T. NISHIMOTO
KODAI MATH. J.
10 (1987), 362—374

CONFLUENT W-K-B APPROXIMATION, 1

BY TOSHIHIKO NISHIMOTO

§1. Introduction.

In this paper, we consider asymptotic expansions of solutions of the second order ordinary differential equations of the form

$$(1.1)\qquad \varepsilon^2\frac{d^2y}{dx^2}=k(x, \alpha)y$$

Here ε is a positive small parameter, x is the complex independent variable and $k(x, \alpha)$ is a polynomial of x which depends on another complex parameter α. The zeros of $k(x, \alpha)$ with respect to x are so-called turning points of the equation. These turning points move as the parameter α varies and some of turning points may coalesce at certain values of the parameter, say α_0 which we call critical values of the parameter. For fixed α, the usual W-K-B theory, see for examples Evgrafov and Fedoryuk [1] and Fröman and Fröman [2], and the matching method for higher order turning point problems, Nishimoto [3], are well applied. But if we want to know asymptotic properties of the solutions and their connection formulas that are uniformly valid with respect to the parameter α in a region containing critical values, then these methods do not work well.

In the several fields of physics we encounter to treat such cases. One of the examples is the inelastic scattering theory of atomic or molecular collisions. In a certain simplified model of the problem, the coefficient function is given by

$$k(x, a, b, \mu)=-\left\{\frac{1}{4}(a^2x^2+b^2)^2-a^2\mu x+\frac{1}{4}\right\},$$

where a^2, b^2 and μ are real parameters, and μ is a small positive number. For physical explanation of the problems, the reader may consult with the book "Molecular Collision Theory" by M. S. Child (1974), Academic press. This problem will be studied in future as an example to which our method can be applied.

Now by the term "confluent W-K-B approximation", we mean the asymptotic expansions and connection formulas of solutions that are uniformly valid with respect to the parameter in certain regions containing critical values. To understand more precisely we consider at first the simplest case having such natures,

Received July 30, 1987.

(1.2) $$\varepsilon^2\frac{d^2u}{dx^2}=(x^2+r^2)u.$$

Here turning points are $x=\pm ri$ and when r tends to zero, these two turning points coalesce. We call this equation as the reduced equation, since more general equations having coalescing two turning points may be approximately reduced to this equation by the so-called Liouville transformation, see for example Olver [5].

Here we divide our problem into three parts. The first part is about the existence theorem of asymptotic expansions of solutions in appropriate regions of the complex (x, α) space as the parameter ε tends to zero. In the course of analysis, the notion of the preservation of the canonical domains is introduced. The second part is to get asymptotic forms of the connection formulas between solutions defined in different regions of the complex x-plane that uniformly valid with respect to the parameter α. The results of this part is briefly anounced in Nishimoto [4]. The last part is to apply our mathematical theory to the inelastic scattering problem of molecular collisions.

In this paper, the first part, that is the existence theorem will be considered when two simple turning points coalesce. The same problem is treated by Olver [5] when x and α are real. But if we want to apply the method to the physical problem, extension to the complex case is indispensable.

In section 2, the Liouville transformation is introduced, and the terms canonical domains and the preservation of the canonical domains are explained. In section 3, the parabolic cylinder function and its asymptotic properties are given. And in the last section, we give the existence theorem of asymptotic expansions of solutions of (1.1). The forms of the asymptotic expansions are the same forms with the usual W-K-B type expasions, and these are uniformly valid with respect to α in a region containing a critical values.

§ 2. The Liouville transformation and the canonical domains.

We assume in this and subsequent papers that the function $k(x, \alpha)$ is a polynomial of x and α. By a linear transformation of x and by introducing new parameter a instead of α, we can assume without loss of generality that two coalescing turning points are $\pm ai$, and so the function $k(x, a)$ is expressed as

$$k(x, a)=(x^2+a^2)k_1(x, a),$$

where the function $k_1(x, a)$ does not vanish at $x=\pm ai$ for all small $|a|$.

The Liouville transformation from x, y to ζ, u is defined by

(2.1) $$x=x(\zeta),$$

$$u(\zeta)=\left(\frac{dx}{d\zeta}\right)^{-1/2}y(x(\zeta)),$$

by which the equation (1.1) becomes

$$\varepsilon^2\frac{d^2u}{d\zeta^2}=\left\{\left(\frac{dx}{d\zeta}\right)^2k(x,\ a)+\varepsilon^2\left(\frac{dx}{d\zeta}\right)^{1/2}\frac{d^2}{d\zeta^2}\left(\left(\frac{dx}{d\zeta}\right)^{-1/2}\right)\right\}u\,. \tag{2.2}$$

Here the transformation $x=x(\zeta)$ is chosen such that

$$\left(\frac{dx}{d\zeta}\right)^2k(x,\ a)=\zeta^2+r^2, \tag{2.3}$$

$$\int_{-a}^{x}k(x,\ a)^{1/2}dx=\int_{-r}^{\zeta}(\zeta^2+r^2)^{1/2}d\zeta,$$

so that we have

$$\varepsilon^2\frac{d^2u}{d\zeta^2}=\{(\zeta^2+r^2)+\varepsilon^2\phi(\zeta,\ r)\}u\,, \tag{2.4}$$

where

$$\phi(\zeta,\ r)=\left(\frac{dx}{d\zeta}\right)^{1/2}\frac{d^2}{d\zeta^2}\left(\left(\frac{dx}{d\zeta}\right)^{-1/2}\right). \tag{2.5}$$

$$=\left[\left\{\left(\frac{k_1''}{4k_1}-\frac{5k_1'^2}{16k_1^2}\right)(a^2+x^2)^2+\frac{k_1'}{4k_1}x(a^2+x^2)-\frac{1}{4}(2a^2+3x^2)\right\}\cdot\frac{r^2+\zeta^2}{k_1(a^2+x^2)^3}\right]+\frac{2r^2+3\zeta^2}{4(r^2+\zeta^2)^2}\,.$$

In the relation (2.3) $x=-ai$ corresponds to $\zeta=-ri$, and to correspond $x=ai$ to $\zeta=ri$ we have

$$\int_{-ai}^{ai}(a^2+x^2)^{1/2}k_1(x,\ a)^{1/2}dx=\int_{-ri}^{ri}(r^2+\zeta^2)^{1/2}d\zeta=\frac{\pi ir^2}{2}\,,$$

that is

$$r^2=\frac{2}{\pi i}\int_{-ai}^{ai}(a^2+x^2)^{1/2}k_1(x,\ a)^{1/2}dx \tag{2.6}$$

$$=\frac{2a^2}{\pi}\int_{-1}^{1}(1-v^2)^{1/2}k_1(aiv,\ a)^{1/2}dv$$

This relation gives us the correspondence between r and a, and as a tends to zero, we have

$$r\sim k_1(0,\ 0)^{1/4}a \qquad (a\to 0)\,.$$

To see what regions in the x-plane and ζ-plane correspond by the transformation (2.3), it is convenient to introduce the notion of the canonical domains with respect to

$$q=\int_{-ai}^{x}k(x,\ a)^{1/2}dx \qquad \text{(in the } x\text{-plane)},$$

and

$$q=\int_{-ri}^{\zeta}(\zeta^2+r^2)^{1/2}d\zeta \qquad \text{(in the } \zeta\text{-plane)}$$

respectively, following to Evgrafov and Fedoryuk [1]. Since the function $k(x, a)$ is a polynomial of x, it is easy to see that every canonical domain in the x-plane corresponds one to one and conformally onto a canonical domain possibly with cuts in the ζ-plane. We express canonical domains in the x-plane by $D_i(a)$ and those in the ζ-plane by $\mathscr{D}_i(r)$, that depend on the parameters a

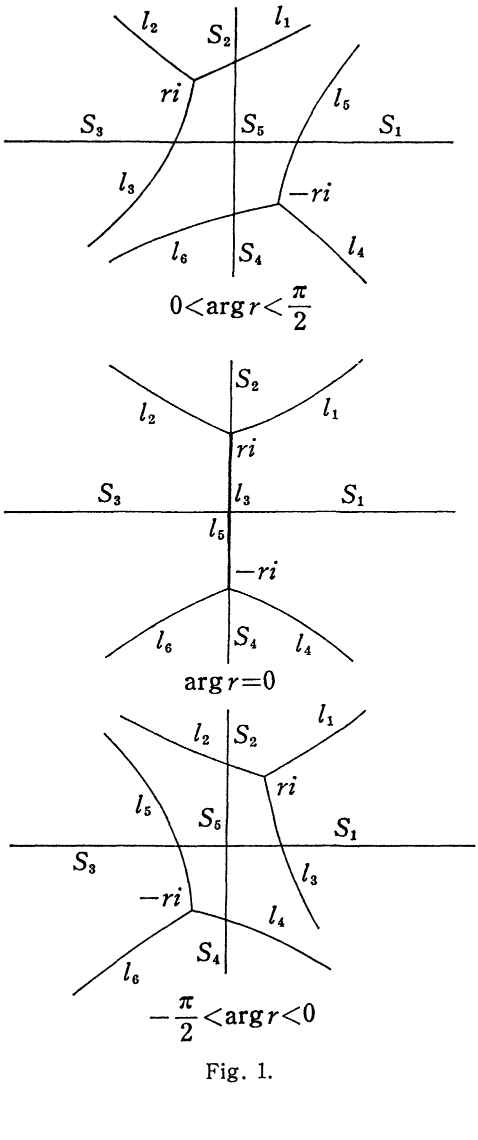

Fig. 1.

and r respectively.

It is assumed in the subsequent analysis that every canonical domain $\mathscr{D}_i(r)$ in the ζ-plane remains canonical when r tends to zero. We explain this more precisely. The Stokes curve configuration in the ζ-plane is illustrated in Fig. 1. The turning points of the reduced equation (1.2) are $\pm ir$, and the Stokes curves are defined as curves starting from $\pm ir$ along which

$$\mathrm{Re}\int_{\pm ri}^{\zeta}(\zeta^2+r^2)^{1/2}d\zeta=0,$$

and we label these curves as l_i ($i=1, 2, \cdots, 6$). The unbounded regions S_i ($i=1, 2, \cdots, 5$), bounded by Stokes curves are called Stokes regions, see Fig. 1.

Then canonical domains are given by the following table.

Table: Canonical domains for the reduced equation

canonical domain	$-\frac{\pi}{2}<\arg r<0$	$\arg r=0$	$0<\arg r<\frac{\pi}{2}$
$\mathscr{D}_1$	$S_1\cup l_1\cup S_2$	$S_1\cup l_1\cup S_2$	$S_1\cup l_5\cup l_1\cup S_2$
$\mathscr{D}_2$	$S_2\cup l_2\cup S_5\cup l_5\cup S_3$	$S_2\cup l_2\cup S_3$	$S_2\cup l_2\cup S_3$
$\mathscr{D}_3$	$S_3\cup l_5\cup S_5\cup l_3\cup S_1$	$S_3\cup l_5\cup S_1$	$S_3\cup l_3\cup S_5\cup l_5\cup S_1$
$\mathscr{D}_4$	$S_3\cup l_6\cup S_4$	$S_3\cup l_6\cup S_4$	$S_3\cup l_3\cup S_5\cup l_6\cup S_4$
$\mathscr{D}_5$	$S_4\cup l_4\cup S_5\cup l_3\cup S_1$	$S_4\cup l_4\cup S_1$	$S_4\cup l_4\cup S_1$
$\mathscr{D}_6$	$S_4\cup l_4\cup S_5\cup l_2\cup S_2$	——	$S_4\cup l_6\cup S_5\cup l_1\cup S_2$

Among these canonical domains, $\mathscr{D}_i(r)$ ($i=1, 2, 4, 5$) are preserved canonical when r tends to zero, while $\mathscr{D}_3(r)$ and $\mathscr{D}_6(r)$ do not remain canonical since they split into two Stokes regions as r tends zero for $-\frac{\pi}{2}<\arg r<\frac{\pi}{2}$. Accordingly the corresponding canonical domains $D_i(a)$ ($i=1, 2, 4, 5$) remain canonical in the x-plane when a becomes zero.

Now we want to establish the existence of asymptotic expansions of solutions of the equation (2.4) in each canonical domain which remains canonical as r tends to zero in the ζ-plane. This will be done in the section 4.

§ 3. Parabolic cylinder functions.

Here we give fundamental properties of the parabolic cylinder functions by which solutions of the reduced equation (1.2) are expressed.

The parabolic cylinder function $U(s, b)$ is defined by the integral

(3.1) $$U(s, b)=\frac{\Gamma\left(\frac{1}{2}-b\right)}{2\pi i}e^{-1/4s^2}\int_C t^{b-1/2}\exp\left(st-\frac{1}{2}t^2\right)dt$$

where the integral path C is described in the Figure 2.

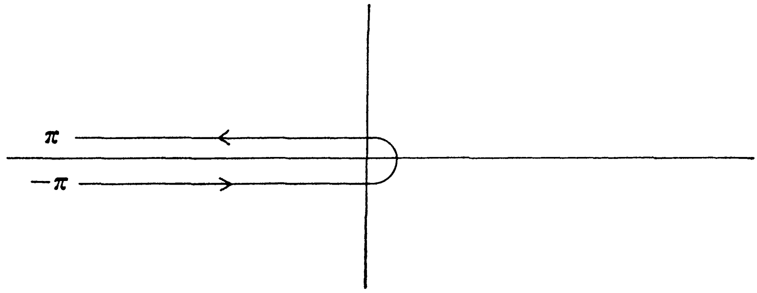

Fig. 2.

The function $U(s, b)$ satisfies the differential equation

(3.2) $$\frac{d^2u}{ds^2}=\left(\frac{1}{4}s^2+b\right)u$$

and has an asymptotic expansion

(3.3) $$U(s, b)\simeq s^{-b-1/2}\left\{1-\frac{1}{2}\left(b+\frac{1}{2}\right)\left(b+\frac{3}{2}\right)s^{-2}+\cdots\right\}\exp\left(-\frac{1}{4}s^2\right)$$

as s tends to infinity in the sector $|\arg s|<\frac{3}{4}\pi$. It is easy to see that the function $U(s, b)$ is subdominant, that is exponentially small as s tends to infinity in the sector $|\arg s|<\frac{1}{4}\pi$ and dominant in others.

The functions defined by $U(-s, b)$, $U(is, -b)$ and $U(-is, -b)$ are also solutions of the equation (3.2). Let M_k be sectors in the complex s-plane such that (see Fig. 3)

$$M_k:\ -\frac{3}{4}\pi+\frac{\pi}{2}k<\arg s<-\frac{\pi}{4}+\frac{\pi}{2}k \qquad (k=1, 2, 3, 4).$$

Then the functions $U(-is, b)$, $U(-s, b)$ and $U(is, -b)$ are subdominant as s tends to infinity in M_2, M_3 and M_4 respectively.

Let us define four functions $U_k(x, r, \varepsilon)$ $(k=1, 2, 3, 4)$ by

(3.4) $$U_1(x, r, \varepsilon)=U\left(\sqrt{\frac{2}{\varepsilon}}x,\ \frac{r^2}{2\varepsilon}\right),$$

$$U_2(x, r, \varepsilon)=U\left(-i\sqrt{\frac{2}{\varepsilon}}x,\ -\frac{r^2}{2\varepsilon}\right),$$

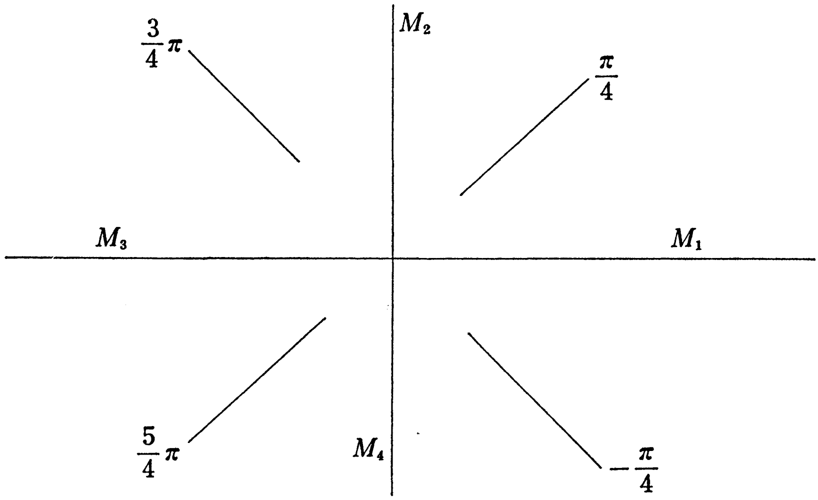

Fig. 3.

$$U_3(x, r, \varepsilon)=U\left(-\sqrt{\frac{2}{\varepsilon}}x, \frac{r^2}{2\varepsilon}\right),$$

$$U_4(x, r, \varepsilon)=U\left(i\sqrt{\frac{2}{\varepsilon}}x, -\frac{r^2}{2\varepsilon}\right).$$

Then these functions are solutions of the reduced equation (1.2). From the asymptotic formulas (3.3) of $U(s, b)$, we have asymptotic expansions of the functions $U_k(x, r, \varepsilon)$ as $\sqrt{2/\varepsilon}\cdot x$ goes to infinity:

$$(3.5) \qquad U_1(x, r, \varepsilon)\simeq\left(\sqrt{\frac{2}{\varepsilon}}x\right)^{-r^2/2\varepsilon-1/2}\exp\left(-\frac{x^2}{2\varepsilon}\right), \qquad |\arg x|<\frac{3}{4}\pi,$$

$$U_2(x, r, \varepsilon)\simeq\left(-i\sqrt{\frac{2}{\varepsilon}}x\right)^{r^2/2\varepsilon-1/2}\exp\left(\frac{x^2}{2\varepsilon}\right), \qquad |\arg(-ix)|<\frac{3}{4}\pi,$$

$$U_3(x, r, \varepsilon)\simeq\left(-\sqrt{\frac{2}{\varepsilon}}x\right)^{-r^2/2\varepsilon-1/2}\exp\left(-\frac{x^2}{2\varepsilon}\right), \qquad |\arg(-x)|<\frac{3}{4}\pi,$$

$$U_4(x, r, \varepsilon)\simeq\left(i\sqrt{\frac{2}{\varepsilon}}x\right)^{r^2/2\varepsilon-1/2}\exp\left(\frac{x^2}{2\varepsilon}\right), \qquad |\arg(ix)|<\frac{3}{4}\pi.$$

On the other hand, the reduced equation (1.2) has solutions whose asymptotic expansions are of the W-K-B types as follows:

$$(3.6) \qquad V_1(x, r, \varepsilon)\simeq(x^2+r^2)^{-1/4}\exp\left\{-\frac{1}{\varepsilon}\int_{ri}^{x}(x^2+r^2)^{1/2}dx\right\},$$

$$V_2(x, r, \varepsilon)\simeq(x^2+r^2)^{-1/4}\exp\left\{\frac{1}{\varepsilon}\int_{ri}^{x}(x^2+r^2)^{1/2}dx\right\},$$

$$V_3(x, r, \varepsilon)\simeq(x^2+r^2)^{-1/4}\exp\left\{-\frac{1}{\varepsilon}\int_{-ri}^{x}(x^2+r^2)^{1/2}dx\right\},$$

$$V_4(x, r, \varepsilon)\simeq(x^2+r^2)^{-1/4}\exp\left\{\frac{1}{\varepsilon}\int_{-ri}^{x}(x^2+r^2)^{1/2}dx\right\}.$$

The solution $V_i(x, r, \varepsilon)$ is subdominant as $\varepsilon\to 0$ for x in M_i of the complex x-plane, and at the same time as x tends to infinity in M_i for fixed ε positive.

Since both functions $U_i(x, r, \varepsilon)$ and $V_i(x, r, \varepsilon)$ are solutions of the reduced equation (1.2) and subdominant as $x\to\infty$ in M_i, we must have

(3.7) $$U_i(x, r, \varepsilon)=h_i(r, \varepsilon)V_i(x, r, \varepsilon) \qquad (i=1, 2, 3, 4).$$

Here the coefficients $h_i(r, \varepsilon)$ can be obtained from

(3.8) $$h_i(r, \varepsilon)=\lim_{x\to\infty,\, x\in M_i}\frac{U_i(x, r, \varepsilon)}{V_i(x, r, \varepsilon)}$$

From the asymptotic formula (3.3) of $U(s, b)$, we have

(3.9) $$U_1(x, r, \varepsilon)\simeq\left(\sqrt{\frac{2}{\varepsilon}}x\right)^{-r^2/2\varepsilon-1/2}\exp\left(-\frac{x^2}{2\varepsilon}\right), \qquad |\arg x|<\frac{3}{4}\pi,$$

And since

$$\int^{x}(x^2+r^2)^{1/2}dx=\frac{1}{2}x(x^2+r^2)^{1/2}+\frac{r^2}{2}\log\{x+(x^2+a^2)^{1/2}\},$$

then for large $|x|$

$$\int_{\pm ri}^{x}(x^2+r^2)^{1/2}dx=\frac{1}{2}x^2+\frac{r^2}{4}+\frac{r^2}{2}\log\frac{2x}{\pm ri}+O\left(\frac{r^4}{x^2}\right).$$

Also it stands for large $|x|$

$$(x^2+r^2)^{-1/4}=x^{-1/2}\left(1+O\left(\frac{r^2}{x^2}\right)\right).$$

Therefore we have as $x\to\infty$,

(3.10) $$V_1(x, r, \varepsilon)\simeq x^{-1/2}\exp\left\{-\frac{1}{\varepsilon}\left(\frac{1}{2}x^2+\frac{r^2}{4}+\frac{r^2}{2}\log\frac{2x}{ri}\right\}.$$

From (3.8), (3.9) and (3.10) we get

(3.11) $$h_1(r, \varepsilon)=\left(\frac{2}{\varepsilon}\right)^{-1/4(r^2/\varepsilon+1)}\left(\frac{2}{ri}\right)^{r^2/2\varepsilon}\exp\left(\frac{r^2}{4\varepsilon}\right).$$

By the same way, other coefficients $h_i(r, \varepsilon)$ $(i=1, 2, 3, 4)$ can be obtained as follows:

$$h_2(r, \varepsilon)=(-i)^{r^2/2\varepsilon-1/2}\left(\frac{2}{\varepsilon}\right)^{1/4(r^2/\varepsilon-1)}\left(\frac{2}{ri}\right)^{-r^2/2\varepsilon}\exp\left(-\frac{r^2}{4\varepsilon}\right),$$

$$h_3(r, \varepsilon)=(-1)^{-r^2/2\varepsilon-1/2}\left(\frac{2}{\varepsilon}\right)^{-1/4(r^2/\varepsilon+1)}\left(-\frac{2}{ri}\right)^{r^2/2\varepsilon}\exp\left(\frac{r^2}{4\varepsilon}\right),$$

$$h_4(r, \varepsilon)=(i)^{r^2/2\varepsilon-1/2}\left(\frac{2}{\varepsilon}\right)^{1/4(r^2/\varepsilon-1)}\left(-\frac{2}{ri}\right)^{-r^2/2\varepsilon}\exp\left(-\frac{r^2}{4\varepsilon}\right).$$

§ 4. Existence theorem.

Let us consider firstly the differential equation (2.4) in one of the canonical domains, say $\mathscr{D}_1(r)$. It is clear that for $|\arg r|<\pi/2$, $\mathscr{D}_1(r)$ remains canonical as r tends to zero.

The differential equation (2.4) is equivalent to the following system of equations

$$\varepsilon\frac{dZ}{d\zeta}=A(\zeta)Z, \tag{4.1}$$

$$A(\zeta)=\begin{bmatrix} 0 & 1 \\ p(\zeta, r) & 0 \end{bmatrix}+\varepsilon^2\begin{bmatrix} 0 & 0 \\ \phi(\zeta, r) & 0 \end{bmatrix}, \quad Z=\begin{bmatrix} u \\ \varepsilon\dfrac{du}{d\zeta} \end{bmatrix}$$

where $p(\zeta, r)=\zeta^2+r^2$. By the transformation

$$Z(\zeta)=T(\zeta)V(\zeta), \tag{4.2}$$

where

$$T(\zeta)=\begin{bmatrix} 1 & 1 \\ \sqrt{p}-\dfrac{\varepsilon p'}{4p} & -\sqrt{p}-\dfrac{\varepsilon p'}{4p} \end{bmatrix}$$

the differential equation (4.1) becomes

$$\varepsilon\frac{dV}{d\zeta}=\left\{T(\zeta)^{-1}A(\zeta)T(\zeta)-\varepsilon T(\zeta)^{-1}\frac{dT(\zeta)}{d\zeta}\right\}V.$$

Evaluating the matrices in the bracket, we obtain

$$\varepsilon\frac{dV}{d\zeta}=\left\{\sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}-\frac{\varepsilon p'}{4p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\varepsilon^2 s(\zeta)\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right\}V, \tag{4.3}$$

where

$$s(\zeta)=\frac{\phi}{2\sqrt{p}}+\frac{p''}{8p\sqrt{p}}-\frac{5}{32}\frac{p'^2}{p^2\sqrt{p}}. \tag{4.4}$$

If we put

(4.5)
$$V=(E+W)p^{-1/4}\begin{bmatrix} \exp\left(\int_{\zeta_0}^{\zeta}\frac{\sqrt{p}}{\varepsilon}d\zeta\right) & 0 \\ 0 & \exp\left(-\int_{\zeta_0}^{\zeta}\frac{\sqrt{p}}{\varepsilon}d\zeta\right) \end{bmatrix},$$

$$W=\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix},$$

where E is the second order unit matrix, then we have for W the differential equation

(4.6)
$$\varepsilon\frac{dW}{d\zeta}=\varepsilon^2 s\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}+\left\{\begin{bmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{bmatrix}+\varepsilon^2 s\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right\}W-W\begin{bmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{bmatrix}.$$

For each component of W, the above equation can be rewritten

$(4.6)_1$
$$\begin{cases} \varepsilon w_{11}'=\varepsilon^2 s+\varepsilon^2 s(w_{11}+w_{21}), \\ \varepsilon w_{21}'=-\varepsilon^2 s-2\sqrt{p}\,w_{21}-\varepsilon^2 s(w_{11}+w_{21}). \end{cases}$$

$(4.6)_2$
$$\begin{cases} \varepsilon w_{12}'=\varepsilon^2 s+2\sqrt{p}\,w_{12}+\varepsilon^2 s(w_{12}+w_{22}), \\ \varepsilon w_{22}'=-\varepsilon^2 s-\varepsilon^2 s(w_{12}+w_{22}). \end{cases}$$

Because the analysis of the differential equations $(4.6)_1$ and $(4.6)_2$ is analogous, we treat only the latter $(4.6)_2$. The equation $(4.6)_2$ is transformed into the following integral equation

(4.7)
$$w_{12}(\zeta,\varepsilon)=-\varepsilon\int_{\gamma(\zeta)}\left\{\exp\int_{\eta}^{\zeta}\frac{2\sqrt{p}}{\varepsilon}d\eta\right\}s(\eta)\{1+w_{12}(\eta)+w_{22}(\eta)\}\,d\eta,$$

$$w_{22}(\zeta,\varepsilon)=\varepsilon\int_{\gamma(\zeta)}s(\eta)\{1+w_{12}(\eta)+w_{22}(\eta)\}\,d\eta,$$

where the integral path $\gamma(\zeta)$ is a continuous and piecewise smooth curve connecting ζ and infinity in $\mathcal{D}_1(r)$ and along $\gamma(\zeta)$ we must have

(4.8)
$$\exp\int_{\eta}^{\zeta}\frac{2\sqrt{p}}{\varepsilon}d\eta\leqq 1,$$

or

$$\mathrm{Re}\int_{\eta}^{\zeta}\frac{2\sqrt{p}}{\varepsilon}d\eta\leqq 0.$$

For every ζ in $\mathcal{D}_1(r)$, it is possible to choose such a curve $\gamma(\zeta)$ because that $\mathcal{D}_1(r)$ is canonical domain with respect to $p(\zeta)$, (see [1] or [3]).

Suppose that the polynomial $k(x, a)$ is of the degree n, then from the relation (2.3) between x and ζ, it is easy to see that $x\sim\zeta^{4/(n+2)}$ as $\zeta\to\infty$ or $\zeta\sim x^{(n+2)/4}$ as $x\to\infty$. The expression (2.5) shows that as $\zeta\to\infty$ the function $\psi(\zeta, r)$ decreases zero by the order $O(\zeta^{-2})$, and so from (4.4) the function $s(\zeta)$

is of the order ζ^{-3} as $\zeta\to\infty$.

From these considerations we can conclude that the total variation of $\varepsilon s(\zeta)$ along $\gamma(\zeta)$ is bounded:

$$V[s(\zeta)]=\int_{\gamma(\zeta)}|\varepsilon s(\eta)||d\eta|<\infty,$$

and moreover for fixed ε, $V[s(\zeta)]=O(\zeta^{-2})$ as $\zeta\to\infty$ and at the same time for fixed ζ, $V[s(\zeta)]=O(\varepsilon)$.

Now it is possible to prove the following lemma.

LEMMA. *The integral equation* (4.7) *has a solution* $w_{12}(\zeta, \varepsilon)$, $w_{22}(\zeta, \varepsilon)$ *in the canonical domain* $\mathcal{D}_1(r)$ *which satisfies*

$$|w_{12}(\zeta, \varepsilon)|,\ |w_{22}(\zeta, \varepsilon)|\leqq\exp\{2V[s(\zeta)]\}-1.$$

Proof. We successively define functions $w_{12}^{(n)}(\zeta)$ and $w_{22}^{(n)}(\zeta)$ as follows.

$$\begin{cases} w_{12}^{(0)}(\zeta, \varepsilon)=-\varepsilon\displaystyle\int_{\gamma(\zeta)}\left\{\exp\int_{\eta}^{\zeta}\frac{2\sqrt{p}}{\varepsilon}d\eta\right\}s(\eta)d\eta\,, \\ w_{22}^{(0)}(\zeta, \varepsilon)=\varepsilon\displaystyle\int_{\gamma(\zeta)}s(\eta)d\eta\,, \end{cases}$$

and

$$\begin{cases} w_{12}^{(n)}(\zeta, \varepsilon)=-\varepsilon\displaystyle\int_{\gamma(\zeta)}\left\{\exp\int_{\eta}^{\zeta}\frac{2\sqrt{p}}{\varepsilon}d\eta\right\}s(\eta)\{w_{12}^{(n-1)}(\eta, \varepsilon)+w_{22}^{(n-1)}(\eta, \varepsilon)\}\,d\eta\,, \\ w_{22}^{(n)}(\zeta, \varepsilon)=\varepsilon\displaystyle\int_{\gamma(\zeta)}s(\eta)\{w_{12}^{(n-1)}(\eta, \varepsilon)+w_{22}^{(n-1)}(\eta, \varepsilon)\}\,d\eta\,. \end{cases}$$

By induction we can prove the inequality

$$|w_{k2}^{(n)}(\zeta, \varepsilon)|\leqq\frac{\{2V[s(\zeta)]\}^{n+1}}{(n+1)!}\,,\qquad (k=1, 2)\,. \tag{4.9}$$

For $n=0$, this is obvious, and if we assume the inequality (4.9) is true for $n-1$, we have by applying (4.8),

$$\begin{aligned} |w_{k2}^{(n)}(\zeta, \varepsilon)| &\leqq\int_{\gamma(\zeta)}\frac{2\{2V[s(\eta)]\}^{n}}{n!}|\varepsilon s(\eta)||d\eta| \\ &\leqq\int_{\gamma(\zeta)}\frac{2\{2V[s(\eta)]\}^{n}}{n!}dV[s(\eta)] \\ &\cdot\frac{\{2V[s(\zeta)]\}^{n+1}}{n+1!}\,. \end{aligned}$$

Then our lemma is proved immediately by using the usual Picard iteration argument to the integral equation (4.7).

Therefore we have obtained the following existence theorem.

THEOREM 1. *Let $\mathcal{D}(r)$ be one of the canonical domains for $\varepsilon=\{r: 0\leqq|r|\leqq r_0, (r_0>0), |\arg r|<\pi/2\}$. Then the differential* (2.4) *has a fundamental system of solutions $u_1(\zeta, \varepsilon)$, $u_2(\zeta, \varepsilon)$ in $\mathcal{D}(r)$ whose asymptotic expansion is uniformly valid in ε with respect to r and has a form*

$$\begin{bmatrix} u_1 & u_2 \\ \varepsilon\dfrac{du_1}{d\zeta} & \varepsilon\dfrac{du_2}{d\zeta} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ \sqrt{p}-\dfrac{\varepsilon p'}{4p} & -\sqrt{p}-\dfrac{\varepsilon p'}{4p} \end{bmatrix} p^{-1/4}\{E+W(\zeta, \varepsilon)\} \cdot\begin{bmatrix} \exp\left(\int_{\zeta_0}^{\zeta}\dfrac{\sqrt{p}}{\varepsilon}d\zeta\right) & 0 \\ 0 & \exp\left(-\int_{\zeta_0}^{\zeta}\dfrac{\sqrt{p}}{\varepsilon}d\zeta\right) \end{bmatrix}, \tag{4.10}$$

where $p=\zeta^2+r^2$, E is the unit matrix of order 2, and $W(\zeta, \varepsilon)=(w_{ij}(\zeta, \varepsilon))$ is the two by two matrix satisfying

$$w_{ij}(\zeta, \varepsilon)=\begin{cases} 0(\varepsilon), & \text{for } \zeta\in\mathcal{D}(r) \text{ as } \varepsilon\to 0, \\ 0(\zeta^{-2}), & \text{for the fixed } \varepsilon \text{ as } \zeta\to\infty \text{ in } \mathcal{D}(r). \end{cases}$$

This theorem shows that the differential equation (2.4) has a fundamental system of solutions whose asymptotic expansions have essentially the same forms with those of the reduced equation, and then it has solutions whose principal terms are expressed by using the parabolic cylinder functions. This fact is important to get connection formulas between the WKB type solutions of the differential equations (2.4) and (1.1).

By considering the inverse Liouville transformation we obtain a existence theorem of solutions of the original equation (1.1). From (2.1) and (2.3) we have

$$y(x)=\left(\frac{dx}{d\zeta}\right)^{1/2}u(\zeta)=\left(\frac{p(\zeta, r)}{k(x, a)}\right)^{1/4}u(\zeta),$$

$$\varepsilon\frac{dy}{dx}=k(x, a)^{1/4}p(\zeta, r)^{-1/4}\varepsilon\frac{du}{d\zeta}-\frac{1}{4}k(x, a)^{-5/4}\frac{dk}{dx}(x, a) \cdot p(\zeta, r)^{1/4}\varepsilon u(\zeta)+\frac{1}{4}k(x, a)^{1/4}p(\zeta, r)^{-5/4}\frac{dp}{d\zeta}(\zeta, r)\varepsilon u(\zeta),$$

$$\int_{x_0}^{x}k(x, a)^{1/2}dx=\int_{\zeta_0}^{\zeta}p(\zeta, r)^{1/2}d\zeta,$$

and by using the asymptotic expansions (4.9) for $u(\zeta)$ and $\varepsilon du(\zeta)/d\zeta$ we get the following.

THEOREM 2. *Let E be the region of the parameter a corresponding to ε by the relation* (2.6) *and let $D(a)$ be the canonical domain in the x-plane corresponding to the canonical domain $\mathcal{D}(r)$. Then the differential equation* (1.1) *with $k(x, a)$ polynomial of x of degree n has solutions $y_1(x, \varepsilon)$ and $y_2(x, \varepsilon)$ in $D(a)$ of the form*

$$(4.11)\quad \begin{cases} y_1(x, \varepsilon)=k(x, a)^{-1/4}(1+r_{11}(x, \varepsilon))\exp\left\{\frac{1}{\varepsilon}\int_{x_0}^{x} k(x, a)^{1/2}dx\right\}, \\ \varepsilon\frac{dy_1}{dx}(x, \varepsilon)=k(x, a)^{-1/4}\Big(k(x, a)^{1/2} \\ \qquad -\frac{k'(x, a)}{4k(x, a)}\Big)(1+r_{12}(x, \varepsilon))\exp\left\{\frac{1}{\varepsilon}\int_{x_0}^{x} k(x, a)^{1/2}dx\right\}, \end{cases}$$

$$\begin{cases} y_2(x, \varepsilon)=K(x, a)^{-1/4}(1+r_{21}(x, \varepsilon))\exp\left\{-\frac{1}{\varepsilon}\int_{x_0}^{x} k(x, a)^{1/2}dx\right\}, \\ \varepsilon\frac{dy_2}{dx}(x, \varepsilon)=k(x, a)^{-1/4}\Big(-k(x, a)^{1/2} \\ \qquad -\frac{k'(x, a)}{4k(x, a)}\Big)(1+r_{22}(x, \varepsilon))\exp\left\{-\frac{1}{\varepsilon}\int_{x_0}^{x} k(x, a)^{1/2}dx\right\} \end{cases}$$

where

$$(4.12)\quad r_{ij}(x, \varepsilon)=\begin{cases} 0(\varepsilon) & \textit{for} \quad x\in D(a) \quad \textit{as} \quad \varepsilon\to 0, \\ 0(x^{-(n+2)/2}) & \textit{for fixed } \varepsilon \textit{ as } x\to\infty \textit{ in } D(a). \end{cases} \qquad (i, j=1, 2).$$

The above asymptotic formulas are valid uniformly with respect to a in E containing the critical value $a=0$.

REFERENCES

[1] EVGRAFOV, E. M. AND M. V. FEDORYUK, Asymptotic behavior as $\lambda\to\infty$ of the solutions of the equation $w''(z)-p(z,\lambda)w(z)=0$ in the complex plane, Russian Math. Surveys **21** (1966), 1-48.

[2] FRÖMAN, M. AND P. O. FRÖMAN, JWKB approximation, North-Holland, Amsterdam (1965).

[3] NISHIMOTO, T., On an extension theorem and its application for turning point problem of large order, Kodai Math. Sem. Rep., **25** (1973), 458-489.

[4] NISHIMOTO, T., Confluent WKB approximation, To appear in Proceeding of France-Japan Colloquium on the theory of differential equations in the complex domain, Herman.

[5] OLVER, F. W. J., Second order linear differential equations with two turning points, Phil. Trans. of the Royal Soc. of London, **278** (1975), 137-174.

[6] Iwanami's Mathematical Dictionary (in Japanese), The Mathematical Society of Japan (1960).

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY