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THE FIXED POINT SET AND THE ITERATIONAL LIMITS OF A HOLOMORPHIC SELF-MAP

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Introduction.

Let D be a domain in the complex Euclidean space $\boldsymbol{C}^n$, and $\mathcal{H}(D)$ denote the space of all holomorphic self-maps of D. $\mathrm{Hol}(D, \boldsymbol{C}^n)$, the space of all holomorphic maps $f: D \to \boldsymbol{C}^n$, is endued with the compact open topology and $\mathcal{H}(D)$ is given the induced topology. For a map $f \in \mathcal{H}(D)$, we denote the m-th iterate of f by f^m, that is $f^1 = f$, $f^m = f \circ f^{m-1}$, $m = 2, 3, \cdots$.

Let $\Gamma(f)$ be the closure of the sequence $\{f^m\}$ in $\mathrm{Hol}(D, \boldsymbol{C}^n)$ with respect to the compact open topology. Then $\Gamma(f)$ is an abelian topological semigroup and contained in $\mathrm{Hol}(D, \bar{D})$.

Let $\Gamma'(f)$ denote the set of all subsequential limits of $\{f^m\}$. If f is a holomorphic retraction of D, $\Gamma(f) = \Gamma'(f) = \{f\}$.

We denote the fixed point set of f in D by $\mathrm{Fix}(f)$. The purpose of this paper is to investigate the relation between $\Gamma(f)$ and $\mathrm{Fix}(f)$. When D is the unit disc U in $\boldsymbol{C}$, the following results are classical;

THEOREM I (A. Denjoy [2], J. Wolff [19]). *Let f be a map in $\mathcal{H}(U)$.*

(1) *If* $\mathrm{Fix}(f)$ *is non-empty, then either $\{f^m\}$ converges to a constant map p^* ($p \in U$) or f is an analytic automorphism of U.*

(2) *If* $\mathrm{Fix}(f)$ *is empty, then there exists a boundary point q such that $\{f^m\}$ converges to the constant map q^*.*

(For a point p in $\boldsymbol{C}^n$, p^* denotes the constant map of which value is p).

This theorem implies that for a holomorphic self-map f of the unit disc U, $\Gamma(f)$ is contained in $\mathcal{H}(U)$ if and only if $\mathrm{Fix}(f)$ is nonempty. Also this theorem has been extended to the case of the unit hyperball B^n in $\boldsymbol{C}^n$ ($n \geqq 2$).

THEOREM II (Y. Kubota [6], B. MacCluer [9]). *Let f be a map in $\mathcal{H}(B^n)$ with* $\mathrm{Fix}(f) \neq \phi$, *then $\Gamma(f)$ contains a holomorphic retraction of B^n. (ϕ denotes the empty set.)*

THEOREM III (B. MacCluer [9]). *For $f \in \mathcal{H}(B^n)$ with* $\mathrm{Fix}(f) = \phi$, *there exists a boundary point q such that $\{f^m\}$ uniformly converges to the constant map q^*.*

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Firstly we shall extend Theorem II to the case of bounded convex domains in C^n.

THEOREM A (see Theorem 3.1). *Let D be a bounded convex domain in C^n and a map f belong to $\mathcal{H}(D)$. If* $\mathrm{Fix}(f)\neq\phi$, *then $\Gamma(f)$ contains a uniqueholo morphic retraction $R: D\to A$, where A is a connected complex submanifold in D, and any element g of $\Gamma'(f)$ can be expressed as $g=T\circ R$ where T is a biholomorphic automorphism of A. Furthermore the dimension of A is d_1+d ($d_1=\dim\mathrm{Fix}(f)$, $d\geqq 0$, see section 3).*

Secondly we consider the inverse problem of Theorem A. That is the following problem; Let D be a bounded convex domain and $f\in\mathcal{H}(D)$. Does f have a fixed point in D provided that $\Gamma(f)$ contains a holomorphic retraction? To solve this problem, we show a theorem with respect to the common fixed point;

THEOREM B (see Theorem 4.1). *Assume that D is a bounded convex domain in C^n. Let f and g be commutative elements of $\mathcal{H}(D)$ with fixed points. If* $\dim\mathrm{Fix}(f)\leqq 1$ *or* $\dim\mathrm{Fix}(g)\leqq 1$, *then f and g have a common fixed point.*

Finally, we have a partial answer to the above problem. Put $\mathcal{H}^*(D)=\mathcal{H}(D)-\{$all holomorphic retractions of $D\}$.

THEOREM C (see Theorem 5.1). *Let D be as above and $f\in\mathcal{H}^*(D)$. Assume that $\Gamma(f)$ contains a holomorphic retraction $R: D\to A$. If* $\dim A\leqq 1$, *then f has a fixed point in D.*

1. Complex geodesics.

Let D be a hyperbolic domain in C^n and TD, TD^* denote the holomorphic tangent bundle of D, $TD-$(zero section) respectively. Then TD can be identified with $D\times C^n$.

For $(p; v)\in TD$, $C_D(p; v)$ and $K_D(p; v)$ denote the Carathéodory metric (C-metric, for short) of D and the Kobayashi metric (K-metric) of D respectively. Their integrated metrics c_D and k_D are called the Carathéodory distance (C-distance) and the Kobayashi distance (K-distance) respectively.

Let U be the unit disc in C with the Poincaré metric $C_U(t; v)=K_U(t; v)=|v|/(1-|t|^2)$. For two points z_1 and z_2 in D, $c_D^*(z_1, z_2)=\sup\{c_U(f(z_1), f(z_2)); f\in\mathrm{Hol}(D, U)\}$ is called the classical Carathéodory distance of D. In general, $c_D^*\leqq c_D\leqq k_D$.

E. Vesentini [13, 14] introduced the complex geodesics;

DEFINITION 1.1. *Let F be a map in* $\mathrm{Hol}(U, D)$.

(1) *If, for any points t_1, t_2 of U, $c_D^*(F(t_1), F(t_2))=c_U(t_1, t_2)$, then F (or its image $F(U)$) is called a complex geodesic for c_D^*.*

(2) *If, for* $(p;v)\in TD^*$, *F satiesfies that* $F(0)=p$, $F'(0)=rv$ *for some* $r>0$, *and* $C_D(F(t);F'(t))=C_U(t;1)$ *for all* $t\in U$, *then F is called a complex geodesic for* $C_D(p;v)$.

Remarks. If C-distance of D is inner, then the complex geodesic for c_D^* and the one for $C_D(p;v)$ coincide. There exists a (non-convex) pseudoconvex domain where C-distance is not inner (see Vigué [15]). On the bounded convex domains, two definitions are equivalent.

LEMMA 1.2. (1) *A map F in* $\mathrm{Hol}(U, D)$ *is a complex geodesic for* c_D^* *if and only if there exist two points* t_1 *and* t_2 *of* U *such that* $c_D^*(F(t_1), F(t_2))=c_U(t_1, t_2)$.

(2) *A map F in* $\mathrm{Hol}(U, D)$ *is a complex geodesic for* $C_D(p;v)$ *if and only if* $F(0)=p$, $F'(0)=rv$ *for some* $r>0$ *and there is a point* t_1 *of* U *such that* $C_D(F(t_1);F'(t_1))=C_U(t_1;1)$.

This lemma is due to E. Vesentini [14]. For a bounded convex domain D, L. Lempert [7, 8] has proved that the C-metric is equal to the K-metric, hence $c_D^*=c_D=k_D$, and there exist the complex geodesics in D (see Vigué [16, 17] and Suzuki [12]).

THEOREM 1.3. *Let D be a bounded convex domain in* $\boldsymbol{C}^n$.

(1) *Given two distinct points x, y in D, there exists a complex geodesic through x and y.*

(2) *For a given* $(p;v)$ *in* TD^*, *there is a complex geodesic F for* $C_D(p;v)$.

Here we remark the uniqueness of the complex geodesic on a strictly convex domain.

LEMMA 1.4. *If D is a bounded strictly convex domain, then the complex geodesics in* (1), (2) *of the Theorem* 1.3 *are unique up to the analytic automorphisms of U.*

Proof. If D is a strongly convex domain with C^6-boundary, then the com plex geodesic coincides with the stationary map in the sence of Lempert [7] We will use his idea. A complex geodesic $F:U\to D$ is a proper embedding and can be extended continuously to the closure $\bar{U}$ of U. Denoting it again by F, we have $F(\partial U)\subset\partial D$.

The case (1); Assume that there exist two distinct complex geodesics F_1, F_2 in $\mathrm{Hol}(U, D)$ which pass through two points x, y. There are $T_j\in\mathrm{Aut}(U)$ $(j=1, 2)$ such that $F_1\circ T_1(0)=F_2\circ T_2(0)=x$. Let $y=F_1\circ T_1(s_1)=F_2\circ T_2(s_2)$ for points s_1 and s_2 in U. Since F_j is the isometry with respect to C-distances c_D^*, c_U,

$$c_D^*(x, y)=c_D^*(F_j\circ T_j(0),\quad F_j\circ T_j(s_j))=c_U(0, s_j),$$

hence we have $c_U(0, s_1)=c_U(0, s_2)$, i.e. $|s_1|=|s_2|$. Let $s_1=e^{i\theta}s_2$. Setting $r(t)=e^{i\theta}t$ on U and $T_1^*(t)=T_1\circ r(t)$, we have $F_1\circ T_1^*(0)=F_1\circ T_1(0)=x$, $F_1\circ T_1^*(s_2)=F_1\circ T_1(s_1)$

$=y$. Let $G_1=F_1\circ T_1^*$, $G_2=F_2\circ T_2$ and $s=s_2$, then G_1 and G_2 are complex geodesics in D such that $G_1(0)=G_2(0)=x$, $G_1(s)=G_2(s)=y$. Since D is convex, $G(t)=(G_1(t)+G_2(t))/2$ is a well-defined map from U to D, and $G(0)=x$, $G(s)=y$. Hence, from Lemma 1.2 (1), $G(t)$ is a complex geodesic passing through x and y.

If $G_1(t')\neq G_2(t')$ for some boundary point t' of U, then $G(t')$ is an interior point of D since D is strictly convex. This contradicts that the complex geodesic is a proper embedding. Thus we have $G_1(t)=G_2(t)$ on ∂U. From the Cauchy formula $G_1(t)=G_2(t)$ on $\bar{U}$. Therefore $F_1\circ T=F_2$ on U, where $T=T_1^*\circ T_2^{-1}$ is an automorphism of U.

For the case (2), we can do as above, so we shall omit the details.

q. e. d.

2. The fixed point set.

Let D be a bounded convex domain in $\boldsymbol{C}^n$. For a map $f\in\mathcal{H}(D)$, J. P. Vigué [16, 17] investigated the structure of $\mathrm{Fix}(f)$. We arrange his results that we shall use later.

THEOREM 2.1. *Let D be a bounded convex domain in $\boldsymbol{C}^n$ and $f\in\mathcal{H}(D)$.*

(1) *For two distinct fixed points x and y of f, there exists a complex geodesic F which passes x, y and $F(U)\subset\mathrm{Fix}(f)$.* [16, *Theorem* 4.1]).

(2) *If there is $(p;v)$ in TD^* such that $f(p)=p$, $f'(p)v=v$, then there exists a complex geodesic for $C_D(p;v)$ which is contained in* $\mathrm{Fix}(f)$. ([16, *Theorem* 4.2]).

(3) $\mathrm{Fix}(f)$ *is a connected complex submanifold in D.* ([16, *Theorem* 4.3]).

(4) *At any point p of $V=\mathrm{Fix}(f)$, a vector v $(\neq 0)$ belongs to $T_p(V)$ (the holomorphic tangent space of V at p) if and only if $f'(p)v=v$.* ([16, *Proposition* 4.4]).

(5) *Let A be a complex submanifold of dimension 1 in D. A is a fixed point set of a map f in $\mathcal{H}(D)$ if and only if there exists a complex geodesic F with $A=F(U)$.* ([16, *Proposition* 7.3]).

Vigué proved (1), (2) of Theorem 2.1 by using Brouwer's fixed point theorem. We remark that (1), (2) can be proved by using Lemma 1.4 and the Schwarz lemma provided that the domain D is strictly convex.

PROPOSITION 2.2. *Let D be a bounded strictly convex domain in $\boldsymbol{C}^n$ and $f\in\mathcal{H}(D)$.*

(1) *For any two distinct points z_1, z_2 of* $\mathrm{Fix}(f)$, *the complex geodesic F which passes through z_1, z_2 is contained in* $\mathrm{Fix}(f)$.

(2) *If $(p;v)$ satisfies that $f(p)=p$, $f'(p)v=v$, then the complex geodesic F for $C_D(p;v)$ is contained in* $\mathrm{Fix}(f)$.

Proof. (1) From Theorem 1.3, there exists a unique complex geodesic $F\in\mathrm{Hol}(U, D)$ which passes through z_1, z_2. Set $F(t_j)=z_j$ $(t_j\in U,\ j=1, 2)$ and $G=f\circ F$. Then $G\in\mathrm{Hol}(U, D)$ and $G(t_j)=f(z_j)=z_j$. Hence we obtain;

$$c_D^*(G(t_1), G(t_2)) = c_D^*(z_1, z_2) = c_D^*(F(t_1), F(t_2)) = c_U(t_1, t_2).$$

Lemma 1.2 (1) implies that G is also a complex geodesic which passes through z_1, z_2. Since D is strictly convex, from Lemma 1.4, there is an automorphism T of U such that $G=F\circ T$. Then $F(t_j)=z_j=G(t_j)=F\circ T(t_j)$, $j=1, 2$. Since F is an embedding of U into D, we have $T(t_j)=t_j$, $j=1, 2$. The Schwarz lemma implies that if a map $S\in\mathcal{H}(U)$ has two fixed points in U, then S is the identity map of U.

Thus T is the identity map of U, that is, $F(t)=f\circ F(t)$ on U. We have that the complex geodesic $F(U)$ is contained in $\mathrm{Fix}(f)$.

(2) We remark that $C_D=K_D$ on TD. From Theorem 1.3, there exists a complex geodesic F for $C_D(p; v)$, which satisfies $F(0)=p$ and $F'(0)=rv$ ($1/r=K_D(p; v)$). Set $G=f\circ F$ on U. Then $G\in\mathrm{Hol}(U, D)$, $G(0)=f(p)=p$ and $G'(0)=f'(p)rv=rv$. From Lemma 1.4 (2), there is an automorphism T of U such that $G=F\circ T$ because G is the complex geodesic for $C_D(p; v)$. From that $F(T(0))=F(0)=p$ and F is an embedding, we get $T(0)=0$. Hence $T(t)=\zeta t$, $|\zeta|=1$. Since $rv=G'(0)=F'(0)T'(0)=r\zeta v$, $\zeta=1$, i.e. T is the identity map of U. Thus we have $F(U)\subset\mathrm{Fix}(f)$. q.e.d.

3. The iterational limits of a holomorphic self-map.

Let D be a bounded convex domain in $\boldsymbol{C}^n$ and f be a holomorphic self-map of D.

If $\mathrm{Fix}(f)$ is nonempty, then $\Gamma(f)$ is contained in $\mathcal{H}(D)$, and has only one idempotent i.e. holomorphic retraction. Furthermore, if $\Gamma(f)$ contains the identity map, then f is an automorphism of D (cf. Wallace [18] or Shields [11]).

Let p be a point of $\mathrm{Fix}(f)$ and $f'(p)$ be the differential of f at p. If λ_ν ($\nu=1, \cdots, n$) are the eigenvalues of $f'(p)$, then $|\lambda_\nu|\leqq 1$. Set $\mathrm{car}\{\nu; \lambda_\nu=1\}=d_1$, $\mathrm{car}\{\nu; |\lambda_\nu|=1, \lambda_\nu\neq 1\}=d$ and $\mathrm{car}\{\nu; |\lambda_\nu|<1\}=j$, where car denotes the cardinal number of the set.

THEOREM 3.1. *Let D be a bounded convex domain in $\boldsymbol{C}^n$, and $f\in\mathcal{H}(D)$ with $V=\mathrm{Fix}(f)\neq\phi$.*

Then $\Gamma(f)$ contains a unique holomorphic retraction $R: D\to A$, where A is a connected complex submanifold in D and any element g of $\Gamma'(f)$ can be expressed as $g=T\circ R$ where $T\in\mathrm{Aut}(A)$.

Furthermore the numbers d_1, d and j do not depend on the choise of the point p and $\dim A=d_1+d=n-j$.

Proof. Since D is taut, $\mathcal{H}(D)$ is a normal family. We will use the method of Bedford ([1, Theorem 1.1]) and Vigué's Theorem 2.1. Take an element g of $\Gamma'(f)$. Then g is in $\mathcal{H}(D)$ since $\mathrm{Fix}(f)$ is non-empty. There is a subsequence $\{f^{m_\nu}\}$ of $\{f^m\}$ such that $\lim f^{m_\nu}=g$. Taking a subsequence if necessary, we may assume that two sequences $k_\nu=m_{\nu+1}-m_\nu$, $l_\nu=k_\nu-m_\nu$ both tend to infinity. Again taking the subsequences if necessary, we obtain two maps $R, h\in\mathrm{Hol}(D, \bar{D})$, where $\lim f^{k_\nu}=R$ and $\lim f^{l_\nu}=h$. Passing to the limit as $\nu\to\infty$ in the equation

$f^{m_\nu+1}=f^{k_\nu}\circ f^{m_\nu}$, we have $g=R\circ g$. Then $R\in\mathrm{Hol}(D, D)=\mathcal{H}(D)$ since $g\in\mathcal{H}(D)$. Taking the limits of the equations $f^{m_\nu+1}=f^{m_\nu}\circ f^{k_\nu}$ and $f^{k_\nu}=f^{l_\nu}\circ f^{m_\nu}$, we have $g=g\circ R$ and $R=h\circ g$. Set $A=\mathrm{Fix}(R)$. Then, from Theorem 2.1 (3) A is a connected submanifold in D and $g(D)\subset A\subset R(D)$ since $g=R\circ g$. From $g=R\circ g$ and $R=h\circ g$, we see that R and g have the same rank and the preimage $R^{-1}(A)$ is n-dimensional submanifold of D. Hence $R(D)=A$. It follows that $R^2=R$ and that R is a holomorphic retraction of D onto A. The uniqueness of R follows from the general theory of Wallace [18].

Setting $T=g|A$, we must show that T is an automorphism of A. We take the limit of $f^{k_\nu}=f^{m_\nu}\circ f^{l_\nu}$ to obtain $R=g\circ h$. Because $R(z)=z$ on A and $R=g\circ h$, $T(h(z))=h(T(z))=z$ on A. This implies that T is one-to-one and $T^{-1}=h$ on A as $g=T\circ R$.

We calculate the dimension of A. $V=\mathrm{Fix}(f)$ is a connected complex submanifold in D and $V\subset A$. Thus the dimension of V is a constant s at all points of V. A non-zero vector v belongs to the tangent space $T_p(V)$ if and only if $f'(p)v=v$. i.e. v is a eigenvector associated to the eigenvalue 1 (Theorem 2.1 (4)). Thus $\dim T_p(V)=d_1$. On the other hand, $s=\dim V=\dim T_p(V)=d_1$ since V is non-singular. Put $f'(p)$ in Jordan's canonical form;

$$\text{Diagonal}\ [1, \cdots, 1, J_1, \cdots, J_{d'}, J'_1, \cdots, J'_{j'}]$$

where $$J_\nu=\begin{bmatrix}\lambda_\nu & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_\nu\end{bmatrix}(|\lambda_\nu|=1,\ \lambda_\nu\neq 1), \qquad J'_\mu=\begin{bmatrix}\lambda_\mu & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_\mu\end{bmatrix}$$

($|\lambda_\mu|<1$) or 0 (if f degenerates). But we remark that $J_\nu=\text{Diagonal}\ [\lambda_\nu, \cdots, \lambda_\nu]$ because D is bounded (see [5, p 75]). Hence we may assume that the Jordan's canonical form of $f'(p)$ is Diagonal $[1, \cdots, 1, \lambda_1, \cdots, \lambda_d, J'_1, \cdots, J'_j]$. It is clear that $\lim (J'_\mu)^{k_\nu}=$(0-matrix) as $k_\nu\to\infty$ where $\{k_\nu\}$ is the above-mentioned subsequense such that $\lim f^{k_\nu}=R$. The eigenvalues of $R'(p)$ are 1 and 0 since R is the holomophic retraction. The eigenvalue 1 is d_1+d-fold. Thus $\mathrm{rank}_p R=d_1+d=\dim_p A$. Because A is a connected complex manifold, d_1+d is constant over A. q.e.d.

4. The common fixed point set.

A.L. Shield [11] showed that if S is a commuting subfamily of $\mathcal{H}(U)\cap C(\bar{U})$, then the elements of S have a common fixed point in U ($C(\bar{U})$ is the family of functions that are continuous on $\bar{U}$). D.J. Eustice [3] proved an analogous result for the bidisc U^2. We extend this to the case of a bounded convex domain.

THEOREM 4.1. *Let D be a bounded convex domain in $\boldsymbol{C}^n$ and f, g be maps in $\mathcal{H}(D)$ such that $f\circ g=g\circ f$ and both have fixed points in D. If* $\dim \mathrm{Fix}(f)\leqq 1$ *or* $\dim \mathrm{Fix}(g)\leqq 1$, *then f and g have a common fixed point.*

Proof. We may assume $\dim \mathrm{Fix}(g) \leqq 1$. $V=\mathrm{Fix}(g)$ is a connected complex submanifold in D. For any $z \in V$, $g(z)=z$, hence $f(z)=f \circ g(z)=g \circ f(z)$. This means $f(z) \in V$ i.e. f maps V into V.

If $\dim V=0$, then V consists of only one point p in D, we see $f(p)=p$, i.e. $p \in \mathrm{Fix}(f)$.

Next we consider the case of $\dim V=1$. From Vigué's Theorem 2.1 (5), $V=\mathrm{Fix}(g)$ agrees with a complex geodesic $F(U)$. Since F is an isometric embedding, the inverse F^{-1} exists on V.

Setting $h=f|V$ (the restriction of f to V), we have $T=F^{-1} \circ h \circ F$ is well-defined on U and $T \in \mathcal{H}(U)$. $T^m=F^{-1} \circ h^m \circ F$. Since f has a fixed point in D, $\Gamma(f)$ is contained in $H(D)$, thus $\Gamma(T) \subset \mathcal{H}(U)$. From the Denjoy-Wolff's Theorem I, T has a fixed point s in U i.e. $T(s)=s$. Hence $h \circ F(s)=F(s)$. The point $F(s) \in V$ is a fixed point of f because $h=f|V$. q.e.d.

5. The inverse problem.

Let D be a bounded convex domain in $\boldsymbol{C}^n$. We showed that if $f \in \mathcal{H}(D)$ has a fixed point in D, then $\Gamma(f)$ is in $\mathcal{H}(D)$ and contains exactly one holomorphic retraction. As mentioned in introduction, we consider the inverse problem to this.

PROBLEM I. Does f have a fixed point in D provided that $\Gamma(f)$ contains a holomorphic retraction?

PROBLEM II. When f is fixed point free, is the image $g(D)$ contained in the boundary of D for any $g \in \Gamma'(f)$?

Problem I and II are in contraposition. We consider Problem I. If f is itself a holomorphic retraction $D \to A$, then $A=\mathrm{Fix}(f)$. Thus the problem is trivial. We may consider only a holomorphic self-map that is not a holomorphic retraction. Set $\mathcal{H}^*(D)=\mathcal{H}(D)-\{\text{all holomorphic retractions of } D\}$.

For Problem I, we have a partial answer.

THEOREM 5.1. *Suppose that D is a bounded convex bomain in $\boldsymbol{C}^n$, $f \in \mathcal{H}^*(D)$ and $\Gamma(f)$ contains a holomorphic retraction $R: D \to A$. If $\dim A \leqq 1$, then f has a fixed point in D.*

Proof. Assume that a subsequence $\{f^{m_\nu}\}$ converges to R. Since $f \circ f^{m_\nu}= f^{m_\nu} \circ f$, $f \circ R=R \circ f$, i.e. f and R are commutative. We remark that $A=\mathrm{Fix}(R)$.

If A consists of a point in D, then the proof is trivial. Let $\dim A=1$. Since $\Gamma(f|A)$ is contained in $\mathcal{H}(A)$, as the proof of Theorem 4.1, Denjoy-Wolff's theorem I leads to that $f|A$ has a fixed point in$A$, thus f has a fixed point in D. q.e.d.

Remark. It is desired that the restriction ($\dim A \leqq 1$) in Theorem 5.1 can be removed. If $R\in\Gamma(f)$ is a holomorphic retraction of D onto A, then $R|A$ is the identity map of A, hence $f|A$ is an automorphism of A, that is, an isometry with respect to the C-metric $C_A=C_D|A$ (they equal to K-metric) of A. The remaining problem is following; When $\Gamma(f|A)$ is contained in $\mathcal{H}(A)$, does $f|A$ have a fixed point?

Finally we consider the case of 2-dimensional Thullen domain. Let $D=\{(z_1, z_2)\in \boldsymbol{C}^2;\ |z_1|^2+|z_2|^{2p}<1\}$. ($p\geqq 1$, integer). Take a map f in $\mathcal{H}(D)$. Suppose that $\Gamma(f)$ contains a holomorphic retraction $R: D\to A$, where A is a connected complex submanifold in D. D is strictly convex. From Vigué's theorem 2.1, A is one of the following; (1) one point in D, (2) a complex geodesic $F(U)$, (3) D itself.

In the cases of (1) and (2), f has a fixed point by Theorem 5.1. In the case (3), R is the identity map of D, thus f must be an automorphism of D. Then $f=(f_1, f_2)$ is

$$f_1(z)=\zeta_1(z_1-a)(\bar{a}z_1-1)^{-1}, \qquad (|a|<1,\ |\zeta_1|=1),$$

$$f_2(z)=\zeta_2(1-|a|^2)^{1/2p}(1-\bar{a}z_1)^{-1/p}z_2, \qquad (|\zeta_2|=1).$$

(see, Ise [4]). From this form, we see $\mathrm{Fix}(f)=(\mathrm{Fix}(f_1), 0)$. If $\mathrm{Fix}(f)=\phi$, then $(f^m)_1$ converges to a boundary point since $(f^m)_1=f_1^m$, where $f^m=((f^m)_1, (f^m)_2)$, this is absurd. Thus f has a fixed point in all cases, provided that $\Gamma(f)$ contains a holomorphic retraction. Therefore, if f has no fixed point, any subsequential limit g of $\{f^m\}$ is a constant map which value is a boundary point.

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