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GLOBAL PROPERTIES OF THE GAUSS IMAGE OF FLAT SURFACES IN R^4

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Let M be a surface of zero Gaussian curvature in R^4 which has flat normal connection. Let $G_{2,4}$ denote the Grassmann manifold consisting of oriented 2-dimensional linear subspaces of R^4. The Gauss map $G: M \to G_{2,4}$ is defined by assigning each point of M to the tangent plane of M at the point. The image of M by G is called the Gauss image of M.

In [2] we studied local properties of the Gauss image of M; if we identify $G_{2,4}$ with $S^2 \times S^2$, then the Gauss image of M is locally the Riemannian product of two curves γ_1 and γ_2, where γ_1 lies in the first factor of $S^2 \times S^2$ and γ_2 lies in the second factor.

In this paper we study some global properties of the Gauss image of M when M is compact. If G is regular at every point of M, the Gauss image is a finite covering of $\gamma_1 \times \gamma_2$, where γ_1 (resp. γ_2) is a closed curve in the first (resp. second) factor of $G_{2,4}$. Then we show that the total curvatures of γ_1 and γ_2 are both zero (Theorem 1). In particular, if γ_i is simple, it divides the factor of $G_{2,4}$ $(=S^2(1/\sqrt{2}))$ into two regions of the same area.

In § 3, we give a method to construct a flat torus whose Gauss image is prescribed. In Theorem 2, we show that if γ_i $(i=1, 2)$ is a closed curve in $S^2(1/\sqrt{2})$ whose total curvature is zero and if the total curvature of any subarc of γ_i is less than $\pi/2$, then there exists a flat torus whose Gauss image is a finite covering of $\gamma_1 \times \gamma_2$.

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§ 1. Local Properties.

In this section, we recall some basic facts on the geometry of the Grassmann manifold $G_{2,4}$ and flat surfaces in R^4. See [2] for details. Let $G_{2,4}$ denote the Grassmann manifold of oriented 2-dimensional linear subspaces of R^4. If $P \in G_{2,4}$, then there exists a positively oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of R^4 such that $P = e_1 \wedge e_2$. By differentiating $e_1 \wedge e_2$, we see that the tangent space of $G_{2,4}$ at P, $T_P G_{2,4}$, is spanned by $e_i \wedge e_\alpha$ $(i=1, 2, \alpha=3, 4)$. It is known that $G_{2,4}$ equipped with the standard invariant metric is isometric to $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$

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([3]). We identify $G_{2,4}$ with $S^2 \times S^2$ and denote the first factor by S_1 and the second by S_2. $\{e_i \wedge e_\alpha;\ i=1, 2,\ \alpha=3, 4\}$ becomes an orthonormal basis of $T_P G_{2,4}$ with respect to this metric, and $T_P S_1$ is spanned by $\{e_1 \wedge e_3 - e_2 \wedge e_4,\ e_1 \wedge e_4 + e_2 \wedge e_3\}$ and $T_P S_2$ is spanned by $\{e_1 \wedge e_3 + e_2 \wedge e_4,\ e_1 \wedge e_4 - e_2 \wedge e_3\}$.

Let M be an oriented surface in R^4. The Gauss map $G: M \to G_{2,4}$ is defined as $G(p) = T_p M$, where $T_p M$ is the tangent plane of M at p. Let $\{e_1, e_2, e_3, e_4\}$ be a positively oriented orthonormal frame field of R^4 which is defined on an open set of M and satisfies $T_p M = e_1(p) \wedge e_2(p)$. Let ω_{AB} $(1 \leqq A, B \leqq 4)$ be the connection form of the standard covariant differentiation of R^4, i.e., $\omega_{AB} = \langle de_A, e_B \rangle$, where $\langle\ ,\ \rangle$ is the standard inner product of R^4. The differential of the Gauss map is given by

$$dG(X) = -\omega_{13}(X) e_2 \wedge e_3 - \omega_{14}(X) e_2 \wedge e_4 + \omega_{23}(X) e_1 \wedge e_3 + \omega_{24}(X) e_1 \wedge e_4 \tag{1.1}$$

for any $X \in T_p M$.

LEMMA 1.1. ([2]) *If the Gaussian curvature of M is identically zero and the normal connection of M is flat, then there exists a local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on M such that $e_1(p) \wedge e_2(p) = T_p M$ for any $p \in M$, $\omega_{14}(X) = \omega_{23}(X) = 0$ for any $X \in T_p M$ and $\omega_{13}(e_2) = \omega_{24}(e_1) = 0$.*

Let $\lambda = \omega_{13}(e_1)$ and $\mu = \omega_{24}(e_2)$. Then (1.1) is written as

$$dG(X) = -\lambda \langle X, e_1 \rangle e_2 \wedge e_3 + \mu \langle X, e_2 \rangle e_1 \wedge e_4 . \tag{1.2}$$

Hence G is regular at p M if and only if $\lambda\mu \neq 0$. Moreover, if $\lambda\mu \neq 0$ on M, then there is no umbilical point on M and it follows from Reckziegel's theorem ([5]) that each e_A $(1 \leqq A \leqq 4)$ becomes a C^∞ vector field, and λ and μ become C^∞ functions on M. Applying Codazzi equations, we obtain the following equalities:

$$e_1 \mu - \mu \omega_{21}(e_2) = 0 \tag{1.3}$$

$$e_2 \lambda - \lambda \omega_{12}(e_1) = 0 \tag{1.4}$$

$$\lambda \omega_{21}(e_2) - \mu \omega_{34}(e_1) = 0 \tag{1.5}$$

$$\mu \omega_{12}(e_1) + \lambda \omega_{34}(e_2) = 0 . \tag{1.6}$$

In the following, we assume that $\lambda\mu \neq 0$ on M. Then there exists a local coordinate system (ξ_1, ξ_2) on M which satisfies

$$\begin{aligned} \partial/\partial\xi_1 &= 1/\sqrt{2}(1/\lambda e_1 - 1/\mu e_2) \\ \partial/\partial\xi_2 &= 1/\sqrt{2}(1/\lambda e_1 + 1/\mu e_2) . \end{aligned} \tag{1.7}$$

Let σ_1 (resp. σ_2) be an integral curve of $1/\sqrt{2}(1/\lambda e_1 - 1/\mu e_2)$ (resp. $1/\sqrt{2}(1/\lambda e_1 + 1/\mu e_2)$). Let $\gamma_i = G(\sigma_i)$ $(i=1, 2)$. Then we have

PROPOSITION 1. ([2]) *γ_i lies in S_i for each $i=1, 2$. Locally, $G(M)$ is the Riemannian product of γ_1 and γ_2.*

Let κ_i denote the geodesic curvature of γ_i ($i=1, 2$) in $S_i=S^2(1/\sqrt{2})$. By Proposition 4.1 in [2], we have

$$\kappa_1=\sqrt{2}\left(e_1\frac{1}{\mu}+e_2\frac{1}{\lambda}\right) \tag{1.8}$$
$$\kappa_2=\sqrt{2}\left(e_1\frac{1}{\mu}-e_2\frac{1}{\lambda}\right)$$

Examples. (i) Let C_1 and C_2 be plane curves. Then the Riemannian product $C_1\times C_2$ is a flat surface in R^4 with flat normal connection. For this surface we have $\kappa_1\equiv 0$ and $\kappa_2\equiv 0$. Thus the Gauss image is locally the product of two great circles. Conversely, if the Gauss image of a flat surface in R^4 is totally geodesic, then it is locally the Riemannian product of two plane curves. This follows from the Chen-Yamaguchi classification theorem for surfaces in R^4 with totally geodesic Gauss image ([1]).
(ii) Let $(z_1(t), z_2(t))$ be a curve parameterized by arc-length in $C^2=R^4$. Suppose $|z_1|^2+|z_2|^2\equiv 1$ so that this curve is contained in $S^3\subset R^4$. In addition, (z_1, z_2) is required to satisfy $z_1'\overline{z_1}+z_2'\overline{z_2}\equiv 0$. Let M be the surface in R^4 defined by $(t, \phi)\to(e^{i\phi}z_1(t), e^{i\phi}z_2(t))$. Then M is a flat surface with flat normal connection. For this surface we have either $\kappa_1\equiv 0$ or $\kappa_2\equiv 0$. (M is called a Hopf torus. Some descriptions of the geometry of Hopf tori are given in [4].)

§ 2. Global Properties.

Let M be a compact, oriented surface in R^4 which has zero Gaussian curvature and flat normal connection at every point. We assume that the Gauss map $G: M\to G_{2,4}=S_1\times S_2$ is regular at every point. Then it follows from Proposition 1 that there exist immersed closed curves $\gamma_1\subset S_1$ and $\gamma_2\subset S_2$ such that the Gauss image $G(M)$ is a finite covering of the Riemannian product of γ_1 and γ_2.

Combining (1.8) with (1.3), (1.4) and (1.7), we obtain

$$\kappa_1(G(p))=-2\omega_{12}(\partial/\partial\xi_1(p)) \tag{2.1}$$
$$\kappa_2(G(p))=2\omega_{12}(\partial/\partial\xi_2(p)),$$

for any $p\in M$.

Since the Gaussian curvature is identically zero, the universal covering $\tilde{M}$ of M is isometric to R^2 equipped with the standard flat metric. Let $\pi:\tilde{M}\to M$ be the projection. Since M is compact, M is isometric to R^2/Γ, where Γ is a properly discontinuous subgroup of the isometry group of R^2. Let (x_1, x_2) be a Cartesian coordinate system on $\tilde{M}$. Suppose that Γ is generated by $(x_1, x_2)\to(x_1+a, x_2)$ and $(x_1, x_2)\to(x_1+b, x_2+c)$ for some real numbers a, b and c. Let

$f_i = d\pi(\partial/\partial x_i)$ $(i=1, 2)$. Then $\{f_1, f_2\}$ is a globally defined C^∞ parallel orthonormal frame field of TM.

There exists a global C^∞ orthonormal frame field $\{\tilde{e}_1, \tilde{e}_2\}$ on $\tilde{M}$ such that $e_1 = d\pi(\tilde{e}_1)$ and $e_2 = d\pi(\tilde{e}_2)$ satisfy the conditions in Lemma 1.1. Let $\tilde{\lambda} = \lambda \circ \pi$ and $\tilde{\mu} = \mu \circ \pi$. Then $\tilde{\lambda}$ and $\tilde{\mu}$ are non-zero C^∞ functions on $\tilde{M}$. Let $\tilde{X}_1 = 1/\sqrt{2}(1/\tilde{\lambda}\tilde{e}_1 - 1/\tilde{\mu}\tilde{e}_2)$ and $\tilde{X}_2 = 1/\sqrt{2}(1/\tilde{\lambda}\tilde{e}_1 + 1/\tilde{\mu}\tilde{e}_2)$. Then we have $[\tilde{X}_1, \tilde{X}_2] = 0$ everywhere on M. Hence there exists a global coordinate system $(\tilde{\xi}_1, \tilde{\xi}_2)$ on $\tilde{M}$ such that $\partial/\partial\tilde{\xi}_1 = \tilde{X}_1$ and $\partial/\partial\tilde{\xi}_2 = \tilde{X}_2$.

We define a C^∞ function α on $\tilde{M}$ by

$$\tilde{e}_1 = \cos\alpha\,\partial/\partial x_1 + \sin\alpha\,\partial/\partial x_2 \qquad (2.2)$$
$$\tilde{e}_2 = -\sin\alpha\,\partial/\partial x_1 + \cos\alpha\,\partial/\partial x_2 .$$

Then

$$\pi^*\omega_{12} = \langle d\tilde{e}_1, \tilde{e}_2\rangle = d\alpha . \qquad (2.3)$$

For a curve γ in a two dimensional oriented Riemannian manifold the total integral of the signed geodesic curvature κ along γ, $\int_\gamma \kappa$, is called the *total curvature* of γ. We denote it by $\tau(\gamma)$.

Let $\tilde{\sigma}_1 = \{(\tilde{\xi}_1, \tilde{\xi}_2) : -\infty < \tilde{\xi}_1 < \infty, \tilde{\xi}_2 \equiv 0\}$ and $\tilde{\sigma}_2 = \{(\tilde{\xi}_1, \tilde{\xi}_2) : \tilde{\xi}_1 \equiv 0, -\infty < \tilde{\xi}_2 < \infty\}$. $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are curves on $\tilde{M}$ and $G \circ \pi$ maps $\tilde{\sigma}_i$ onto a closed curve γ_i in S_i for $i=1, 2$. Since G is regular everywhere on M and M is compact, $G(M)$ is a finite covering of $\gamma_1 \times \gamma_2$. Let k be the degree of G as a map from M onto $\gamma_1 \times \gamma_2$. There exists a closed curve σ_i on M such that $\tilde{\sigma}_i$ is mapped onto σ_i by π and $G(\sigma_i)$ is a k-fold covering of γ_i.

LEMMA 2.1. *For $i=1, 2$, we have*

$$\tau(\sigma_i) = \frac{(-1)^i k}{2} \tau(\gamma_i) .$$

Proof. Let $\tilde{\sigma}_i'$ be a subarc of $\tilde{\sigma}_i$ which is mapped bijectively onto σ_i by π. Then we have $\tau(\tilde{\sigma}_i') = \tau(\sigma_i)$.

On the other hand, by (2.1) and (2.3), we have

$$\tau(\gamma_i) = \int_{\gamma_i} \kappa_i = \frac{(-1)^i 2}{k} \int_{\tilde{\sigma}_i'} d\alpha = \frac{(-1)^i 2}{k} \tau(\tilde{\sigma}_i') . \qquad \text{Q.E.D.}$$

LEMMA 2.2. *Let σ be a smooth closed curve in a flat torus $M = R^2/\Gamma$. Let $\tilde{\sigma}$ be a complete curve in the universal covering $\tilde{M} = R^2$ of M whose image by the projection $\pi : \tilde{M} \to M$ is σ.*

If the total curvature of σ is not zero, then $\tilde{\sigma}$ has a self-intersection.

Proof. Let s be an arc-length parameter of σ and let L be the total length of σ. Let s also denote an arc-length parameter of $\tilde{\sigma}$. For any s and any integer m we have $\pi\circ\tilde{\sigma}(s+mL)=\pi\circ\tilde{\sigma}(s)$. Suppose that Γ is generated by $(x_1, x_2)\to(x_1+a, x_2)$ and $(x_1, x_2)\to(x_1+b, x_2+c)$. Since $\pi\circ\tilde{\sigma}(s)=\pi\circ\tilde{\sigma}(s+L)$, if $\tilde{\sigma}(s)=(x_1, x_2)$, there exist integers p and q such that $\tilde{\sigma}(s+L)=(x_1+pa+qb, x_2+qc)$.

Let $\kappa(s)$ $(0\leqq s<L)$ be the signed geodesic curvature of σ. We extend κ to a function defined on $(-\infty, \infty)$ by setting $\kappa(s)=\kappa(s')$ if $s\equiv s'$ (mod L) and $s'\in[0, L)$. Then $\kappa(s)$ gives the signed geodesic curvature of $\tilde{\sigma}$. We define a function $F(s)$ for $s\in(-\infty, \infty)$ by

$$F(s)=\int_0^s \kappa(t)dt$$

so that the total curvature of $\{\tilde{\sigma}(s): s_1\leqq s\leqq s_2\}$ is given by $F(s_1)-F(s_2)$. In particular, for any s, we have $F(s+L)-F(s)=\tau(\sigma)$.

Let $\Lambda_1=\min\{F(s): 0\leqq s\leqq L\}$ and $\Lambda_2=\max\{F(s): 0\leqq s\leqq L\}$. Suppose $\tau(\sigma)=A\neq 0$. Then for any s_1 and s_2 $(s_1<s_2)$ we have

$$mA+\Lambda_1\leqq F(s_2)-F(s_1)\leqq mA+\Lambda_2,$$

where m is the integer such that $m\leqq s_2-s_1<m+1$. Therefore, there exist numbers S_1 and S_2 such that

$$|F(s_2)-F(s_1)|>2\pi \tag{2.4}$$

for any $s_1<S_1$ and $s_2>S_2$.

Since $\{\tilde{\sigma}(s): S_1<s<S_2\}$ is bounded, there exists a circle C in $\tilde{M}$ which contains $\{\tilde{\sigma}(s): S_1<s<S_2\}$ inside. If $\tilde{\sigma}(s)=(x_1, x_2)$, we have $\tilde{\sigma}(s+nL)=(x_1+npa+nqb, x_2+nqc)$. We see from this that the whole $\tilde{\sigma}$ is not bounded and there exist numbers $S_3<S_1$ and $S_4>S_2$ such that $\tilde{\sigma}(S_3)$ and $\tilde{\sigma}(S_4)$ lie outside C. Then σ must intersect C at some $\tilde{\sigma}(T_1)$ and $\tilde{\sigma}(T_2)$, where $S_3<T_1<S_1$ and $S_2<T_2<S_4$.

If $\tilde{\sigma}$ does not have a self-intersection, then $\{\tilde{\sigma}(s): T_1\leqq s\leqq T_2\}$ divides the domain bounded by C into two simply connected subdomains. By the Gauss-Bonnet theorem, the total curvature of $\{\sigma(s): T_1\leqq s\leqq T_2\}$ is smaller than 2π. Thus we obtain $|F(T_2)-F(T_1)|<2\pi$, which contradicts (2.4). Hence $\tilde{\sigma}$ must have a self-intersection. Q. E. D.

THEOREM 1. *Let M be a compact, oriented surface in R^4 which has zero Gaussian curvature and flat normal connection. Suppose that the Gauss map G is regular at every point of M. Then there exist closed curves $\gamma_1\subset S_1$ and $\gamma_2\subset S_2$ such that the Gauss image $G(M)$ is a finite covering of the Riemannian product of γ_1 and γ_2 and the total curvature of each γ_i is zero for $i=1, 2$.*

Proof. Since $(\tilde{\xi}_1, \tilde{\xi}_2)$ defines a global coordinate system on $\tilde{M}$, $\tilde{\sigma}_i=\{(\tilde{\xi}_1, \tilde{\xi}_2): \tilde{\xi}_i\equiv 0\}$ does not have a self-intersection. Since $\sigma_i=\pi\circ\tilde{\sigma}_i$ is closed, we use Lemma 2.2 to see that $\tau(\sigma_i)=0$. Now we have $\tau(\gamma_i)=0$ by Lemma 2.1. Q. E. D.

COROLLARY. *If γ_i is a simple closed curve in S_i, γ_i divides S_i into two domains of the same area.*

Proof. Since the total curvature of γ_i is zero, the Gauss-Bonnet theorem implies the corollary.

Remark. If M is the product of two plane curves $C_1\times C_2$, then $G(M)$ is a double covering of the product of two great circles. If M is a Hopf torus, then $G(M)$ is a double covering of the product of a great circle in one of S_1 and S_2 and a closed curve in the other.

§ 3. Flat Tori with Prescribed Gauss Images.

In this section, we prove the following theorem. Again, we identify $G_{2,4}$ with $S_1\times S_2$, where S_1 and S_2 are isometric to a round 2-sphere of radius $1/\sqrt{2}$.

THEOREM 2. *Let γ_i be a regular closed curve in S_i $(i=1, 2)$. Suppose that the total curvature of γ_i is zero and the total curvature of any subarc of γ_i is less than $\pi/2$. Then there exists a compact immersed surface in R^4 whose Gauss image is a finite covering of the Riemannian product of γ_1 and γ_2.*

Remark. Local existence of a surface in R^4 with prescribed Gauss image is studied by J. Weiner ([6]).

Let $\gamma_i(\xi_i)$ be a closed curve on S_i which is parameterized by arc-length. Let l_i be the total length of γ_i. We assume that $\gamma_i(\xi_i)$ is defined for all ξ_i so that $\gamma_i(\xi_i+l_i)=\gamma_i(\xi_i)$ for any ξ_i. We denote the signed geodesic curvature of γ_i by κ_i. Then our conditions for γ_i are written as

$$\int_{\gamma_i}\kappa_i(\xi_i)d\xi_i=0 \tag{3.1}$$

$$\left|\int_{\gamma_i'}\kappa_i(\xi_i)d\xi_i\right|<\frac{\pi}{2}, \tag{3.2}$$

where γ_i' is any subarc of γ_i.

We define a map $P: R^2\to G_{2,4}$ by $P(\xi_1, \xi_2)=(\gamma_1(\xi_1), \gamma_2(\xi_2))\in S_1\times S_2=G_{2,4}$. Since $\gamma_i(\xi_i+ml_i)=\gamma_i(\xi_i)$ for any integer m, we have

$$P(\xi_1+m_1l_1, \xi_2+m_2l_2)=P(\xi_1, \xi_2) \tag{3.3}$$

for any integers m_1, m_2 and any ξ_1, ξ_2.

Let $\{\vec{e}_A(\xi_1, \xi_2): A=1, 2, 3, 4\}$ be a set of R^4-valued functions which are defined on R^2 and satisfy the following conditions;

(3.4) $\langle \bar{e}_A(\xi_1, \xi_2), \bar{e}_B(\xi_1, \xi_2)\rangle = \delta_{AB}$ for all $A, B=1, \cdots, 4$

(3.5) $P(\xi_1, \xi_2)=\bar{e}_1(\xi_1, \xi_2)\wedge \bar{e}_2(\xi_1, \xi_2)$

(3.6) $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$ is positively oriented in R^4

for all (ξ_1, ξ_2) in R^2.

Let $\bar{\omega}_{AB}$ be a 1-form on R^2 which is defined by $\bar{\omega}_{AB}=\langle d\bar{e}_A, \bar{e}_B\rangle$

LEMMA 3.1.

$$\bar{\omega}_{23}(\partial/\partial\xi_1)=\bar{\omega}_{14}(\partial/\partial\xi_1), \quad \bar{\omega}_{24}(\partial/\partial\xi_1)=-\bar{\omega}_{13}(\partial/\partial\xi_1)$$

$$\bar{\omega}_{23}(\partial/\partial\xi_2)=-\bar{\omega}_{14}(\partial/\partial\xi_2), \quad \bar{\omega}_{24}(\partial/\partial\xi_2)=\bar{\omega}_{13}(\partial/\partial\xi_2).$$

Proof. Differentiating $\bar{e}_1\wedge\bar{e}_2$, we have

$$d(\bar{e}_1\wedge\bar{e}_2)=\bar{\omega}_{23}\bar{e}_1\wedge\bar{e}_3+\bar{\omega}_{24}\bar{e}_1\wedge\bar{e}_4-\bar{\omega}_{13}\bar{e}_2\wedge\bar{e}_3-\bar{\omega}_{14}\bar{e}_2\wedge\bar{e}_4.$$

Since $d(\bar{e}_1\wedge\bar{e}_2)(\partial/\partial\xi_1)$ is tangent to γ_1 in S_1, $d(\bar{e}_1\wedge\bar{e}_2)(\partial/\partial\xi_1)$ is a linear combination of $\bar{e}_1\wedge\bar{e}_3-\bar{e}_2\wedge\bar{e}_4$ and $\bar{e}_1\wedge\bar{e}_4+\bar{e}_2\wedge\bar{e}_3$. This yields the first two equations. Similarly, the last two equations follow from the fact that $d(\bar{e}_1\wedge\bar{e}_2)(\partial/\partial\xi_2)$ is tangent to S_2. Q. E. D.

Set

(3.7)
$$e_1=\cos\theta\bar{e}_1+\sin\theta\bar{e}_2, \quad e_2=-\sin\theta\bar{e}_1+\cos\theta\bar{e}_2$$
$$e_3=\cos\phi\bar{e}_3+\sin\phi\bar{e}_4, \quad e_4=-\sin\phi\bar{e}_3+\cos\phi\bar{e}_4.$$

Let $\omega_{AB}=\langle de_A, e_B\rangle$. Then we have the following lemma by an easy computation.

LEMMA 3.2.

$$\omega_{14}(\partial/\partial\xi_1)=-\bar{\omega}_{13}(\partial/\partial\xi_1)\sin(\theta+\phi)+\bar{\omega}_{14}(\partial/\partial\xi_1)\cos(\theta+\phi)$$

$$\omega_{14}(\partial/\partial\xi_2)=\bar{\omega}_{13}(\partial/\partial\xi_2)\sin(\theta-\phi)+\bar{\omega}_{14}(\partial/\partial\xi_2)\cos(\theta-\phi).$$

We need the following lemma to prove Lemma 3.4. The proof is also easy.

LEMMA 3.3. *Let $f(\xi_1, \xi_2)$ and $g(\xi_1, \xi_2)$ be C^∞ functions on R^2. If f^2+g^2 never vanishes on R^2, then there exists a C^∞ function $\alpha(\xi_1, \xi_2)$ defined on R^2 such that*

$$f(\xi_1, \xi_2)\cos\alpha(\xi_1, \xi_2)+g(\xi_1, \xi_2)\sin\alpha(\xi_1, \xi_2)=0$$

for all (ξ_1, ξ_2) in R^2.

LEMMA 3.4. *There exist C^∞ maps $e_A: R^2\to R^4$ $(A=1, \cdots, 4)$ which satisfy the following conditions:*

(3.8) $\langle e_A(\xi_1, \xi_2), e_B(\xi_1, \xi_2)\rangle=\delta_{AB}$ *for all* $A, B=1, \cdots, 4$

(3.9) $P(\xi_1, \xi_2)=e_1(\xi_1, \xi_2)\wedge e_2(\xi_1, \xi_2)$

(3.10) $\{e_1, e_2, e_3, e_4\}$ *is positively oriented in* R^4

(3.11) $\omega_{14}\equiv 0$.

Proof. Since $d(e_1\wedge e_2)(\partial/\partial\xi_i)\neq 0$, $(\bar\omega_{13}(\partial/\partial\xi_i))^2+\bar\omega_{14}(\partial/\partial\xi_i))^2\neq 0$ by Lemma 3.1. Hence, by Lemma 3.3, there exist C^∞ functions $\alpha_i(\xi_1, \xi_2)$ defined on R^2 such that

(3.12)
$$-\bar\omega_{13}(\partial/\partial\xi_1)\sin\alpha_1+\bar\omega_{14}(\partial/\partial\xi_1)\cos\alpha_1\equiv 0$$
$$\bar\omega_{13}(\partial/\partial\xi_2)\sin\alpha_2+\bar\omega_{14}(\partial/\partial\xi_2)\cos\alpha_2\equiv 0.$$

Set $\theta=(\alpha_1+\alpha_2)/2$ and $\phi=(\alpha_1-\alpha_2)/2$. Then $\{e_A(\xi_1, \xi_2): A=1, \cdots, 4\}$ defined by (3.7) is a set of C^∞ maps from R^2 to R^4 which satisfies the conditions (3.8)-(3.11). Q.E.D.

Such a special set of vectors as $\{e_A: A=1, \cdots, 4\}$ in Lemma 3.4 is not unique but very limited as we see in the following lemma;

LEMMA 3.5. *Let* $\{e_A: A=1, \cdots, 4\}$ *be a set of* R^4*-valued functions defined on* R^2 *which satisfies the conditions in Lemma* 3.4. *Let* $\{e'_A: A=1, \cdots, 4\}$ *be any other set of* R^4*-valued functions satisfying* (3.8)-(3.11). *Then* $\{e'_1, e'_2, e'_3, e'_4\}$ *must be one of the followings*:

(i)	$\{e_1, e_2, e_3, e_4\}$	(ii)	$\{e_1, e_2, -e_3, -e_4\}$
(iii)	$\{e_2, -e_1, e_4, -e_3\}$	(iv)	$\{e_2, -e_1, -e_4, e_3\}$
(v)	$\{-e_1, -e_2, e_3, e_4\}$	(vi)	$\{-e_1, -e_2, -e_3, -e_4\}$
(vii)	$\{-e_2, e_1, e_4, -e_3\}$	(viii)	$\{-e_2, e_1, -e_4, e_3\}$

Proof. We may write

$$e'_1=\cos\theta e_1+\sin\theta e_2, \qquad e'_2=-\sin\theta e_1+\cos\theta e_2$$
$$e'_3=\cos\phi e_3+\sin\phi e_4, \qquad e'_4=-\sin\theta e_3+\cos\theta e_4.$$

Then, by Lemma 3.2, we see that $\sin(\theta+\phi)=0$ and $\sin(\theta-\phi)=0$. Hence $\theta-\phi=m\pi$, $\theta+\phi=n\pi$ for some integers m, n. Q.E.D.

LEMMA 3.6. *There exists a* C^∞ *function* $\beta(\xi_1, \xi_2)$ *defined on* R^2 *which satisfies* $d\beta=\omega_{34}$.

Proof. Let $\{e_A: A=1, \cdots, 4\}$ be a set of R^4-valued functions given in Lemma 3.4. By Lemma 3.1, the condition $\omega_{14}\equiv 0$ implies $\omega_{23}\equiv 0$. Using the structure equation, we obtain

$$d\omega_{34}=\omega_{31}\wedge\omega_{14}+\omega_{32}\wedge\omega_{24}\equiv 0.$$

Since ω_{34} is a globally defined 1-form on R^2, there exists a function β on R^2 such that $d\beta=\omega_{34}$. Q. E. D.

LEMMA 3.7. *Let $\{e_A(\xi_1, \xi_2): A=1, \cdots, 4\}$ be a set of R^4-valued functions satisfying the conditions* (3.8)-(3.11) *in Lemma* 3.4. *Let $\kappa_i(\xi_i)$ be the signed geodesic curvature of γ_i. Then we have*

$$\kappa_i=-2(-1)^i\omega_{34}(\partial/\partial\xi_i)=-2(-1)^i\partial\beta/\partial\xi_i.$$

Proof. By Lemma 3.1, we have $d(e_1\wedge e_2)(\partial/\partial\xi_1)=\omega_{24}(\partial/\partial\xi_1)(e_1\wedge e_4+e_2\wedge e_3)$. Since ξ_1 is an arc-length parameter of γ_1, $\|d(e_1\wedge e_2)(\partial/\partial\xi_1)\|=1$. This implies $\omega_{24}(\partial/\partial\xi_1)=\pm 1/\sqrt{2}$. Changing parameter from ξ_1 to $-\xi_1$ if necessary, we may assume that

$$\omega_{24}(\partial/\partial\xi_1)=-1/\sqrt{2}. \tag{3.13}$$

Using Lemma 3.1 again, we have

$$\omega_{13}(\partial/\partial\xi_1)=1/\sqrt{2}. \tag{3.14}$$

A similar argument shows that

$$\omega_{24}(\partial/\partial\xi_2)=\omega_{13}(\partial/\partial\xi_2)=1/\sqrt{2}. \tag{3.15}$$

Thus if we set $v_i=d(e_1\wedge e_2)(\partial/\partial\xi_i)$, then $v_1=-1/\sqrt{2}(e_1\wedge e_4+e_2\wedge e_3)$, $v_2=1/\sqrt{2}(e_1\wedge e_4-e_2\wedge e_3)$. Note that v_i is a unit tangent vector of γ_i. Let $\tilde{D}$ be the Riemannian connection on $G_{2,4}$ associated with the standard invariant metric. Using Lemma 3.1 in [2], we obtain $\tilde{D}v_1=1/\sqrt{2}(\omega_{12}+\omega_{34})(e_1\wedge e_3-e_2\wedge e_4)$ and $\tilde{D}v_2=1/\sqrt{2}(\omega_{12}-\omega_{34})(e_1\wedge e_3+e_2\wedge e_4)$. This gives

$$\begin{aligned}\kappa_1&=\langle\tilde{D}_{\partial/\partial\xi_1}v_1,\ 1/\sqrt{2}(e_1\wedge e_3-e_2\wedge e_4)\rangle\\&=\omega_{12}(\partial/\partial\xi_1)+\omega_{34}(\partial/\partial\xi_1)\\\kappa_2&=\langle\tilde{D}_{\partial/\partial\xi_2}v_2,\ 1/\sqrt{2}(e_1\wedge e_3+e_2\wedge e_4)\rangle\\&=\omega_{12}(\partial/\partial\xi_2)-\omega_{34}(\partial/\partial\xi_2).\end{aligned} \tag{3.16}$$

On the other hand, since v_i is tangent to S_i, we have $\tilde{D}_{\partial/\partial\xi_2}v_1=0$ and $\tilde{D}_{\partial/\partial\xi_1}v_2=0$. This gives

$$\begin{aligned}&\omega_{12}(\partial/\partial\xi_2)+\omega_{34}(\partial/\partial\xi_2)=0\\&\omega_{12}(\partial/\partial\xi_1)-\omega_{34}(\partial/\partial\xi_1)=0.\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\kappa_1 = 2\omega_{34}(\partial/\partial\xi_1) = 2\partial\beta/\partial\xi_1$$

$$\kappa_2 = -2\omega_{34}(\partial/\partial\xi_2) = -2\partial\beta/\partial\xi_2 . \qquad \text{Q. E. D.}$$

By Lemma 3.7, the total curvature $\tau(\gamma_1)$ of γ_1 is given by

$$\tau(\gamma_1) = \int_{\xi_1}^{\xi_1+l_1} 2\partial\beta/\partial\xi_1 d\xi_1 = 2(\beta(\xi_1+l_1, \xi_2) - \beta(\xi_1, \xi_2)),$$

where (ξ_1, ξ_2) is any point in R^2. Similarly, $\tau(\gamma_2) = -2(\beta(\xi_1, \xi_2+l_2) - \beta(\xi_1, \xi_2))$. Thus we have the following lemma.

LEMMA 3.8. *$\tau(\gamma_1)=0$ if and only if $\beta(\xi_1+l_1, \xi_2)=\beta(\xi_1, \xi_2)$. $\tau(\gamma_2)=0$ if and only if $\beta(\xi_1, \xi_2+l_2)=\beta(\xi_1, \xi_2)$.*

We define a C^∞ map x of R^2 into S^3 by $x(\xi_1, \xi_2) = -\cos(\beta(\xi_1, \xi_2))e_3(\xi_1, \xi_2) + \sin(\beta(\xi_1, \xi_2))e_4(\xi_1, \xi_2)$, where $\beta(\xi_1, \xi_2)$ is the function on R^2 in Lemma 3.6 and $\{e_3, e_4\}$ is a set of R^4-valued functions defined on R^2 which is given in Lemma 3.4. Since $\gamma_1(\xi_1+l_1)=\gamma_1(\xi_1)$, $P(\xi_1+l_1, \xi_2)=P(\xi_1, \xi_2)$. Hence, by Lemma 3.5, we have only four possibilities for $\{e_3(\xi_1+l_1, \xi_2), e_4(\xi_1+l_1, \xi_2)\}$ as follows:

$$\{e_3(\xi_1+l_1, \xi_2), e_4(\xi_1+l_1, \xi_2)\} = \{e_3(\xi_1, \xi_2), e_4(\xi_1, \xi_2)\} \qquad \text{(a)}$$

$$\text{or} \quad = \{-e_3(\xi_1, \xi_2), -e_4(\xi_1, \xi_2)\} \qquad \text{(b)}$$

$$\text{or} \quad = \{e_4(\xi_1, \xi_2), -e_3(\xi_1, \xi_2)\} \qquad \text{(c)}$$

$$\text{or} \quad = \{-e_4(\xi_1, \xi_2), e_3(\xi_1, \xi_2)\} \qquad \text{(d)}$$

Note that, by continuity, if any of (a)-(d) holds for some (ξ_1, ξ_2), it must hold for all (ξ_1, ξ_2).

If we have (b), then $\{e_3(\xi_1+2l_1, \xi_2), e_4(\xi_1+2l_1, \xi_2)\} = \{e_3(\xi_1, \xi_2), e_4(\xi_1, \xi_2)\}$.

If we have (c) or (d), then $\{e_3(\xi_1+4l_1, \xi_2), e_4(\xi_1+4l_1, \xi_2)\} = \{e_3(\xi_1, \xi_2), e_4(\xi_1, \xi_2)\}$.

Summarizing these, we see that

$$\{e_3(\xi_1+m_1l_1, \xi_2), e_4(\xi_1+m_1l_1, \xi_2)\} = \{e_3(\xi_1, \xi_2), e_4(\xi_1, \xi_2)\} \tag{3.18}$$

holds for any (ξ_1, ξ_2), where m_1 is 1, 2 or 4 and constant for all (ξ_1, ξ_2).

Similarly, it can be shown that

$$\{e_3(\xi_1, \xi_2+m_2l_2), e_4(\xi_1, \xi_2+m_2l_2)\} = \{e_3(\xi_1, \xi_2), e_4(\xi_1, \xi_2)\} \tag{3.19}$$

holds for any (ξ_1, ξ_2), where m_2 is 1, 2 or 4 and constant for all (ξ_1, ξ_2).

If the total curvature of γ_i is zero for $i=1, 2$, we have

$$\begin{aligned} \beta(\xi_1+l_1, \xi_2) &= \beta(\xi_1, \xi_2) \\ \beta(\xi_1, \xi_2+l_2) &= \beta(\xi_1, \xi_2), \end{aligned} \tag{3.20}$$

by Lemma 3.8.

Using (3.18), (3.19) and (3.20), we see that

$$x(\xi_1+m_1l_1,\ \xi_2)=x(\xi_1,\ \xi_2) \tag{3.21}$$

and

$$x(\xi_1,\ \xi_2+m_2l_2)=x(\xi_1,\ \xi_2) \tag{3.22}$$

(3.21) and (3.22) show that x defines a C^∞ map from a torus R^2/Γ into S^3, where Γ is a subgroup of Isom(R^2) which is generated by $(\xi_1, \xi_2)\rightarrow(\xi_1+m_1l_1, \xi_2)$ and $(\xi_1, \xi_2)\rightarrow(\xi_1, \xi_2+m_2l_2)$.

LEMMA 3.9. *Let x be the C^∞ map from a torus into S^3 which is constructed above. Then x is regular at (ξ_1, ξ_2) if and only if $\beta(\xi_1, \xi_2)\neq k\pi/2$ for any integer k.*

Proof. The differential of x is given by $dx=d(-\cos\beta e_3+\sin\beta e_4)=\sin\beta\, d\beta e_3-\cos\beta(\omega_{31}e_1+\omega_{32}e_2+\omega_{34}e_4)+\cos\beta\, d\beta e_4+\sin\beta(\omega_{41}e_1+\omega_{42}e_2+\omega_{43}e_3)$.

Since $\omega_{23}(\partial/\partial\xi_i)=\omega_{14}(\partial/\partial\xi_i)=0$ by Lemma 3.4 and Lemma 3.1, $-\omega_{24}(\partial/\partial\xi_1)=\omega_{13}(\partial/\partial\xi_1)=1/\sqrt{2}$ and $\omega_{24}(\partial/\partial\xi_2)=\omega_{13}(\partial/\partial\xi_2)=1/\sqrt{2}$ by (3.13), (3.14) and (3.15), and $d\beta=\omega_{34}$ by Lemma 3.6, we have

$$dx(\partial/\partial\xi_1)=1/\sqrt{2}(\cos\beta e_1+\sin\beta e_2) \tag{3.23}$$

and

$$dx(\partial/\partial\xi_2)=1/\sqrt{2}(\cos\beta e_1-\sin\beta e_2)\,. \tag{3.24}$$

From this, we see that $dx(\partial/\partial\xi_1)$ and $dx(\partial/\partial\xi_2)$ are linearly independent if and only if $\sin\beta\cos\beta\neq0$ at (ξ_1, ξ_2). Q. E. D.

LEMMA 3.10. *Suppose the condition* (3.2) *holds for any subarc γ_i' of γ_i for $i=1, 2$. Then a C^∞ function $\beta(\xi_1, \xi_2)$ in Lemma* 3.6 *can be chosen in such a way that $0<\beta(\xi_1, \xi_2)<\dfrac{\pi}{2}$ for all (ξ_1, ξ_2) in R^2.*

Proof. By Lemma 3.7, we have

$$\beta(\xi_1'', \xi_2)-\beta(\xi_1', \xi_2)=-\frac{1}{2}\int_{\xi_1'}^{\xi_1''}\kappa_1(\xi_1)d\xi_1 \quad \text{for any } \xi_1', \xi_1'', \xi_2$$

and

$$\beta(\xi_1, \xi_2'')-\beta(\xi_1, \xi_2')=\frac{1}{2}\int_{\xi_2'}^{\xi_2''}\kappa_2(\xi_2)d\xi_2 \quad \text{for any } \xi_1, \xi_2', \xi_2''\,.$$

Since the total curvature of γ_i is zero, the condition (3.2) implies that we have

$$\left|\int_{\xi_1'}^{\xi_1''}\kappa_1(\xi_1)d\xi_1\right|<\frac{\pi}{2} \quad \text{and} \quad \left|\int_{\xi_2'}^{\xi_2''}\kappa_2(\xi_2)d\xi_2\right|<\frac{\pi}{2}$$

for any $\xi_1', \xi_1'', \xi_2', \xi_2''$.

Let $Q=\{(\xi_1, \xi_2): 0\leqq\xi_1\leqq l_1,\ 0\leqq\xi_2\leqq l_2\}$ and let $\beta_1=\min\{\beta(\xi_1, \xi_2): (\xi_1, \xi_2)\in Q\}$ and $\beta_2=\max\{\beta(\xi_1, \xi_2): (\xi_1, \xi_2)\in Q\}$. Then, by the periodicity (3.20), we see that $\beta_1=\min\{\beta(\xi_1, \xi_2): (\xi_1, \xi_2)\in R^2\}$ and $\beta_2=\max\{\beta(\xi_1, \xi_2): (\xi_1, \xi_2)\in R^2\}$. Suppose that $\beta(\xi_1', \xi_2')=\beta_1$ and $\beta(\xi_1'', \xi_2'')=\beta_2$. Then

$$\begin{aligned}\beta_2-\beta_1&=\beta(\xi_1'', \xi_2'')-\beta(\xi_1', \xi_2')\\&=\beta(\xi_1'', \xi_2'')-\beta(\xi_1', \xi_2'')+\beta(\xi_1', \xi_2'')-\beta(\xi_1', \xi_2')\\&=-\frac{1}{2}\int_{\xi_1'}^{\xi_1''}\kappa_1(\xi_1)d\xi_1+\frac{1}{2}\int_{\xi_2'}^{\xi_2''}\kappa_2(\xi_2)d\xi_2\\&\leqq\frac{1}{2}\left|\int_{\xi_1'}^{\xi_1''}\kappa_1(\xi_1)d\xi_1\right|+\frac{1}{2}\left|\int_{\xi_2'}^{\xi_2''}\kappa_2(\xi_2)d\xi_2\right|\\&<\frac{\pi}{2}.\end{aligned}$$

We define a new function $\bar{\beta}(\xi_1, \xi_2)$ by $\bar{\beta}(\xi_1, \xi_2)=\beta(\xi_1, \xi_2)-\frac{1}{2}(\beta_1+\beta_2)+\frac{\pi}{4}$. Since $\bar{\beta}$ differs from β by a constant, $\bar{\beta}$ also satisfies $d\bar{\beta}=\omega_{34}$. It is easy to check that $0<\bar{\beta}<\frac{\pi}{2}$. Q.E.D.

Proof of Theorem 2. By Lemma 3.9 and Lemma 3.10, a C^∞ map x becomes an immersion of a torus into S^3 in R^4 if the condition (3.2) is satisfied. (3.23) and (3.24) show that the tangent plane of the image of x at each point is $e_1\wedge e_2=P(\xi_1, \xi_2)$. Hence the Gauss image of x is locally the product of γ_1 and γ_2. Q.E.D.

Remark 1. Let M be the image of x in R^4. Since $d\omega_{12}=\omega_{13}\wedge\omega_{32}+\omega_{14}\wedge\omega_{42}\equiv 0$, the Gaussian curvature of M is identically zero.

Since M lies in S^3, the normal connection of M as a surface in R^4 is flat.

Remark 2. From the way of construction of M we see that the Gauss image of M is a k-fold covering of $\gamma_1\times\gamma_2$, where $k=1, 2$ or 4.

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