)+\dot{\alpha}_1(s)\sin\alpha_3(s)\cos\alpha_3(s))$$

for any $s\in[0, L]$. If $u(s)=\cos\alpha_1(s)$ for any $s\in[0, L]$, then

$$\dot{\alpha}_1(s)=-\frac{\dot{u}(s)}{\sqrt{1-u(s)^2}}$$

for all $s\in[0, L]$. Hence, by the formula (3.3), we have

$$(1.9)\qquad |\dot{c}_2(s)|=f_s(s, u(s))+f_u(s, u(s))u(s)=(l_1(s)k(s)+l_1(s)\dot{\alpha}_1(s)-\sin\alpha_1(s))\frac{1}{\sin\alpha_2(s)}$$

$$(1.10)\qquad |c_3(s)|=f_s(s, -u(s))+f_u(s, -u(s))(-\dot{u}(s))=(l_3(s)k(s)-l_3(s)\dot{\alpha}_1(s)-\sin\alpha_1(s))\frac{1}{\sin\alpha_3(s)}$$

for any $s\in[0, L]$, where $k(s)$ is the curvature of the curve c at $c(s)$. If we substitute (1.7), (1.8), (1.9) and (1.10) to the first and third equation in (1.5), then we have the following differential equation.

$$(1.11)\qquad \left(-1+\frac{\sin\alpha_2}{\sin\alpha_1\cos\alpha_3}\right)\dot{\alpha}_1+\frac{\cos\alpha_1}{\cos\alpha_2\cos\alpha_3}\dot{\alpha}_2=k-\frac{2\sin\alpha_1\sin\alpha_2\sin\alpha_3}{R\cos\alpha_2\cos\alpha_3}$$

$$(1.12)\qquad \left(1-\frac{\sin\alpha_3}{\sin\alpha_1\cos\alpha_2}\right)\dot{\alpha}_1-\frac{\cos\alpha_1}{\cos\alpha_2\cos\alpha_3}\dot{\alpha}_3=k-\frac{2\sin\alpha_1\sin\alpha_2\sin\alpha_3}{R\cos\alpha_2\cos\alpha_3}$$

It should be noted that the right hand sides of the equation coincide and the left hand sides have different signs. We will obtain a special solution of the equation and prove that it gives the examples of the convex curves we want.

2. Construction.

Let K be a unit circle in $\boldsymbol{E}^2$ and let $\varDelta ABC$ be an equilateral triangle whose vertices A, B, C are on K. The length of the sides is $\sqrt{3}$. Fix positive ε

and ε' which are less than $\pi/3$. Let $c: [0, \lambda] \to \boldsymbol{E}^2$ be a C^∞ curve with $|\dot{c}|=1$, $c(0)=A$, $c(\lambda)=B$, $c([0, \varepsilon]\cup[\lambda-\varepsilon', \lambda])\subset K$. If $k(s)$, $0\leqq s\leqq\lambda$, is the curvature of c at $c(s)$, then $k(s)=1$ for any $s\in[0, \varepsilon]\cup[\lambda-\varepsilon', \lambda]$. We assume that $k(s)$ is sufficiently close to 1 for any $s\in[0, \lambda]$ but not constant.

We consider the following equation for each $s\in[0, \lambda]$ which implies that the right hand sides of (1.11) and (1.12) is zero.

$$x+2y=\pi\,;\ \frac{2\sin x\sin^2 y}{3\sqrt{3}\cos^2 y}=k(s), \tag{2.1}$$

i. e.,

$$x+2y=\pi\,;\ \frac{4}{3\sqrt{3}}\sin^3 y=k(s)\cos y\,.$$

We denote the solution (x, y) by $(\alpha_1(s), \alpha_2(s))$ for each $s\in[0, \lambda]$. Put $\alpha_3(s)=\alpha_2(s)$ for all $s\in[0, \lambda]$. Then, $(\alpha_1(s), \alpha_2(s), \alpha_3(s))$, $0\leqq s\leqq\lambda$, is a solution of the equation (1.11) and (1.12), since

$$\dot{\alpha}_1(s)=-2\dot{\alpha}_2(s)$$

and

$$\left(-1+\frac{\sin\alpha_2(s)}{\sin\alpha_1(s)\cos\alpha_2(s)}\right)(-2\dot{\alpha}_2(s))+\frac{\cos\alpha_1(s)}{\cos^2\alpha_2(s)}\dot{\alpha}_2(s)=0$$

for any $s\in[0, \lambda]$. And, we see that

$$\alpha_1(s)=\alpha_2(s)=\alpha_3(s)=\pi/3 \tag{2.2}$$

for any $s\in[0, \varepsilon]\cup[\lambda-\varepsilon', \lambda]$. We define the lengths of the sides and vertices of the triangles as follows:

$$\begin{aligned} l_1(s)&=\frac{3\sqrt{3}\cos\alpha_3(s)}{2\sin\alpha_1(s)\sin\alpha_2(s)}\\ l_2(s)&=\frac{3\sqrt{3}\cos\alpha_1(s)}{2\sin\alpha_2(s)\sin\alpha_3(s)}\\ l_3(s)&=\frac{3\sqrt{3}\cos\alpha_2(s)}{2\sin\alpha_3(s)\sin\alpha_1(s)}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned} c_1(s)&=c(s)\\ c_2(s)&=c_1(s)+l_1(s)(\cos\alpha_1(s)e_1(s)+\sin\alpha_1(s)e_2(s))\\ c_3(s)&=c_1(s)+l_3(s)(-\cos\alpha_1(s)e_1(s)+\sin\alpha_1(s)e_2(s))\end{aligned} \tag{2.4}$$

for any $s\in[0, \lambda]$, where $\{e_1, e_2\}$ is the orthonormal frame along c.

The curve $D=c_1\cup c_2\cup c_3([0, \lambda])$ is a C^∞ convex curve because the curvature of c is close to 1. Since $l_1(s)=l_2(s)=l_3(s)=\sqrt{3}$ for any $s\in[0, \varepsilon]\cup[\lambda-\varepsilon', \lambda]$ (by (2.2) and (2.3)), we see that the triangles in $s\in[0, \varepsilon]\cup[\lambda-\varepsilon', \lambda]$ are congruent to the triangle $\varDelta ABC$, and, hence, $c_1\cup c_2\cup c_3([0, \varepsilon]\cup[\lambda-\varepsilon', \lambda])\subset K$. This implies that D is a closed curve. We must prove that the angle of reflection with D is equal to the angle of incidence. Let $\tilde{\alpha}_2(s)$, $0\leqq s\leqq\lambda$, be the angle of $c_2([0, \lambda])$ with the straight line through $c_1(s)$ and $c_2(s)$. Then, by (1.5) and (1.9), we have

$$\begin{aligned} l_1'(s) &= |\dot{c}_2(s)|\cos\tilde{\alpha}_2(s)-\cos\alpha_1(s) \\ &=(l_1(s)k(s)+l_1(s)\dot{\alpha}_1(s)-\sin\alpha_1(s))\frac{\cos\tilde{\alpha}_2(s)}{\sin\tilde{\alpha}_2(s)}-\cos\alpha_1(s) \end{aligned}$$

for all $s\in[0, \lambda]$. Since $(\alpha_1(s), \alpha_2(s), \alpha_3(s))$, $0\leqq s\leqq\lambda$, is a solution of the differential equation (1.11) and (2.1), we can have

$$l_1'(s)=(l_1(s)k(s)+l_1(s)\dot{\alpha}_1(s)-\sin\alpha_1(s))\frac{\cos\alpha_2(s)}{\sin\alpha_2(s)}-\cos\alpha_1(s)$$

for all $s\in[0, \lambda]$. From these equations we get

$$\alpha_2(s)=\tilde{\alpha}_2(s)$$

for all $s\in[0, \lambda]$. Similarly, $\alpha_3(s)=\tilde{\alpha}_3(s)$ holds for any $s\in[0, \lambda]$. Since the angles of the triangles are $\pi-2\alpha_1(s)$, $\pi-2\alpha_2(s)$, $\pi-2\alpha_3(s)$ at the vertices $c_1(s)$, $c_2(s)$, $c_3(s)$ for all $s\in[0, \lambda]$, respectively, the angle of reflection with D is equal to the angle of incidence. Therefore, the curve D we obtained has the property that each point of D is a vertex of a billiard triangle.

3. Appendix: Geometry of chords.

Let $c: [0, L]\to\boldsymbol{E}^2$ be a C^∞ closed convex curve with $|\dot{c}|=1$ and let $\{e_1, e_2\}$ be the orthonormal frame field along c with $\dot{c}=e_1$. By the Frenet formula we have

$$\dot{e}_1(s)=k(s)e_2(s)\,;\ \dot{e}_2(s)=-k(s)e_1(s)$$

for any $s\in[0, L]$, where $k(s)$, $0\leqq s\leqq L$, is the curvature of the curve c at $c(s)$. For any $(s, u)\in[0, L]\times(-1, 1)$ let $m_{(s,u)}: (-\infty, \infty)\to\boldsymbol{E}^2$ be the straight line given by

$$m_{(s,u)}(t)=c(s)+t(ue_1(s)+\sqrt{1-u^2}\ e_2(s))$$

for any $t\in(-\infty, \infty)$. We define a map $\varphi=(f, g)$: $[0, L]\times(-1, 1)\to[0, L]\times(-1, 1)$ as follows: $f(s, u)$ is the parameter of c other than s where $m_{(s,u)}$ intersects c. $g(s, u)$ is the cosine of the angle of $e_1(f(s, u))$ with $ue_1(s)+\sqrt{1-u^2}\ e_2(s)$ (the tangent vector of $m_{(s,u)}$ at $c(f(s, u))$). Let $l(s, u)$, $(s, u)\in[0, L]\times(-1, 1)$, be the length of the chord connecting $c(s)$ and $c(f(s, u))$, i.e., $l(s, u)=|c(f(s, u))-c(s)|$. Then we have the following formulas.

$$
\begin{aligned}
&f(f(s,u),-g(s,u))=s\\
(3.1)\qquad &g(f(s,u),-g(s,u))=-u\\
&l(s,u)=l(f(s,u),-g(s,u))
\end{aligned}
$$

$$
\begin{aligned}
(3.2)\qquad &l_s(s,u)=f_s(s,u)g(s,u)-u\\
&l_u(s,u)=f_u(s,u)g(s,u)
\end{aligned}
$$

$$
\begin{aligned}
&f_s(s,u)=-\frac{\sqrt{1-u^2}}{\sqrt{1-g(s,u)^2}}+\frac{k(s)l(s,u)}{\sqrt{1-g(s,u)^2}}\\
&f_u(s,u)=-\frac{l(s,u)}{\sqrt{(1-u^2)(1-g(s,u)^2)}}<0\\
(3.3)\qquad &g_s(s,u)=k(s)\sqrt{1-g(s,u)^2}+k(f(s,u))\sqrt{1-u^2}\\
&\qquad -k(s)k(f(s,u))l(s,u)\\
&g_u(s,u)=-\frac{\sqrt{1-g(s,u)^2}}{\sqrt{1-u^2}}+\frac{k(f(s,u))l(s,u)}{\sqrt{1-u^2}}
\end{aligned}
$$

$$(3.4)\qquad \det d\varphi(s,u)=f_s(s,u)g_u(s,u)-f_u(s,u)g_s(s,u)=1$$

$$(3.5)\qquad g_u(s,u)=f_s(f(s,u),-g(s,u))$$

We prove the following lemma.

LEMMA. *A C^∞ closed convex curve $c\colon[0, L]\to \boldsymbol{E}^2$ with $|c|=1$ is a circle if and only if c satisfies one of the following conditions:*

(1) *There exists a $u_0\in(-1, 1)$ such that both $g(s, u_0)$ and $f_s(s, u_0)$ are constants in $s\in[0, L]$.*

(2) *$l(s, u)$ depends only on the parameter $u\in(-1, 1)$.*

(3) *$g(s, u)$ depends only on the parameter $u\in(-1, 1)$.*

(4) *$f_s(s, u)$ depends only on the parameter $u\in(-1, 1)$.*

Proof. After proving that (2)-(4) are equivalent, we show that (1) characterizes a circle.

(2)→(3): Put $l(s, u)=L(u)$ for any $(s, u)\in[0, L]\times(-1, 1)$. By (3.2) and (3.3), we have

$$L'(u)=\frac{L(u)g(s,u)}{\sqrt{(1-u^2)(1-g(s,u)^2)}}$$

for any $(s, u)\in[0, L]\times(-1, 1)$. Hence,

$$g(s,u)^2=\frac{(1-u^2)L'(u)^2}{(1-u^2)L'(u)^2+L(u)^2}$$

for any $(s, u) \in [0, L] \times (-1, 1)$. This implies that $g(s, u)$ is a function of $u \in (-1, 1)$.

(3)→(4): Put $g(s, u) = G(u)$ for any $(s, u) \in [0, L] \times (-1, 1)$. Then, by (3.4), we have

$$\det d\varphi(s, u) = f_s(s, u) g_u(s, u) = 1$$

for any $(s, u) \in [0, L] \times (-1, 1)$. This implies that $f_s(s, u)$ is a function of $u \in (-1, 1)$.

(4)→(3): Put $f_s(s, u) = F(u)$ for any $(s, u) \in [0, L] \times (-1, 1)$. Hence,

$$f(s, u) = sF(u) + a(u)$$

for any $(s, u) \in [0, L] \times (-1, 1)$, where $a(u)$ is a suitable function of $u \in (-1, 1)$. Since c is a closed curve, i.e.,

$$a(u) = f(0, u) = f(L, u) = LF(u) + a(u) \quad \mathrm{mod}\, L\,,$$

we have that $F(u) = 1$ for any $u \in (-1, 1)$. Thus, $f_s(s, u) = 1$ for any $(s, u) \in [0, L] \times (-1, 1)$. By (3.5), we know that $g_u(s, u) = 1$ for any $(s, u) \in [0, L] \times (-1, 1)$. It follows from (3.4) and $f_u(s, u) < 0$ that $g_s(s, u) = 0$ for any $(s, u) \in [0, L] \times (-1, 1)$.

(3) and (4)→(2): By (3.2), we see that $l_s(s, u)$ depends only on the parameter $u \in (-1, 1)$. This implies (2).

Now, we prove that c is a circle if c satisfies (1). Put $g(s, u_0) = a$ and $f_s(s, u_0) = b$ for any $s \in [0, L]$. Since $l_s(s, u) = ab - u_0$,

$$l(s, u_0) = (ab - u_0)s + d$$

for any $s \in [0, L]$, where d is a constant. Since $l(s, u)$ is periodic for s, we see that $l(s, u_0) = d$ for any $s \in [0, L]$. By (3.3), we have

$$f_s(s, u_0) = -\frac{\sqrt{1-u_0^2}}{\sqrt{1-a^2}} + \frac{k(s)d}{\sqrt{1-a^2}} = b$$

for any $s \in [0, L]$. This implies that $k(s)$ is constant in $s \in [0, L]$, and, therefore, c is a circle. The lemma is proved.

As an application of the lemma we can give a characterization of circles by the existence of a family of billiard triangles.

PROPOSITION. *Let $c: [0, L] \to \boldsymbol{E}^2$ be a C^∞ closed convex curve. If each point $c(s)$, $s \in [0, L]$, is a vertex of billiard equilateral triangles and if these equilateral triangles depend differentiably on the parameter $s \in [0, L]$, then c is a circle.*

Proof. We use the notation in Section 1. By (1.6) we know

$$l_1(s) + l_2(s) + l_3(s) = \text{const.}$$

for any $s \in [0, L]$. Since the triangles are equilateral,

$$l_1(s)=l_2(s)=l_3(s)=\text{const.}$$

for any $s\in[0, L]$. It follows from (3.2) that if $c_2(s)=c(f(s, 1/2))$, $f_s(s, 1/2)=1$ for any $s\in[0, L]$, because $g(s, 1/2)=1/2$ for all $s\in[0, L]$. Thus (1) in Lemma is satisfied by putting $u_0=1/2$. This completes the proof.

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