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ON TWO DIMENSIONAL ISOSYSTOLIC INEQUALITIES

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1. Introduction and statements of results.

Let M be a closed (i. e. compact without boundary) Riemannian manifold of dimension two. We denote by Area(M) the area of M, by sys(M) the length of the shortest noncontractible closed geodesic in M. Put

$$\Sigma_M = \inf \frac{\mathrm{Area}(M)}{\mathrm{sys}(M)^2},$$

where infimum is taken over all metrics on M. The explicit value of Σ_M is known when M is a real projective space, torus and Klein bottle.

THEOREM 1 (Pu [9]). *When M is a real projective space PR^2,*

$$\Sigma_{PR^2} = \frac{2}{\pi},$$

where the infimum is attained by the metric of constant curvature.

THEOREM 2 (Loewner [2]). *When M is a torus T^2,*

$$\Sigma_{T^2} = \frac{\sqrt{3}}{2}.$$

The infimum is attained by R^2/Γ with flat metric, where Γ is a lattice generated by $(1, 0)$, $(1/2, \sqrt{3}/2)$.

THEOREM 3 (Bavard [1]). *When M is a Klein bottle K,*

$$\Sigma_K = \frac{2\sqrt{2}}{\pi},$$

where the infimum is attained by a metric of positive constant curvature with singularities.

When M is of large genus, it seems difficult to find Σ_M, but some lower bounds are known.

THEOREM 4 (Gromov [6]). *Let M be as above, with the first Betti number*

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$b_1(M)$. *Then*

(1) $$\Sigma_M \geqq \max\left(\frac{3}{4}, \frac{\sqrt{b_1(M)}}{16\sqrt{2}} + \frac{27}{64}\right),$$

(2) $$40\Sigma_M \cdot 5^{3\sqrt{\log 40\Sigma_M}} \geqq b_1(M).$$

Remark. For higher dimensional isosystolic inequalities, see [5], [6].

In this paper, we will prove an inequality analogous to theorem 4(1) by a different method from that of [6], and show some relations between the isosystolic constants of dimension two and that of dimension one.

Before stating our results, we define some isosystolic constants of dimension one. Let Γ be a finite graph. We denote by length Γ the whole sum of the length of edges, by sys Γ the length of the shortest closed curve in Γ. Let G_g be the set of all finite graphs of Euler number $-g$ such that at each vertex there are more than one edges. Put

$$C_g = \inf_{\Gamma \in G_g} \frac{\text{length}\,\Gamma}{\text{sys}\,\Gamma}.$$

Let G'_g be the set of all finite graphs of Euler number $-g$ such that at each vertex there are even numbers of edges. Put

$$C'_g = \inf_{\Gamma \in G'_g} \frac{\text{length}\,\Gamma}{\text{sys}\,\Gamma}.$$

For an orientable closed Riemannian manifold M of dimension two with genus g, we put $\Sigma_g = \Sigma_M$.

Our results are the following.

THEOREM A. *For* $g \geqq 1$,

(1) $$C_{g-1} \geqq \Sigma_g \geqq \frac{C_{2g-1}^2}{12(2g+1)},$$

(2) $$C'_{g-1} \geqq \Sigma_g \geqq \frac{C'^2_{2g-1}}{48g}.$$

PROPOSITION B. *For* $g \geqq 1$,

$$\Sigma_g \geqq \left(\frac{\sqrt{g}}{36\sqrt{2}} + \frac{343}{648}\right).$$

By theorem 4(2), for any $0 < \vartheta < 1$, Σ_g can be written as $\Sigma_g \geqq c_\vartheta g^\vartheta$, where c_ϑ is a constant depending on ϑ. But theorem A says that if C_g could be written as $C_g \geqq \alpha g + \beta$ with constants α, β then Σ_g would be linear. One sees our inequality of proposition B is worse than that of theorem 4, but our method to prove proposition B, using Morse function and counting the number of the critical points, is interesting itself. The author wishes to express his hearty thanks to Professor S. Tanno for continuous encouragements and valuable suggestions. He also would like to thank the referee for valuable advices.

2. Notations and definitions.

In this paper, we adopt the following notations and definitions.

(1) For a set (a topological space) A, we denote by $^{\#}A$ the number of elements (resp. components) in A.

(2) Assume that M is orientable, and an orientation is given. If a Morse function f is given on M, then grad f determines a direction on $f^{-1}(t)$ for all $t\in R$ excepting critical points of index 0 or 2 and, for $x_1, x_2\in f^{-1}(t)$, determines a directed segment $\overrightarrow{x_1x_2}$ in $f^{-1}(t)$ from x_1 to x_2. For a path τ in M, τ^{-1} means the inversely directed path of τ.

(3) We put

$$M(t)=f^{-1}(t),\quad M[t, t']=f^{-1}[t, t'],\quad M[t, \infty)=f^{-1}[t, \infty),$$

$$M_x[t, t']=\text{the component of } M[t, t'] \text{ containing } x\in M.$$

For an arbitrary subset B of M, put

$$B(t)=M(t)\cap B,\quad B[t, t']=M[t, t']\cap B,\quad B[t, \infty)=M[t, \infty)\cap B,$$

and

$$B_x[t, t']=\text{the component of } B[t, t'] \text{ containing } x\in M.$$

3. Some preliminaries on finite graphs.

Let $G_g^{(3)}$ be the set of all finite graphs with Euler number $-g$ such that at each vertex there are three or two edges. Then put

$$C_g^{(3)}=\inf_{\Gamma\in G_g^{(3)}}\frac{\operatorname{length}\Gamma}{\operatorname{sys}\Gamma}.$$

Similarly let $G_g^{(4)}$ be the set of all finite graphs with Euler number $-g$ such that at each vertex there are four or two edges. Then put

$$C_g^{(4)}=\inf_{\Gamma\in G_g^{(4)}}\frac{\operatorname{length}\Gamma}{\operatorname{sys}\Gamma}.$$

For convienience, we ignore the vertices v of $\Gamma\in G_g$, $g\geqq 1$, such that there meet exactly two edges e_1, e_2 at v and regard $e_1\cup e_2$ as one edge. Therefore we can assume that at each vertex of $\Gamma\in G_g$ there are more than two edges. In particular, if $\Gamma\in G_g^{(3)}$ then Γ has $2g$ vertices and $3g$ edges, and if $\Gamma\in G_g^{(4)}$ then Γ has g vertices and $2g$ edges. Throughout this section, we use $0<\varepsilon, \varepsilon'$ as sufficiently small numbers. We verify the following.

LEMMA 3.1. *For $g, g'\geqq 0$,*

$$C_g^{(3)}=C_g,\qquad C_g^{(4)}=C'_g, \tag{3.1}$$

(3.2) $$C_{g+1} \geqq C_g, \qquad C'_{g+1} \geqq C'_g,$$

(3.3) $$C_{g'} + C_g \geqq C_{g'+g+2}, \qquad C'_{g'} + C'_g \geqq C'_{g'+g+2}.$$

Proof. First we prove lemma for the cases of C_g. To see (3.1), let us take $\Gamma_1 \in G_g$ such that length Γ_1/sys $\Gamma_1 \leqq C_g + \varepsilon$. If there are $d(v)$ edges $e_1, e_2, \cdots, e_{d(v)}$, $d(v) \geqq 4$, at vertex v, then move each edge e_i, $i=4, \cdots, d(v)$, sufficiently near v so that we get $\Gamma_i \in G_g^{(3)}$ with $C_g^{(3)} \leqq$ length Γ_i/sys $\Gamma_i \leqq C_g + \varepsilon'$. To see (3.2), let us take $\Gamma_2 \in G_{g+1}$ such that length Γ_2/sys $\Gamma_2 \leqq C_{g+1} + \varepsilon$, and remove one edge from Γ_2. Then we get $\Gamma'_2 \in G_g$ so that $C_{g+1} + \varepsilon \geqq$ length Γ'_2/sys $\Gamma'_2 \geqq C_g$. To see (3.3), let us take $\Gamma'_3 \in G_{g'}$, $\Gamma_3 \in G_g$ such that length Γ_3/sys $\Gamma_3 \leqq C_g + \varepsilon$, length Γ'_3/sys $\Gamma'_3 \leqq C_{g'} + \varepsilon$ and sys $\Gamma'_3 =$ sys Γ_3. Choose two points p_1, p_2 in Γ_3 (q_1, q_2 in Γ'_3) such that the distance between p_1 and p_2 (resp. q_1 and q_2) is not less than sys $\Gamma_3/2$. And join p_1 and q_1 (p_2 and q_2) by adding short edges. Then we get $\Gamma''_3 \in G_{g+g'+2}$ so that

$$C_{g'} + C_g + 2\varepsilon + 2\varepsilon' \geqq \text{length}\,\Gamma''_3/\text{sys}\,\Gamma''_3 \geqq C_{g'+g+2}.$$

We can prove the cases of C'_g by modifying the above proof as follows: for (3.1), moving one edge is replaced by moving a pair of two edges. For (3.2), removing one edge is replaced by removing one vertex, that is making one vertex with four edges into two vertices at which there meet two edges. For (3.3), joinning two points by a new short edge is replaced by identifying the two points. q. e. d.

M. Gromov ([6], p. 63.) constructed a two dimensional Riemannian manifold of genus g with Area$(M)=$length Γ sys Γ, sys$(M)=$sys Γ, but with singularities, from any $\Gamma \in G_{g-1}^{(3)}$, $g \geqq 1$. Therefore $C_{g-1}^{(3)} \geqq \Sigma_g$, and with (3.1) we get the first inequalities of theorem A(1), (2).

Remark. For more details, we can verify

$$\frac{1}{2} \geqq C_{g+1} - C_g \geqq \frac{C_{g+1}}{3(g+1)}.$$

Thus $(g+2)/2 = g/2 + C_0 \geqq C_g$. In fact, take $\Gamma \in G_g$ such that length Γ/sys $\Gamma + \varepsilon \geqq C_g$, choose two points p_1, p_2 such that the distance between p_1 and p_2 is not less than sys $\Gamma/2$, and join p_1 and p_2 by a new edge of length sys $\Gamma/2$. Then we get $\Gamma' \in G_{g+1}$ with sys $\Gamma' =$ sys Γ and

$$C_{g+1} \leqq \frac{\text{length}\,\Gamma'}{\text{sys}\,\Gamma'} = \frac{\text{length}\,\Gamma + \text{sys}\,\Gamma/2}{\text{sys}\,\Gamma} = C_g + \frac{1}{2}.$$

Next, in the proof of (3.2), Γ_2 has $3(g+1)$ edges. Therefore we can remove an edge whose length is not less than length $\Gamma_2/3(g+1)$, and get Γ'_2 with

$$C_g \leqq \frac{\text{length}\,\Gamma'_2}{\text{sys}\,\Gamma'_2} \leqq \frac{\text{length}\,\Gamma_2 - \text{length}\,\Gamma_2/3(g+1)}{\text{sys}\,\Gamma_2}$$

$$=(C_{g-1}+\varepsilon)\Big(1-\frac{1}{3(g+1)}\Big).$$

LEMMA 3.2. *For* $\Gamma\in G_g^{(4)}$, $g\geqq 0$,

(1) Γ *is a union of simple circuits, and each two simple circuits intersect at two vertices at most,*

(2) *the number of simple circuits mentioned above is not less than* $1/2+\sqrt{g+1/4}$, *and so,*

$$\operatorname{length}\Gamma\geqq\Big(\frac{1}{2}+\sqrt{g+\frac{1}{4}}\Big)\operatorname{sys}\Gamma.$$

Proof. (1) First we can assume $\Gamma=\bigcup_{i=1}^{m} c_i$, where c_i $(i=1,\cdots,m)$ is a closed curve in Γ passing each edge of c_i only one time and $c_i\cap c_j$ $(i\neq j)$ consists of the vertices of Γ. In fact as such c_i we can take an Euler circuit of Γ, that is a closed curve in Γ passing every edges of Γ exactly one time. Next, if this $\bigcup_{i=1}^{m} c_i$ satisfies the conditions of (1) of the lemma then our proof is completed, but if not then we can increase the number of closed curves keeping the above conditions by the following two types of operations until we get a desired union of simple circuits.

(i) If c_i is self-intersected at vertex v_0, then divide c_i into two closed curves at v_0.

(ii) If $c_i\cap c_j$ $(i\neq j)$ contains three vertices v_1, v_2, v_3:

$$c_i=\overline{v_1v_2}\cup\overline{v_2v_3}\cup\overline{v_3v_1},\qquad c_j=\overline{\overline{v_1v_2}}\cup\overline{\overline{v_2v_3}}\cup\overline{\overline{v_3v_1}},$$

then decompose $c_i\cup c_j$ into three closed curves:

$$\overline{v_1v_2}\cup\overline{\overline{v_2v_1}},\qquad \overline{v_2v_3}\cup\overline{\overline{v_3v_2}},\qquad \overline{v_3v_1}\cup\overline{\overline{v_1v_3}}.$$

Finally we get $\Gamma=\bigcup_{i=1}^{m+u+v}\bar{c}_i$ satisfying the conditions of (1) of lemma, after u times of operation (i) and v times of operation (ii).

(2) Since Γ has g vertices and each pair of two simple circuits shares at most two vertices, the number N of simple circuits satisfies

$$\frac{N(N-1)}{2}2\geqq g. \qquad \text{q.e.d.}$$

Next, we consider finite graphs of the following type. Suppose there are m circles $c_1, c_2, \cdots, c_m$ $(c_i=S^1)$ and g intervals $I_1, I_2, \cdots, I_g$ $(I_j=[0, 1])$, and $2g$ points v_j^a $(j=1,\cdots,g,\ a=0, 1)$ on $\bigcup_{i=1}^{m} c_i$. Attaching the end points $I_j(0)$, $I_j(1)$ to v_j^0, v_j^1, respectively, we get a finite graph $\Gamma\in G_g^{(3)}$. Then the next lemma holds.

LEMMA 3.3.

$$\sum_{i=1}^{m}\operatorname{length} c_i+\sum_{j=1}^{g}\operatorname{length} I_j\geqq C_g\operatorname{sys}\Gamma. \tag{3.4}$$

(3.5) $$\sum_{i=1}^{m} \text{length } c_i + 2\sum_{j=1}^{g} \text{length } I_j \geqq C'_g \operatorname{sys} \Gamma .$$

We can take integers u, v, w such that

(3.6) $$\sum_{i=1}^{m} \text{length } c_i + 2\sum_{j=1}^{w} \text{length } I_{i_j} \geqq (m+u+v) \operatorname{sys} \Gamma ,$$

$$m+u+v \geqq \frac{1}{2} + \sqrt{g+\frac{1}{4}} ,$$

$$u+3v \geqq w .$$

Proof. (1) Since the left term of (3.4) is length Γ, this is nothing but the definition of C_g.

(2) To see (3.5), let us take a copy $\bar{I}_j$ of I_j $(j=1, 2, \cdots, g)$, identify $\bar{I}_j(1/2)$, $\bar{I}_j(0)$, $\bar{I}_j(1)$, $I_j(0)$, $I_j(1)$ with $I_j(1/2)$, $\bar{v}_j(0)$, $\bar{v}_j(1)$, $v_j(0)$, $v_j(1)$, respectively, and remove $\overline{\bar{v}_j(b) v_j(b)}$, where $\bar{v}_j(b)$, $v_j(b)$ are points near v_j^b $(b=0, 1)$. Then we get a finite graph $\Gamma' \in G_g^{(4)}$ with vertices $I_j(1/2)$, $j=1, \cdots, g$. Since $\operatorname{sys} \Gamma' \geqq \operatorname{sys} \Gamma - \varepsilon$, we get

$$\text{length } \Gamma' = \sum_{i=1}^{m} \text{length } c_i - \varepsilon + 2\sum_{j=1}^{g} \text{length } I_j \geqq C'_g \operatorname{sys} \Gamma'$$
$$\geqq C'_g (\operatorname{sys} \Gamma - \varepsilon) .$$

(3) By identifying v_j^0 with v_j^1 in $\bigcup_{i=1}^{m} c_i$, we get a finite graph $\Gamma'' \in G_g^{(4)}$ with new vertices v_j $(j=1, \cdots, g)$ and through the operations of type (i), (ii) in the proof of lemma 3.2(1), we get $\Gamma'' = \bigcup_{i=1}^{m+u+v} \bar{c}_i$, $m+u+v \geqq \frac{1}{2}\sqrt{g+\frac{1}{4}}$, after u times of the operation of type (i) and v times of the operation of type (ii), satisfying the conditions of lemma 3.2(1). Each $\bar{c}_i$ can be written as

$$\bar{c}_i = \overline{v_{i_1} v_{i_2}} \cup \overline{v_{i_2} v_{i_3}} \cup \cdots \cup \overline{v_{i_{s(i)}} v_{i_1}} ,$$

where $\overline{v_{i_j} v_{i_{j+1}}}$ is connected in some c_l and $\overline{v_{i_j} v_{i_{j+1}}} \cup \overline{v_{i_{j+1}} v_{i_{j+2}}}$ is not connected in any c_l. Then

$$\bar{c}_i = \overline{v_{i_1} v_{i_2}} \cup I_{i_2} \cup \overline{v_{i_2} v_{i_3}} \cup I_{i_3} \cup \cdots \cup \overline{v_{i_{s(i)}} v_{i_1}} \cup I_{i_1}$$

is a simple closed curve in Γ. Now we consider the number of jump points v_{i_j} $(i=1, \cdots, m+u+v,\ j=1, \cdots, s(i))$ appearing in $\bar{c}_i$. Since each jump point v_{i_j} belongs to exactly two closed curves $\bar{c}_i$ and $\bar{c}_{i'}$, the total number of jump points is $w = \sum_{i=1}^{m+u+v} s(i)/2$. Since each operation of type (i) creates at most one jump point and each operation of type (ii) creates at most three jump points, we get $(u+3v) \geqq w$. Summing up the length of $\bar{c}_i$, we obtain

$$\sum_{i=1}^{m+u+v} \text{length}\,\bar{c}_i = \sum_{i=1}^{m+u+v} \sum_{j=1}^{s(i)} \text{length}\,\overline{v_{i_j} v_{i_{j+1}}} + \sum_{i=1}^{m+u+v} \sum_{i=1}^{s(i)} \text{length}\, I_{i_j}$$

$$= \sum_{i=1}^{m} \text{length}\, c_i + 2 \sum_{j=1}^{w} \text{length}\, I_{i_j} \geqq (m+u+v)\, \text{sys}\, \Gamma . \qquad \text{q.e.d.}$$

4. An approximation of the distance function by a Morse function.

Let M be a closed orientable Riemannian manifold of dimension two with genus g. For a point p of M, we denote by $d=d(p, *)$ the distance function from p, by C_p the cut locus of p, and by $B(p, r)$ the metric ball of radius r centered at p. For $x \in M$ we denote by $\pi(x)$ the cut point of p along minimal geodesic from p to x. We define an ε-neighbarhood of C_p as

$$C_p^{\varepsilon} = \{x \in M \mid d(x, \pi(x)) < \varepsilon\}.$$

We approximate d by a Morse function $f=f_{\varepsilon}$ as follows,

$$|d-f| \leqq \varepsilon \qquad \text{on } M, \tag{4.1}$$

$$|\text{grad}\, d - \text{grad}\, f| \leqq \varepsilon \qquad \text{on} \quad M \backslash \{C_p^{\varepsilon} \cup B(p, \varepsilon)\}. \tag{4.2}$$

Furthermore we can assume that

(*1) f has a critical point of index 0 only at $p=p_0$ with $t_0=f(p_0)=0$,

(*2) for each critical value t_i ($t_i<t_{i+1}$, $i=1, 2, \cdots, n$) of f, there is exactly one corresponding critical point p_i, $t_i=f(p_i)$.

We denote by Q_i ($i=0, 1, 2$) the set of all critical points of index i. Then by the definition of f, we get

$$^{\#}Q_0 = 1, \tag{4.3}$$

$$1 - {}^{\#}Q_1 + {}^{\#}Q_2 = 2-2g. \tag{4.4}$$

Let us take a small $\delta>0$ such that $t_{i+1}-t_i>2\delta$ for all i. We assign $I(p_i)=\pm 1$ to each p_i, if

$$^{\#}\{M_{p_i}[t_i, t_i+\delta] \backslash M(t_i)\} = {}^{\#}\{M_{p_i}[t_i-\delta, t_i] \backslash M(t_i)\} \pm 1 .$$

Obviously, $I(p_0)=1$ and $I(p_i)=-1$ for $p_i \in Q_2$. We denote by $Q_1^{\pm}$ the set of all critical points p_i of index 1 with $I(p_i)=\pm 1$. Since M is orientable, we can verify

$$Q_1^+ \cap Q_1^- = \phi, \qquad Q_1 = Q_1^+ \cup Q_1^- . \tag{4.5}$$

In fact, for $p_i \in Q_1$, $M_{p_i}(t_i)$ consists of two two-sided circles c_i^1, c_i^2, and $M_{p_i}[t_i-\delta, t_i+\delta] \backslash M_{p_i}(t_i)$ consists of three cylinders. Therefore

$$\sum_{i=1}^{n} I(p_i) = 1 + {}^{\#}Q_1^+ - {}^{\#}Q_1^- - {}^{\#}Q_2 = 0 . \tag{4.6}$$

Combining with (4.3), (4.4), (4.5) and (4.6), we get

(4.7) $$^{\#}Q_0+{}^{\#}Q_1^+={}^{\#}Q_1^-+{}^{\#}Q_2,$$

(4.8) $$1-{}^{\#}Q_1^-={}^{\#}Q_2-{}^{\#}Q_1^+=1-g,$$

(4.9) $$^{\#}Q_1^-=g.$$

For the later arguments, we need to remove some disks from M. For $p_i\in Q_1$, $M_{p_i}[t_i, \infty)\backslash M_{p_i}(t_i)$ consists of one or two components. If such a component D contains only the critical points of Q_1^+ or Q_2 then D is a disk and we remove it from M. Let $\{D_j\}_{j=1}^m$ be the set of all such D's. Put

$$\tilde{M}=M\backslash\bigcup_{j=1}^m D_j,$$

$$\tilde{Q}_1^{\pm}=\tilde{M}\cap Q_1^{\pm},\quad \partial Q_1^{\pm}=\partial\tilde{M}\cap Q_1^{\pm},\quad \operatorname{int} Q_1^{\pm}=\operatorname{int}\tilde{M}\cap Q_1^{\pm},$$

where $\partial\tilde{M}$ ($\operatorname{int}\tilde{M}$) is the boundary (resp. interiour) of $\tilde{M}$. In this case, we also get Morse equality,

$$^{\#}\partial Q_1^+ + {}^{\#}\partial Q_1^- = m,$$

$$1-({}^{\#}\tilde{Q}_1^+ + {}^{\#}\tilde{Q}_1^-)+m=2-2g.$$

Calculations similar to (4.7), (4.8) and (4.9) lead us to

$$1+{}^{\#}\tilde{Q}_1^+={}^{\#}\tilde{Q}_1^-+m,$$

$$m-{}^{\#}\tilde{Q}_1^+=1-g$$

and therefore

(4.10) $$^{\#}\operatorname{int} Q_1^+={}^{\#}\partial Q_1^-+(g-1)\geqq g.$$

For brevity let us put $Q^+=\operatorname{int} Q_1^+$, $Q^-=\tilde{Q}_1^-$, $\partial Q^-=\partial Q_1^-$. Then above argument shows the following.

LEMMA 4.1.

(4.11) $$^{\#}Q^-=g,$$

(4.12) $$^{\#}Q^+={}^{\#}\partial Q^-+g-1\geqq g.$$

5. The behavior of the level curves near the cut locus.

Throughout this section we assume that M is a real analytic closed Riemannian manifold of dimension two whose genus is not less than one. We fix a point $p\in M$ and approximate the distance function $d(p, *)$ from p by a Morse function $f=f_\varepsilon$ satisfying the conditions (4.1), (4.2).

For $t\in[0, \max f]$ let A_q be a compoment of $f^{-1}(t)\cap C_p^\varepsilon$ which contains $q\in f^{-1}(t)\cap C_p^\varepsilon$.

This section is devoted to prove the following proposition.

PROPOSITION 5.1. *Assume that M is real analytic and its genus is not less than one. Then for a sufficiently small $\varepsilon>0$, we can take $\alpha(\varepsilon)>0$, which goes to 0 as ε goes to 0, such that*

$$A_q \subset B(q, \alpha(\varepsilon)),$$

where $\alpha(\varepsilon)$ depends only on ε, M and does not depend on the choice of f_ε.

S. Myers [8] showed that if M is real analytic, then C_p is a finite graph and $h=d|_{C_p}$ is analytic on each edge of C_p with respect to arc length parameter s. Futhermore, M. Berger [3] and J. Hebda [7] showed that, if the genus of M is not less than one then h is nonconstant and $\partial B(p, t)$ is an Euler graph with vertices $C_p \cap \partial B(p, t)=\{r_i(t)\}_{i=1}^{m_t}$ and real analytic edges. The number m of the segments of $C_p \backslash \{a_i\}_{i=1}^{m'}$ is not less than m_t for all t, where $\{a_i\}_{i=1}^{m'}$ is the set of all vertices of C_p and all points at which $dh/ds(a_i)=0$. In fact, the analyticy of h on each edge of C_p ensures that on each segment of $C_p \backslash \{a_i\}_{i=1}^{m'}$ h is strictly monotonous and there exists at most one $r_i(t)$.

We denote by ρ be the distance on C_p and at $q \in C_p$ put

$$D(q, \delta)=\{q' \in C_p \mid \rho(q, q') \leqq \delta\}.$$

CLAIM 5.2. *For all $\varepsilon'>0$, we can take δ_1, $\varepsilon_1=\varepsilon_1(\delta_1)<\varepsilon'$ such that, if $|h(s)-t|<\varepsilon_1$ then*

$$s \in \bigcup_{i=1}^{m_t} D(r_i(t), \delta_1) \cup \bigcup_{i=1}^{m'} D(a_i, \delta_1).$$

Proof. Since h is strictly monotonous on each segment of $C_p \backslash \{a_i\}_{i=1}^{m'}$, the measure of

$$h^{-1}[\min h+(k-1)\tau, \min h+k\tau] \qquad (k=1, 2, \cdots, N),$$

goes to 0 when N goes to ∞, where $\tau=(\max h-\min h)/N$. Then by taking a large N, we can choose small ε_1, δ_1 such that
measure of $h^{-1}[\min h+(k-1)\tau, \min h+k\tau]<\delta_1/10$, $\varepsilon_1<\tau/10$
and for all $t \in [\min h+(k-1)\tau, \min h+k\tau]$,

$$s \in h^{-1}[t-\varepsilon_1, t+\varepsilon_1] \subset h^{-1}[\min h+(k-2)\tau, \min h+(k+1)\tau]$$
$$\subset \bigcup_{i=1}^{m_t} D(r_i(t), \delta_1) \cup \bigcup_{i=1}^{m'} D(a_i, \delta_1).$$

q. e. c.

Proof of Proposition 5.1. We can easily see

$$A_q \subset C_p^\varepsilon \cap \{B(p, t+\varepsilon) \backslash B(p, t-\varepsilon)\}$$
$$\subset \varepsilon\text{-neigborhood of } C_p^\varepsilon \cap \{B(p, t+2\varepsilon) \backslash B(p, t-2\varepsilon)\}.$$

Assume $\varepsilon<\varepsilon_1/2$, where ε_1, δ_1, ε' are those given in claim 5.2. Then we have

$$C_p^\varepsilon \cap \{B(p, t+2\varepsilon) \setminus B(p, t-2\varepsilon)\} \subset \left\{ \bigcup_{i=1}^{m_t} D(r_i(t), \delta_1) \cup \bigcup_{i=1}^{m'} D(a_i, \delta_1) \right\},$$

and so, at some $q'(=r_i(t)$ or $a_j)$

$$A_q \subset B(q', (2\delta_1+\varepsilon)(m_t+m')) \subset B(q, \alpha(\varepsilon)),$$

where $\alpha(\varepsilon)=\delta_1+\varepsilon+(2\delta_1+\varepsilon)(m+m')$. When ε goes to 0, we can choose ε' such that $\alpha(\varepsilon)$ goes to 0. q.e.d.

6. Some finite graphs in M.

Let $t_i=f(p_i)$, $i=1, 2, \cdots, n$, $0<t_i<t_{i+1}$, be the critical values of f at $p_i \in Q^+ \cup Q^-$, as defined in section 4, where $n={}^\#Q^+ + {}^\#Q^- \geqq 2g$.

LEMMA 6.1. *Let us denote by* sys(M, p) *the length of the shortest noncontractible closed curve in* M *passing through* $p \in M$. *Then*

$$\left| t_1 - \frac{\text{sys}(M, p)}{2} \right| < 2\varepsilon.$$

Proof. Since $B(p, \text{sys}(M, p)/2-\varepsilon) \supset M[0, \text{sys}(M, p)/2-2\varepsilon]$ is contractible, sys$(M, p)/2-2\varepsilon<t_1$. Since $B(p, \text{sys}(M, p)/2+\varepsilon) \subset M[0, \text{sys}(M, p)/2+2\varepsilon]$ is noncontractible, sys$(M, p)/2+2\varepsilon>t_1$. q.e.d.

By (4.1) and (4.2), we see that $p_i \in C_p^\varepsilon$. Since M is assumed to be orientable, $M_{p_i}(t_i)$ consists of two circles c_i^1, c_i^2. Let us take a point p_i^1 (p_i^2) on c_i^1 (resp. c_i^2) such that p_i^1 (resp. $p_i^2) \in \partial C_p^\varepsilon$ and $\tau_i^1=\overrightarrow{p_i^1 p_i}$ (resp. $\tau_i^2=\overrightarrow{p_i^2 p_i}$), except p_i^1 (resp. p_i^2), is contained in C_p^ε. Let γ_i^j $(j=1, 2)$ be the distance minimizing geodesics from p_i^j to p. Put

$$\gamma_i = \gamma_i^{1^{-1}} \cup \tau_i^1 \cup \tau_i^{2^{-1}} \cup \gamma_i^2.$$

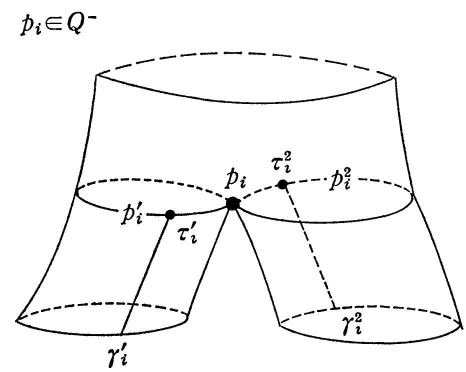

Fig. 1.

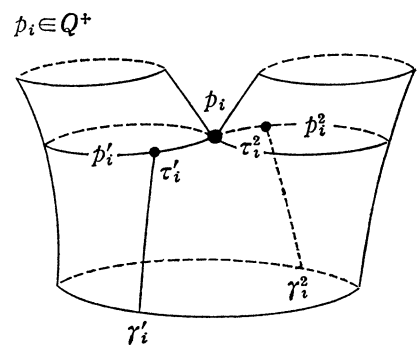

Fig. 2.

Put $\Lambda=\bigcup_{i=1}^{n}\gamma_i$, $\Lambda(t)=\Lambda\cap f^{-1}(t)=\bigcup_{i=1}^{n}\gamma_i(t)$, $\Lambda[t, t']=\Lambda\cap f^{-1}[t, t']=\bigcup_{i=1}^{n}\gamma_i[t, t']$ for $0<t<t'\leqq\infty$. In this section, we consider the next finite graph in M:

$$\Gamma(t)=\tilde{M}(t)\cup\Lambda[t, \infty).$$

LEMMA 6.2. *If t is a regular value of f, then any closed curve $c\subset\Gamma(t)$ is not contractible in M, except $M(t)$ for $t<t_1$.*

Proof. Assume c is contractible. Then c divides M into two parts M_1, M_2 such that $p\in M_1$ and M_2 is a disk.

case 1; $c\subset M(t)$, $t>t_1$. Then $M_2=M[t, \infty)$ is a disk. But this cannot occur from the definition of $\tilde{M}$.

case 2; $\max f(c)=t_i=f(p_i)$, $p_i\in Q^+$. Then one of c_i^1 and c_i^2 must belong to M_2 and be contractible. But this can not occur.

case 3; $\max f(c)=t_i=f(p_i)$, $p_i\in Q^-$. At p_i each side of c can be joined by c_i^1. This is a contradiction. q. e. d.

7. Proof of theorem A (second inequalities) and proposition B.

We prove our results only for the real analytic cases, because we can approximate a smooth metric by a real analytic one and $\mathrm{sys}(M)$, $\mathrm{area}(M)$ change continuously. In section 6, we constructed some finite graphs on M. With (3.4), (3.5) and (3.6), we can estimate the length of $\partial B(p, t)$ and, with the co-area formula, $\mathrm{Area}(M)$.

Let $\hat{\tau}_i^j$ $(i=1, 2, \cdots, n,\ j=1, 2)$ be a distance minimizing geodesic segment from p_i^j to p_i. Since $\hat{\tau}_i^j\subset B(p_i, \alpha(\varepsilon))$, by Proposition 5.1, $\gamma_i[t, \infty)$ is homotopic to $\hat{\gamma}_i[t, \infty)$, which is obtained from γ_i by replacing τ_i^j to $\hat{\tau}_i^j$ $(j=1, 2)$, keeping end points $\gamma_i^j(t)$ $(j=1, 2)$ fixed, and

$$\text{length}\,\hat{\gamma}_i[t, \infty)\leqq 2(t_i-t)+4(\varepsilon+\alpha(\varepsilon)). \tag{7.1}$$

By the co-area formula, we get

$$\mathrm{Area}(M\setminus C_p^{\varepsilon})=\int_0^{\infty}dt\int_{f^{-1}(t)\setminus C_p^{\varepsilon}}\frac{d\mu(t)}{|\nabla f|},$$

where $d\mu(t)$ is the meaure of $f^{-1}(t)$. Now let ε tend to 0. Then we obtain

$$\mathrm{Area}(M)=\int_0^{\infty}\text{length}\,M(t)dt\geqq\int_0^{\infty}\text{length}\,\tilde{M}(t)dt, \tag{7.2}$$

by (7.1),

$$\text{length}\,\hat{\gamma}_i[t, \infty)=2(t_i-t), \tag{7.3}$$

and by lemma 6.1,

$$t_1=\mathrm{sys}(M, p)/2\geqq\mathrm{sys}(M)/2. \tag{7.4}$$

Proof of theorem A(1) (*second inequality*). Put

$$I_k=[t_1+(k-1)\tau,\ t_1+k\tau],\qquad k=0,1,2,\cdots,K,$$
$$t_1\in I_1,\quad t_n\in I_K,\quad \tau=C_{2g-1}\operatorname{sys}(M)/6(2g+1),$$

where $K\geqq(t_n-t_1)/\tau>K-1$. We can take g_k critical points $p_{k,i}$ $(i=1,2,\cdots,g_k)$ in $M[t_1+(k-1)\tau,\ t_1+k\tau]$ such that

$$p_{1,1}=p_1,\qquad \sum_{k=1}^{K}g_k=2g.$$

If $k\geqq 2$ then by lemma 6.2 any closed curve in

$$\Gamma_k(t)=\tilde{M}(t)\cup\Big(\bigcup_{i=1}^{g_k}\gamma_{k,i}[t,\infty)\Big),\qquad\text{for}\quad t\in I_{k-1},$$

is not contractible in M, and so,

$$\text{(7.5)}\qquad \operatorname{length}\tilde{M}(t)+\sum_{i=1}^{g_k}2(t_{k,i}-t)\geqq C_{g_k}\operatorname{sys}\Gamma_k(t)$$
$$\geqq C_{g_k}\operatorname{sys}(M).$$

Integrating (7.5) on each I_{k-1} $(k=2,\cdots,K)$, we get

$$\int_{I_{k-1}}\operatorname{length}\tilde{M}(t)dt+2\sum_{i=1}^{g_k}\int_{I_{k-1}}(t_{k,i}-t)dt\geqq C_{g_k}\operatorname{sys}(M)\tau.$$

Since

$$\int_{I_{k-1}}(t_{k,i}-t)dt\leqq\frac{3}{2}\tau^2,$$

we get

$$\text{(7.6)}\qquad \operatorname{Area}(M[t_1,\infty))\geqq\sum_{k=2}^{K}\int_{I_{k-1}}\operatorname{length}\tilde{M}(t)dt$$
$$\geqq\sum_{k=2}^{K}(C_{g_k}\operatorname{sys}(M)\tau-3g_k\tau^2).$$

Next if $k=1$ then $\tilde{M}(t)=M(t)$ $(t\in I_0)$ is contractible in M, and so, we can not get (7.5) for $t\in I_0$. But, for $t<t_1$, $\gamma_1(t)$ devides $M(t)$ into two parts $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_1(t)\cup\gamma_1[t,\infty)$, $\sigma_2(t)\cup\gamma_1[t,\infty)$ are not contractible in M. Take a copy $\hat{\gamma}_1[t,\infty)$ of $\gamma_1[t,\infty)$ and regard $\sigma_1(t)\cup\gamma_1[t,\infty)$ and $\sigma_2(t)\cup\hat{\gamma}_1[t,\infty)$ as disjoint two closed curves. Then any closed curve in

$$\tilde{\Gamma}_1(t)=\{\sigma_1(t)\cup\gamma_1[t,\infty)\}\cup\{\sigma_2(t)\cup\hat{\gamma}_1[t,\infty)\}\cup\Big\{\bigcup_{i=2}^{g_1}\gamma_{1,i}[t,\infty)\Big\},$$

for $t\in I_0$, is not contractible in M. We apply (3.4) and (7.3) to $\tilde{\Gamma}_1(t)$. Then we get

$$\text{(7.5)}'\quad \operatorname{length}M(t)+4(t_1-t)+\sum_{i=2}^{g_1}2(t_{1,i}-t)\geqq C_{g_1-1}\operatorname{sys}\tilde{\Gamma}_1(t)\geqq C_{g_1-1}\operatorname{sys}(M),$$

and integrating on I_0,

(7.6)′ $$\mathrm{Area}(M[t_1-\tau, t_1])\geqq C_{g_1-1}\,\mathrm{sys}(M)\tau-6\tau^2-3(g_1-1)\tau^2.$$

With (7.6) and (7.6)′, we get

$$\begin{aligned}\mathrm{Area}(M)&\geqq\Big(\sum_{k=2}^{K} C_{g_k}+C_{g_1-1}\Big)\mathrm{sys}(M)\tau-3\Big(\sum_{k=2}^{K} g_k+2+g_1-1\Big)\tau^2\\ &\geqq C_{2g-1}\,\mathrm{sys}(M)\tau-3(2g+1)\tau^2\geqq\frac{C_{2g-1}^2}{12(2g+1)}\mathrm{sys}(M)^2. \qquad \text{q. e. d.}\end{aligned}$$

Proof of theorem A(2) (*second inequality*). In the above proof, we only need to replace $\tau=C_{2g-1}\mathrm{sys}(M)/6(2g+1)$ to $\tau=C'_{2g-1}\mathrm{sys}(M)/24g$ and apply (3.5) instead of (3.4). q. e. d.

Proof of proposition B. In this case we put

$$I_k=[t_1+(k-1)\tau,\ t_1+k\tau],\qquad k=0,1,\cdots,K,$$
$$\tau=\mathrm{sys}(M)/36.$$

By applying (3.6), (7.1) and lemma 6.2 to

$$\Gamma_k(t)=\hat{M}(t)\cup\bigcup_{j=1}^{g_k}\hat{\gamma}_{k,j}[t,\infty),\qquad \text{for}\quad t\in I_{k-1},$$

we get

(7.7) $$\mathrm{length}\,\tilde{M}(t)+\sum_{j=1}^{w_t}4(t_{k,j}-t)\geqq(s_t+u_t+v_t)\,\mathrm{sys}(M)$$

$$s_t+u_t+v_t\geqq\frac{1}{2}+\sqrt{g_k+\frac{1}{4}},$$

$$u_t+3v_t\geqq w_t,$$

where $s_t\geqq 1$ is the number of the components of $\tilde{M}(t)$. Integrating (7.7) on I_{k-1}, we get

(7.8) $$\begin{aligned}\int_{I_{k-1}}\mathrm{length}\,\tilde{M}(t)dt&\geqq(s_t+u_t+v_t)\,\mathrm{sys}(M)\tau-6w_t\tau^2\\ &\geqq\Big\{s_t+u_t+v_t-\frac{w_t}{6}\Big\}\frac{\mathrm{sys}(M)^2}{36}\\ &\geqq\Big\{\frac{3s_t+2u_t+(u_t+3v_t-w_t)}{6}+\frac{s_t+u_t+v_t}{2}\Big\}\frac{\mathrm{sys}(M)^2}{36}\\ &\geqq\Big\{\frac{1}{2}+\frac{1}{2}\Big(\frac{1}{2}+\sqrt{g_k+\frac{1}{4}}\Big)\Big\}\frac{\mathrm{sys}(M)^2}{36}\\ &\geqq\Big(\frac{1}{48}+\frac{\sqrt{g_k}}{72}\Big)\mathrm{sys}(M)^2.\end{aligned}$$

Since $\sigma_i(t)\cup\hat{\gamma}_1[t, \infty)$ ($i=1, 2$), for $t<t_1$, is noncontractible in M, we obtain

$$\text{length}\,\sigma_i+\text{length}\,\hat{\gamma}_1[t, t_1]\geqq \text{sys}(M), \qquad i=1, 2,$$

and integrating this on $[0, t_1-\tau]$,

$$\int_0^{t_1-\tau}\text{length}\,M(t)dt\geqq\int_{t_1-\text{sys}(M)/2}^{t_1-\tau}(2\,\text{sys}(M)-4(t_1-t))dt\geqq\frac{289}{648}\text{sys}(M)^2, \tag{7.9}$$

where we have used (7.4). With (7.8) and (7.9), we obtain

$$\begin{aligned}\text{Area}(M)&\geqq\sum_{k=0}^{K}\int_{I_k}\text{length}\,\tilde{M}(t)dt+\int_0^{t_1-\tau}\text{length}\,M(t)dt\\&\geqq\frac{\text{sys}(M)^2K}{48}+\frac{\text{sys}(M)^2}{72}\sum_{k=1}^{K}\sqrt{g_k}+\frac{289}{648}\text{sys}(M)^2\\&\geqq\left(\frac{\sqrt{g}}{36\sqrt{2}}+\frac{343}{648}\right)\text{sys}(M)^2 \qquad \text{q.e.c.}\end{aligned}$$

Remark. In the proof of proposition B, we need not an argument for the contractibility of $M(t)$ for $t<t_1$ as in the proof of theorem A. Recalling the proof of (3.6) we can verify $w_t\geqq 1$ and so (7.7) still holds for $t<t_1$.

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