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INSTABILITY OF SPHERES WITH DEFORMED RIEMANNIAN METRICS

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§1. Introduction.

Let (M, g) be a compact Riemannian manifold. Then (M, g) is said to be unstable, if the identity map id_M of (M, g) is unstable as a harmonic map; that is, the Jacobi operator J coming from the second variation of the energy functional at id_M has negative eigenvalues. The standard sphere (S^m, g_0) of constant curvature 1 is unstable for $m \geqq 3$. Furthermore, unstable, simply connected compact (irreducible) symmetric spaces were determined (Smith [10], Nagano [5], Ohnita [7], Urakawa [17]).

In this note, as a class of homogeneous Riemannian manifolds which are not symmetric nor Einstein, we study $(S^m, g(t))$ with $m=2n+1$. Here $g(t)$ is defined as follows: For $m=2n+1$, we have the Hopf fibration $\pi: (S^m, g_0) \to (CP^n, h_0)$, where (CP^n, h_0) denotes the complex projective n-space with the Fubini-Study metric of constant holomorphic sectional curvature 4. Let ξ be a vector field on S^m which is tangent to the fibers and of unit length. ξ is a Killing vector field with respect to g_0 and the 1-form η dual to ξ with respect to g_0 defines a canonical contact structure on S^m. Then a 1-parameter family of Riemannian metrics $g(t)$ on S^m is defined by

$$g(t)=t^{-1}g_0+t^{-1}(t^m-1)\eta\otimes\eta \tag{1.1}$$

where $0<t<\infty$ (Urakawa [16], Tanno [13]). With respect to these Riemannian metrics, the volume element is unchanged.

We prove the following.

THEOREM. *For $m=2n+1\geqq 3$ and $t\in(t_0(m), \infty)$, $(S^m, g(t))$ is unstable, where $t_0(m)=[[(m^2-4)^{1/2}-1]/(m^2-5)]^{1/m}$ and $t_0(3)=0.67\cdots<t_0(m)<1$. For each eigenfunction f corresponding to the first eigenvalue m of the Laplacian of (S^m, g_0),*

$$f\xi+[\{m-2t^m+[(2t^m-1)^2+m^2-1]^{1/2}\}/2(m-1)]\nabla_{\mathrm{grad}\, f}\xi$$

is an eigen vector corresponding to the negative eigenvalue $\mu(t)$ (cf. (3.4)) of the Jacobi operator $J(t)$.

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§ 2. Preliminaries.

Let η be a canonical contact structure on S^m, $m=2n+1\geqq 3$, and ξ be its dual with respect to $g=g_0$. In this section, we write g instead of g_0 for simplicity. Then (ϕ, ξ, η, g) is a Sasakian structure, where $\phi=-\nabla\xi$. The structure tensors satisfy the following relations:

$$\phi\xi=0, \qquad \eta\cdot\phi=0, \qquad \eta(\xi)=1,$$

$$\phi\phi X=-X+\eta(X)\xi,$$

$$g(X, Y)=g(\phi X, \phi Y)+\eta(X)\eta(Y),$$

$$(\nabla_X\phi)(Y)=g(X, Y)\xi-\eta(Y)X,$$

where X and Y are vector fields on S^m.

If $m=4r+3$, then we have Killing vector fields $\xi_{(\alpha)}$, $\alpha=1, 2, 3$, which are orthonormal and satisfy

$$[\xi_{(\alpha)}, \xi_{(\beta)}]=2\xi_{(\gamma)},$$

$$\phi_{(\alpha)}\xi_{(\beta)}=-\phi_{(\beta)}\xi_{(\alpha)}=\xi_{(\gamma)},$$

$$\phi_{(\alpha)}\phi_{(\beta)}-\xi_{(\alpha)}\otimes\eta_{(\beta)}=-\phi_{(\beta)}\phi_{(\alpha)}+\xi_{(\beta)}\otimes\eta_{(\alpha)}=\phi_{(\gamma)},$$

where (α, β, γ) is a cyclic permutation of (1, 2, 3), and $\phi_{(\alpha)}$ and $\eta_{(\alpha)}$ are defined analogously.

Let λ_k be the k-th eigenvalue of the Laplacian Δ acting on functions on (S^m, g) with multiplicity $\nu(k)$. Then

$$\mathrm{Spec}(S^m, g)=\{\lambda_k=k(m+k-1)\,;\ k=0, 1, 2, \cdots\}$$

$\nu(0)=1$, $\nu(1)=m+1$ and $\nu(k)={}_{m+k}C_k-{}_{m+k-2}C_{k-2}$ for $k\geqq 2$. Let V_k denote the eigenspace corresponding to the eigenvalue λ_k. Then we have the orthogonal decomposition of V_k;

$$V_k=V_{k,k}+V_{k,k-2}+\cdots+V_{k,k-2[k/2]},$$

where $[k/2]$ is the integral part of $k/2$, and for $\varphi\in V_{k,\theta}$

$$L_\xi L_\xi\varphi+(k-2p)^2\varphi=0$$

holds for $\theta=k-2p$, $0\leqq p\leqq[k/2]$ (Tanno [13], p. 182). Here L_ξ denotes the Lie derivation by ξ. Let $\varphi\in V_{k,0}$. Then $L_\xi L_\xi\varphi=0$ implies $L_\xi\varphi=0$ and φ is constant along each fiber of the Hopf fibration.

If $m=3$, then $V_{2,0}$ is 3-dimensional and V_1 is 4-dimensional. By $\{f_{(i)}\}$ we denote a base of V_1 or $V_{2,0}$ (cf. [15], p. 122).

PROPOSITION 2.1. *The vector space of all Killing vector fields on* (S^3, g) *is* *spanned by vector fields dual to*

$$\eta_{(1)}=\eta, \quad \eta_{(2)}, \quad \eta_{(3)}, \tag{2.1}$$

$$2f_{(l)}\eta+df_{(l)}\cdot\phi \qquad f_{(l)}\in V_{2,0},\ l=1, 2, 3. \tag{2.2}$$

These 1*-forms are coclosed eigen* 1*-forms corresponding to the eigenvalue* 4 *of the Laplacian.*

The vector space of all conformal Killing vector fields on (S^3, g) *is spanned by vector fields dual to* (2.1), (2.2) *and*

$$f_{(l)}\eta+df_{(l)}\cdot\phi \qquad f_{(l)}\in V_1,\ l=1, 2, 3, 4. \tag{2.3}$$

These 1*-forms in* (2.3) *are closed eigen* 1*-forms corresponding to the eigenvalue* 3 *of the Laplacian.*

Proof. As for eigen 1-forms, see Lemma 2.5 and Proposition 3.1 in [15]. Here, we identified $\nabla_\xi df$ with $df\cdot\phi$ for $f\in V_{k,0}$. A direct method to see that 1-forms in (2.2) define Killing vector fields is to use $\phi=-\nabla\xi$ and the differential equation

$$\nabla_k\nabla_j\nabla_i f+2\nabla_k f g_{ij}+\nabla_j f g_{ik}+\nabla_i f g_{jk}=0$$

satisfied by $f\in V_2$ (cf. Obata [6], Tanno [12]). To verify that 1-forms in (2.3) define conformal Killing vector fields, we use the fact that each f in V_1 satisfies $\nabla_i\nabla_j f=-fg_{ij}$. q. e. d.

Let $\mathfrak{X}M$ be the set of all smooth vector fields and Λ^1M the set of all smooth 1-forms on a smooth manifold M. By Q we denote the Ricci operator;

$$Q: \mathfrak{X}M\to\mathfrak{X}M \qquad (X=(X^j)\to QX=(R^i_jX^j)),$$

$$Q: \Lambda^1M\to\Lambda^1M \qquad (w=(w_k)\to Qw=(R^k_jw_k)),$$

where (R_{jk}) denotes the Ricci tensor of a Riemannian manifold (M, g).

Let J: $\mathfrak{X}M\to\mathfrak{X}M$ be the Jacobi operator of the identity map as a harmonic map of (M, g) onto (M, g) (Smith [10]). By the natural correspondence between $\mathfrak{X}M$ and Λ^1M, in the following we use $J=-\Delta-2Q: \Lambda^1M\to\Lambda^1M$.

$Q=2I$ holds on (S^3, g), where I denotes the identity. If w is one of 1-forms in (2.1) and (2.2), then $Jw=0$ holds. If w is one of 1-forms in (2.3), then $Jw=-w$ holds. The index Ind(id) is equal to 4 and (2.3) gives a base for the space of eigen 1-forms corresponding to the negative eigenvalues of J. The nullity Null(id) is equal to 6 and (2.1) and (2.2) give a base for the nullity space of J. The decomposition in Proposition 2.1 is naturally related to the changing eigen 1-forms of J corresponding to the deformation (1.1) of the Riemannian metrics on S^3. This situation is explained in Theorem 3.8 in the next section.

The following (i)~(v) are proved in [15];

(i) If $\Delta f+\lambda f=0$ holds on (S^m, g), then

$$\Delta(f\eta)=-(\lambda+2m-2)f\eta+2df\cdot\phi,$$

$$\Delta(df\cdot\phi)=2\lambda f\eta-(\lambda+2)df\cdot\phi+2\nabla_\xi df.$$

(ii) If w is $f\eta$ or $df\cdot\phi$, where $f\in V_1$ for (S^m, g), then $L_\xi L_\xi w=-w$ holds.

(iii) Let $f\in V_{2,0}$ for (S^m, g). Then,

$$L_\xi L_\xi(f\eta)=L_\xi L_\xi(df\cdot\phi)=0.$$

(iv) For a function f on (S^m, g)

$$\phi^{rs}\nabla_r(f\eta_s)=(m-1)f,$$

$$\phi^{rs}\nabla_r(\phi_s^h\nabla_h f)=\Delta f-L_\xi L_\xi f.$$

(v) On $(S^m, g(t))$ with (1.1) the inverse $(g(t)^{rs})$ of $(g(t)_{sj})$, the Christoffel's symbols $\Gamma(t)_{jk}^i$, the Ricci curvature tensor $(R_{jk}^{(t)})$ and the Laplacian $\Delta^{(t)}$ are given by

$$g(t)^{rs}=tg^{rs}-t(1-t^{-m})\xi^r\xi^s, \tag{2.4}$$

$$\Gamma(t)_{jk}^i-\Gamma_{jk}^i=(1-t^m)(\phi_j^i\eta_k+\phi_k^i\eta_j), \tag{2.5}$$

$$R_{jk}^{(t)}=R_{jk}-2(t^m-1)g_{jk}+(t^m-1)(m+1+(m-1)t^m)\eta_j\eta_k, \tag{2.6}$$

$$\Delta^{(t)}w=t\Delta w-t(1-t^{-m})L_\xi L_\xi w-2t(t^m-1)(\phi^{rs}\nabla_r w_s)\eta, \tag{2.7}$$

$$\Delta^{(t)}\eta=-2(m-1)t^{m+1}\eta, \tag{2.8}$$

$$\Delta^{(t)}\eta_{(\alpha)}=-[2(m-3)t+4t^{1-m}]\eta_{(\alpha)} \qquad \alpha=2, 3, \tag{2.9}$$

where $w\in\Lambda^1S^m$.

§3. The Jacobi operator $J(t)$.

LEMMA 3.1. *The Ricci operator $Q^{(t)}$ on $(S^m, g(t))$ satisfies the following;*

$$Q^{(t)}\eta=(m-1)t^{m+1}\eta,$$

$$Q^{(t)}w=t(m+1-2t^m)w,$$

for $w\in\Lambda^1S^m$ such that $w(\xi)=0$.

Proof. By (2.4) and (2.6) we obtain

$$R^{(t)r}{}_i=t(m+1-2t^m)\delta_i^r+(m+1)t(t^m-1)\xi^r\eta_i, \tag{3.1}$$

from which proof is completed.

LEMMA 3.2. *The Jacobi operator $J(t)$ on $(S^m, g(t))$ is given by*

$$J(t)w=-t\Delta w+t(1-t^{-m})L_\xi L_\xi w+2t(t^m-1)(\phi^{rs}\nabla_r w_s)\eta \tag{3.2}$$
$$-2t(m+1-2t^m)w-2(m+1)t(t^m-1)w(\xi)\eta$$

for $w\in\Lambda^1 S^m$.

Proof. (3.2) follows from (2.7), (3.1) and the definition of $J(t)$. q. e. d.

LEMMA 3.3. *Let* $f\in V_1$ *for* (S^m, g) *and put*

$$w(t)=f\eta+a(t)df\cdot\phi, \tag{3.3}$$

where

$$a(t)=\{m-2t^m+[(2t^m-1)^2+m^2-1]^{1/2}\}/2(m-1)t^m.$$

Then, $J(t)w(t)=\mu(t)w(t)$ *holds on* $(S^m, g(t))$, *where*

$$\mu(t)=2t^{m+1}+t^{1-m}-t-t[(2t^m-1)^2+m^2-1]^{1/2}. \tag{3.4}$$

Proof. $J(t)w(t)=\mu(t)w(t)$ is verified by direct calculation, using (3.2), $\nabla_j\nabla_i f=-fg_{ij}$, and relations (i), (ii) and (iv) in § 2. q. e. d.

LEMMA 3.4. *With respect to* $\mu(t)$ *of* (3.4), $\mu(t)<0$ *holds for* $t\in(t_0(m), \infty)$, *where* $t_0(m)$ *satisfies*

$$t_0(m)^m=[(m^2-4)^{1/2}-1]/(m^2-5)$$

and $t_0(3)<t_0(m)<1$. *For example,* $t_0(3)=0.676\cdots$, $t_0(5)=0.708\cdots$, $t_0(7)=0.746\cdots$, *etc.*

Proof. The solution $t_0(m)$ of $\mu(t)=0$ is obtained by calculation. For $1<t$, $\mu(t)<0$ is verified by taking the squares of the both sides of

$$2t^{m+1}+t^{1-m}-t<t[(2t^m-1)^2+m^2-1]^{1/2}.$$

LEMMA 3.5. *Let* $f\in V_{2,0}$ *for* (S^m, g) *and put*

$$w(t)=2f\eta+t^{-m}df\cdot\phi. \tag{3.5}$$

Then, $J(t)w(t)=0$ *holds on* $(S^m, g(t))$. *Furthermore,* $w(t)$ *is coclosed and* $w(t)$ *defines a Killing vector field.*

Proof. $J(t)w(t)=0$ is verified by (3.2) and relations (i), (iii) and (iv) in § 2. Coclosedness of $w(t)$ is verified by (2.4), (2.5) and $\xi f=0$. To verify that $w(t)$ defines a Killing vector field, it suffices to apply the classical integral formula:

$$\langle Jw, w\rangle+\langle\delta w, \delta w\rangle=(1/2)\langle L_X g, L_X g\rangle,$$

where $\langle\ ,\ \rangle$ denotes the global inner product and X denotes the vector field corresponding to w. q. e. d.

LEMMA 3.6. *η on $(S^m, g(t))$ or $\eta_{(\alpha)}$ on $(S^{4r+3}, g(t))$ satisfies the following;*

(i) $J(t)\eta=0$,

(ii) $J(t)\eta_{(\alpha)}=4t(t^m-2+t^{-m})\eta_{(\alpha)} \qquad \alpha=2, 3.$

Proof. (i) corresponds to the fact that ξ is a Killing vector field with respect to $g(t)$ for any $t\in(0, \infty)$. To verify (ii), we apply (2.9) and Lemma 3.1 to $J(t)\eta_{(\alpha)}$. q.e.d.

Summarizing the above we obtain the following.

THEOREM 3.7. *$(S^m, g(t))$, $m=2n+1\geqq 3$, is unstable for $t\in(t_0(m), \infty)$, where $t_0(m)^m=[(m^2-4)^2-1]/(m^2-5)$ and*

$$t_0(3)=0.67\cdots<t_0(m)<1.$$

1-forms given in (3.3) are eigen forms corresponding to the negative eigenvalue $\mu(t)$ of $J(t)$.

The contravariant from of (3.3) is obtained by using (2.4); the result is given in the Theorem in the introduction.

If $m=3$, by the deformation $g\to g(t)$, the eigen forms of $J(0)$ given in Proposition 2.1 are changing as follows;

THEOREM 3.8. *On S^3, as $g\to g(t)$*

(i) *η remains to be an eigen form corresponding to the eigenvalue 0 of $J(t)$,*

(ii) *$\eta_{(\alpha)}(\alpha=2, 3)$ are eigen forms corresponding to the eigenvalue $4t(t^3-2+t^{-3})$ of $J(t)$, which vanishes only at $t=1$,*

(iii) *$2f\eta+t^{-3}df\cdot\phi$, $f\in V_{2,0}$, is an eigen form corresponding to the eigenvalue 0 of $J(t)$,*

(iv) *$4t^3f\eta+\{3-2t^3+[(2t^3-1)^2+8]^{1/2}\}df\cdot\phi$, $f\in V_1$, is an eigen 1-form corresponding to the eigenvalue $2t^4+t^{-2}-t-t[(2t^3-1)^2+8]^{1/2}$ of $J(t)$.*

COROLLARY 3.9. *$Null(id)=6$ for (S^3, g), $Null(id)=4$ for $(S^3, g(t))$ with t near 1 and $t\neq 1$, and $Null(id)\geqq 8$ for $(S^3, g(t_0(3)))$.*

Remark. To understand the situation of the negative eigenvalue of $J(t)$, it may be helpful to know the range of the sectional curvature $K_{(t)}(X, Y)$ of $(S^m, g(t))$. The range is given by the following;

$$t^{m+1}\leqq K_{(t)}(X, Y)\leqq t(4-3t^m) \qquad \text{for} \quad t<1 \tag{3.6}$$

$$t(4-3t^m)\leqq K_{(t)}(X, Y)\leqq t^{m+1} \qquad \text{for} \quad t>1. \tag{3.7}$$

In fact, with respect to a D-homothetic deformation $g\to g^*(\alpha)=\alpha g+(\alpha^2-\alpha)\eta\otimes\eta$, the sectional curvature $K^*_{(\alpha)}(X, Y)$ satisfies

$$1 \leqq K^*_{(\alpha)}(X, Y) \leqq H \quad \text{for} \quad \alpha < 1,$$

$$H \leqq K^*_{(\alpha)}(X, Y) \leqq 1 \quad \text{for} \quad \alpha > 1,$$

where $H=(4-3\alpha)/\alpha$ (cf. Lemma 6.4, (12.1) of [11]). We put $\alpha=t^m$. By a homothetic change $g^*(\alpha)\to t^{-m-1}g^*(t^m)$, we get $g(t)$. Then, the inequalities (3.6) and (3.7) are verified.

For example, if $m=3$, then $(S^3, g(t_0(3))$ is δ-pinched, where $\delta=0.1005\cdots$.

Remark. As for stability or instability of (harmonic mappings of) various Riemannian manifolds, see Howard [1], Howard and Wei [2], Leung [3], [4], Nagano [5], Ohnita [7], Okayasu [8], Pan [9], Urakawa [17], [18], Xin [19], and so on.

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