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REMARKS ON GEOMETRIC PROPERTIES OF CERTAIN COEFFICIENT ESTIMATES

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0. Introduction.

Let Σ_0 denote the class of functions, analytic and univalent in $|z|>1$ with the expansion

$$f(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}.$$

On some coefficient problems, extremal functions are odd or have real coefficients. Leung and Schober [5, Lemma 2.3] proved that the extremal function for the problem $\max_{\Sigma_0} \mathrm{Re}(b_3+\lambda b_1)$ must be odd. And Jenkins [2, §6] presented a very simple proof of the inequality $|A_3|\leqq 3$ in the familiar class S by showing that the extremal function for the problem $\max_S \mathrm{Re}\, A_3$ has real coefficients.

We represent such facts in terms of quadratic differentials by making use of Jenkins' General Coefficient Theorem. Then we give two applications. One is a simple proof of the fact that the third Ozawa number $B_3=3$ [1], [3] where $B_3=\inf\{t:\mathrm{Re}(tb_1-b_3)\leqq t$ for all $f\in\Sigma_0\}$ [6]. The other is the coefficient inequality for the coefficient functional $b_3+(1/2)b_1^2+\lambda b_2$ with real λ.

1. Quadratic differentials.

We use the following two special cases of Jenkins' General Coefficient Theorem (e.g. [7, Theorem 8.12]).

LEMMA 1.1. *Let $\phi(w)=w+a_1w^{-1}+a_2w^{-2}+\cdots$ be univalent and admissible for the quadratic differential $Q(w)dw^2=(A_0w+A_1)dw^2$, $(A_0\neq 0)$. Then*

$$\mathrm{Re}(A_0a_2+A_1a_1)\leqq 0. \tag{1.1}$$

If equality holds in (1.1), *then $a_1=0$.*

LEMMA 1.2. *Let $\phi(w)=w+a_1w^{-1}+a_2w^{-2}+a_3w^{-3}+\cdots$ be univalent and admissible for the quadratic differential $Q(w)dw^2=(A_0w^2+A_1w+A_2)dw^2$, $(A_0\neq 0)$.*

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Then

$$\operatorname{Re}\left(A_0a_3+A_1a_2+A_2a_1+\frac{1}{2}A_0a_1^2\right)\leqq 0. \tag{1.2}$$

If equality holds in (1.2), *then* $2A_0a_2+A_1a_1=0$.

The lemma of Leung-Schober [5, Lemma 2.3] can be slightly generalized as follows.

THEOREM 1.3. *Let* $f(z)=z+\sum_{n=1}^{\infty}b_nz^{-n}$ *be in* Σ_0. *If* $\hat{C}-f(|z|>1)$ *is on the trajectory arcs of the quadratic differential* $Q(w)dw^2=(A_0w^2+A_2)dw^2$, $(A_0\neq 0)$, *then* $f(z)$ *is odd.*

Since this can be proved by the same technique as theirs, we omit the proof.

COROLLARY 1.4. *Put the coefficient functional*

$$L(f)=\sum_{m=1}^{M}\alpha_mb_1^m+(\sum_{n=0}^{N}\beta_nb_1^n)b_3,$$

$f(z)=z+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+\cdots$, *where* α_m *and* β_n *are complex constants. Then*

$$\max_{\Sigma_0}\operatorname{Re} L(f)=\max_{f:\mathrm{odd}\in\Sigma_0}\operatorname{Re} L(f).$$

Proof. Let $g(z)=z+\sum_{n=1}^{\infty}b_nz^{-n}$ be an extremal function for the problem $\max_{\Sigma_0}\operatorname{Re} L(f)$. If $\sum_{n=0}^{N}\beta_nb_1^n=0$, then $\max_{\Sigma_0}\operatorname{Re} L(f)=\operatorname{Re} L(g)=\operatorname{Re}(\sum_{m=1}^{M}\alpha_mb_1^m)$. Putting $\rho=|b_1|^{-1/2}$ and $\theta=-(1/2)\arg(b_1)$, we have $\rho^{-1}e^{-i\theta}k(\rho e^{i\theta}z)=z+\rho^{-2}e^{-2i\theta}z^{-1}=z+b_1z^{-1}$, where $k(z)=z+z^{-1}$. Hence we have $\max_{\Sigma_0}\operatorname{Re} L(f)=\operatorname{Re} L(g)=\operatorname{Re} L(\rho^{-1}e^{-i\theta}k(\rho e^{i\theta}z))=\max_{f:\mathrm{odd}\in\Sigma_0}\operatorname{Re} L(f)$. Now a ssume that $\sum_{n=0}^{N}\beta_nb_1^n\neq 0$. Then the Gâteaux differential of $L(\cdot)$ at g is given by

$$l(h)=(\sum_{n=0}^{N}\beta_nb_1^n)c_3+(\sum_{m=1}^{M}m\alpha_mb_1^{m-1}+b_3(\sum_{n=1}^{N}n\beta_nb_1^{n-1}))c_1,$$

$h(z)=z+c_1z^{-1}+c_2z^{-2}+c_3z^{-3}+\cdots$. Thus the omitted set of $w=g(z)$ lies on the trajectory arcs of the quadratic differential

$$l\left(\frac{1}{g-w}\right)dw^2=[(\sum_{n=0}^{N}\beta_nb_1^n)(w^2-b_1)+(\sum_{m=1}^{M}m\alpha_mb_1^{m-1}+b_3(\sum_{n=1}^{N}n\beta_nb_1^{n-1}))]dw^2$$

(e.g. [8]).

Hence we know that $g(z)$ is an odd function by the above theorem. Thus we have the desired result.

Next we give the real coefficients case.

THEOREM 1.5. *Let* $f(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$ *be in* Σ_0. *Let* $\hat{\boldsymbol{C}}-f(|z|>1)$ *be on the trajectory arcs of the quadratic differential* $Q(w)dw^2=(A_0w^2+A_1w+A_2)dw^2$, $(A_0, A_1, A_2 \in \boldsymbol{R})$. *If one of the following conditions is satisfied, then* $\overline{f(\bar{z})}=f(z)$.

1) $A_0 \geqq 0$

2) $A_1=0$ *and* $|A_2| \geqq 4|A_0|$

3) $A_2=0$ *and* $|A_1| \geqq 4|A_0|$

Proof. Case 1.1) $A_0=0$ and $A_1=0$. Then $Q(w)dw^2=A_2dw^2$. It is easy to see that $f(z)=z+z^{-1}$ when $A_2>0$ and $f(z)=z-z^{-1}$ when $A_2<0$.

Case 1.2) $A_0=0$ and $A_1 \neq 0$. Then $Q(w)dw^2=(A_1w+A_2)dw^2$. By the assumption and Schwarz reflection principle we have

$$(1.3) \qquad (A_1f(z)+A_2)z^2f'(z)^2$$
$$=A_1z^3+A_2z^2-b_1A_1z-(3b_2A_1+2b_1A_2)-\bar{b}_1A_1z^{-1}+A_2z^{-2}+A_1z^{-3}.$$

We put $\phi(w)=\overline{f(\overline{f^{-1}(w)})}=w+(b_1-\bar{b}_1)w^{-1}+(b_2-\bar{b}_2)w^{-2}+(b_3-\bar{b}_3+\bar{b}_1(b_1-\bar{b}_1))w^{-3}+\cdots$. Applying Lemma 1.1 to the pair of $\phi(w)$ and $(A_1w+A_2)dw^2$, we have

$$b_1=\bar{b}_1.$$

Hence the coefficients of the right hand side of (1.3) are real. Comparing the coefficients of both sides of (1.3) we know that all b_n are real.

Case 1.3) $A_0>0$. By the assumption and Schwarz reflection principle we have

$$(1.4) \qquad \begin{aligned}(A_0f(z)^2+A_1f(z)+A_2)z^2f'(z)^2&=A_0z^4+A_1z^3+A_2z^2-(2b_2A_0+b_1A_1)z\\ &-(4b_3A_0+3b_2A_1+2b_1A_2+2b_1^2A_0)-(2\bar{b}_2A_0+\bar{b}_1A_1)z^{-1}+A_2z^{-2}+A_1z^{-3}+A_0z^{-4}.\end{aligned}$$

We denote the right hand side of (1.4) by $z^{-4}q(z)$. Applying Lemma 1.2 to the pair of $\phi(w)$ (see Case 1.2)) and $(A_0w^2+A_1w+A_2)dw^2$, the left hand side of (1.2) becomes $\mathrm{Re}(A_0(b_3-\bar{b}_3)+A_1(b_2-\bar{b}_2)+A_2(b_1-\bar{b}_1)+(1/2)A_0(b_1^2-\bar{b}_1^2))=0$. Thus

$$2b_2A_0+b_1A_1=2\bar{b}_2A_0+\bar{b}_1A_1.$$

This means that the coefficients of $q(z)$ are all real. By $w=f(z)$ (1.4) becomes

$$(1.5) \qquad (A_0w^2+A_1w+A_2)dw^2=z^{-6}q(z)dz^2.$$

It follows from this equation that $(-\infty, -1)$ and $(1, +\infty)$, the components of the real axis in $|z|>1$, are mapped by $w=f(z)$ onto trajectory or orthogonal trajectory arcs of $(A_0w^2+A_1w+A_2)dw^2$ and that $w=f(z)$ is on a trajectory of $(A_0w^2+A_1w+A_2)dw^2$ for all sufficiently large real z because $A_0>0$. Since the conformal center $\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta})d\theta=0$, the omitted set $\Gamma=f(|z|=1)$ must contain the origin. In addition, $f((-\infty, -1))$ and $f((1, +\infty))$ are running from Γ to ∞. These can be possible only when $f((-\infty, -1))$ and $f((1, +\infty))$ are on the real axis. Hence b_n are all real.

Case 2) $A_0<0$, $A_1=0$ and $|A_2|\geqq 4|A_0|$. Then $Q(w)dw^2=(A_0w^2+A_2)dw^2$. The distance between the critical points $\pm\sqrt{-A_2/A_0}$ is $2|\sqrt{-A_2/A_0}|\geqq 4$. Moreover, $f(z)$ is odd by Theorem 1.3. Hence it follows that $f(z)=z+z^{-1}$ when $A_2>0$ and $f(z)=z-z^{-1}$ when $A_2<0$.

Case 3) $A_0<0$, $A_2=0$ and $|A_1|\geqq 4|A_0|$. Then $Q(w)dw^2=(A_0w^2+A_1w)dw^2$. Since $\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta})d\theta=0$ and $|-A_1/A_0|\geqq 4$, the omitted set $\Gamma=f(|z|=1)$ contains the origin and does not contain $-A_1/A_0$. There is z_0 on the real axis in $|z|>1$ such that $f(z_0)=-A_1/A_0$ because the right hand side of (1.5) has real coefficients. Let J be one of the components of real axis in $|z|>1$ such that $z_0\in J$. Then $f(J)$ is a smooth Jordan arc which is always on trajectory or orthogonal trajectory arcs of $Q(w)dw^2$ and goes from Γ to ∞ via $-A_1/A_0$. This can be possible only when $f(J)$ is on the real axis. Hence b_n are all real. This completes the proof.

COUNTEREXAMPLE. Suppose that $Q(w)dw^2=(A_0w^2+A_2)dw^2$ with $A_0, A_2\in \boldsymbol{R}$, $A_0<0$ and $|A_2|<4|A_0|$. Let Γ be a continuum symmetric with respect to the origin and not symmetric with respect to the real axis which consists of the segment $[-\sqrt{-A_2/A_0}, \sqrt{-A_2/A_0}]$, which may degenerate, together with the trajectory arcs from $\pm\sqrt{-A_2/A_0}$. Since the distance between the critical points $\pm\sqrt{-A_2/A_0}$ is $2|\sqrt{-A_2/A_0}/<4$, we can take Γ with transfinite diameter 1. Hence there is an odd and not real coefficient function in the class Σ_0 whose omitted set is on the trajectory arcs of the quadratic differential $Q(w)dw^2$. The necessity of its oddness is known by Theorem 1.3, too.

2. Coefficient estimates.

It is known that the third Ozawa number $B_3=3$ by the results of Garabedian and Schiffer [1] and Kirwan and Schober [3]. Now we give its direct proof by making use of Löwner's method (e.g. [7, Chapter 6]).

THEOREM 2.1.

$$\operatorname*{Max}_{\Sigma_0} \operatorname{Re}(\lambda b_1-b_3)=\begin{cases} \lambda & \textit{for } 3\leqq\lambda \\ \lambda(t_0+1)e^{-t_0}+\dfrac{1}{2}+\left(\dfrac{1}{2}t_0^2-t_0-\dfrac{1}{2}\right)e^{-2t_0} & \textit{for } 0\leqq\lambda<3\,, \end{cases}$$

where t_0 *is the root of* $(3-t)e^{-t}=\lambda$.

Proof. It is sufficient to examine only odd functions in Σ_0 by Corollary 1.4. If $f(z)=z+b_1z^{-1}+b_3z^{-3}+\cdots$ is an odd function in Σ_0, then $f(z^{-1/2})^{-2}=z-2b_1z^2-(2b_3-3b_1^2)z^3+\cdots$ belongs to the familiar class S. We assume that this function has Löwner's coefficient representations, that is to say,

$$-2b_1=-2\int_0^\infty e^{-t}e^{i\theta(t)}dt \quad \text{and}$$

$$-(2b_3-3b_1^2)=-2\int_0^\infty e^{-2t}e^{2i\theta(t)}dt+4\left(\int_0^\infty e^{-t}e^{i\theta(t)}dt\right)^2$$

where $\theta(t)$ is a continuous function on $(0, \infty)$. Then we have

$$\begin{aligned}\operatorname{Re}(\lambda b_1-b_3)&=\lambda\int_0^\infty e^{-t}\cos\theta(t)dt-\int_0^\infty e^{-2t}\cos 2\theta(t)dt\\ &\quad+\frac{1}{2}\left(\left(\int_0^\infty e^{-t}\cos\theta(t)dt\right)^2-\left(\int_0^\infty e^{-t}\sin\theta(t)dt\right)^2\right)\\ &\leqq\lambda\int_0^\infty e^{-t}\cos\theta(t)dt+\frac{1}{2}-2\int_0^\infty e^{-2t}\cos^2\theta(t)dt+\frac{1}{2}\left(\int_0^\infty e^{-t}\cos\theta(t)dt\right)^2.\end{aligned}$$

If we put $\int_0^\infty e^{-2t}\cos^2\theta(t)dt=\left(t+\frac{1}{2}\right)e^{-2t}$ for some t, $0\leqq t<\infty$, then it follows from Valiron-Landau Theorem [4] that

$$\operatorname{Re}(\lambda b_1-b_3)\leqq\lambda(t+1)e^{-t}+\frac{1}{2}-2\left(t+\frac{1}{2}\right)e^{-2t}+\frac{1}{2}(t+1)^2e^{-2t}\equiv\phi(t)\,.$$

Then $(d/dt)\phi(t)=te^{-t}((3-t)e^{-t}-\lambda)$. Hence $(d/dt)\phi(t)\leqq 0$ for all $t\geqq 0$ if $3\leqq\lambda$. Thus we have $\operatorname{Re}(\lambda b_1-b_3)\leqq\phi(t)\leqq\phi(0)=\lambda$ for $3\leqq\lambda$. Assume that $0\leqq\lambda<3$. In this case $\phi(t)\leqq\phi(t_0)$ for t_0 such that $(3-t_0)e^{-t_0}-\lambda=0$. Hence we have

$$\operatorname{Re}(\lambda b_1-b_3)\leqq\lambda(t_0+1)e^{-t_0}+\frac{1}{2}+\left(\frac{1}{2}t_0^2-t_0-\frac{1}{2}\right)e^{-2t_0} \qquad \text{for } 0\leqq\lambda<3\,.$$

If we take a piecewise continuous function $\nu(t)$ such that

$$\cos\nu(t)=\begin{cases} e^{t-t_0} & \text{for } 0\leqq t\leqq t_0 \\ 1 & \text{for } t_0<t<\infty \end{cases} \quad \text{and} \quad \sin\nu(t)=\begin{cases} (1-e^{2(t-t_0)})^{1/2} & \text{for } 0\leqq t<t_1 \\ -(1-e^{2(t-t_0)})^{1/2} & \text{for } t_1\leqq t\leqq t_0 \\ 0 & \text{for } t_0<t<\infty \end{cases}$$

where t_1 is determined by the condition $\int_0^\infty e^{-t}\sin\nu(t)dt=0$, then $e^{i\nu(t)}$ generates a function h, which belongs to the class S, whose square root inversion transformation $h(z^{-2})^{-1/2}$ is an extremal function for $\max_{\Sigma_0}\mathrm{Re}(\lambda b_1-b_3)=\phi(t_0)$. This completes the proof.

Next we give an application of real coefficients case. It is well known that $|b_3+(1/2)b_1^2|\leqq 1/2$ and the extremal functions are odd. The oddness of them is found in Corollary 1.4, too. The following estimate complements it in a sense.

THEOREM 2.2. *Let $\lambda>0$. Then*

$$\max\mathrm{Re}\Big(b_3+\frac{1}{2}b_1^2+\lambda b_2\Big)=\frac{17}{864}\lambda^4-\frac{4}{27}\lambda^3+\frac{2}{9}\lambda^2+\frac{8}{27}\lambda+\frac{11}{54}$$
$$-\frac{\lambda^4}{64}\log\Big\{\frac{1}{3\lambda}(\lambda-4+2\sqrt{\lambda^2-2\lambda+4})\Big\}-\Big(\frac{17}{864}\lambda^3-\frac{7}{72}\lambda^2+\frac{1}{9}\lambda-\frac{4}{27}\Big)\sqrt{\lambda^2-2\lambda+4}\,.$$

Extremal function is unique.

Proof. Let $f(z)$ be an extremal function. Then its omitted set $\hat{\boldsymbol{C}}-f(|z|>1)$ is on the trajectory arcs of the quadratic differential $Q(w)dw^2=w(w+\lambda)dw^2$. So $f(z)$ must have real coefficients by Theorem 1.5 and its omitted set consists of three arcs emanating from the origin. Thus it follows by Schwarz reflection principle that $f(z)(f(z)+\lambda)z^2f'(z)^2$ has double zeros at the points 1, $e^{i\alpha}$ and $e^{-i\alpha}$, for some real α, which correspond to the three tips, and simple zeros at the points $-r$ and $-r^{-1}$, for some $r>1$, which correspond to the point $-\lambda$. So we can put

$$\begin{aligned}&f(z)(f(z)+\lambda)z^2f'(z)^2\\ &=z^4+\lambda z^3-(\lambda b_1+2b_2)z-2\Big(2b_3+b_1^2+\frac{3}{2}\lambda b_2\Big)-(\lambda b_1+2b_2)z^{-1}+\lambda z^{-3}+z^{-4}\\ &=z^{-4}[(z-1)(z-e^{i\alpha})(z-e^{-i\alpha})]^2(z+r)(z+r^{-1})\end{aligned}\tag{2.1}$$

for some real α and r $(r>1)$. A comparison of coefficients gives

$$\lambda=-4\cos\alpha+2(R-1)\,,\tag{2.2}$$

$$\cos^2\alpha-2(R-1)\cos\alpha-(R-1)=0\,,\tag{2.3}$$

$$\lambda b_1+2b_2=-4\cos\alpha-2(R-1)(2\cos\alpha+1)(2\cos\alpha+3)\quad\text{and}\tag{2.4}$$

$$2b_3+b_1^2+\frac{3}{2}\lambda b_2=4\cos^2\alpha+1+2(R-1)(4\cos^2\alpha+4\cos\alpha+2)\tag{2.5}$$

with $R=(r+r^{-1})/2$.

We integrate (2.1) by using the correspondence $0=f(-1)$,

$$\int_0^w \sqrt{w(w+\lambda)}dw=\int_{-1}^z z^{-3}(z-1)(z-e^{i\alpha})(z-e^{-i\alpha})\sqrt{(z+r)(z+r^{-1})}dz\,.$$

Then it follows that

$$\frac{1}{4}(\lambda+2w)\sqrt{w(w+\lambda)}+\frac{\lambda^2}{8}\log\{(\sqrt{w+\lambda}-\sqrt{w})/(\sqrt{w+\lambda}+\sqrt{w})\}$$
$$=F(z+\sqrt{(z+r)(z+r^{-1})})-F(-1+\sqrt{(-1+r)(-1+r^{-1})}), \tag{2.6}$$

where

$$F(t)=-\frac{a}{2}(t-1)^{-2}-b(t-1)^{-1}+A\log(t-1)$$
$$-\frac{m}{2}(t+1)^{-2}-n(t+1)^{-1}+B\log(t+1)$$
$$-\frac{p}{16}(t+R)^{-2}-\frac{q}{4}(t+R)^{-1}+\frac{C}{2}\log(2(t+R))+\frac{t^2}{8}+kt$$

with

$$a=-(R+1)^2,\quad b=\frac{1}{2}(R+1)(4\cos\alpha-R+1),\quad A=2(R-1)\cos\alpha+\frac{1}{2}(R+1)^2-2,$$

$$m=(R-1)^2,\quad n=\frac{1}{2}(R-1)(4\cos\alpha-R+3),\quad B=-\frac{1}{2}(R-1)(4\cos\alpha+R+3)\,,$$

$$p=2(R^2-1)^2,\quad q=-2(R^2-1)(2\cos\alpha+1),\quad C=-(R-1)(4\cos\alpha+R+3)$$

and

$$k=-\frac{1}{4}(4\cos\alpha-R+2)\,.$$

By (2.3) $\cos^2\alpha=(R-1)(1+2\cos\alpha)$. Since $R=(r+r^{-1})/2>1$, we have $\cos\alpha>-1/2$. Hence it follows from (2.2) and (2.3) that

$$\cos\alpha=(-\lambda-2+\sqrt{\lambda^2-2\lambda+4})/6 \quad\text{and}\quad R=(\lambda+2+2\sqrt{\lambda^2-2\lambda+4})/6\,. \tag{2.7}$$

We substitute $w=z+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+\cdots$ into the left hand side of (2.6) and expand both sides of it around $z=\infty$. Then we obtain, using (2.7),

$$b_1=-\frac{\lambda^2}{12}+\frac{2}{3}\lambda-\frac{1}{3}+\frac{\lambda^2}{8}\log\left\{\frac{1}{3\lambda}(\lambda-4+2\sqrt{\lambda^2-2\lambda+4})\right\}-\frac{1}{3}\left(1-\frac{\lambda}{4}\right)\sqrt{\lambda^2-2\lambda+4}$$

by comparing the constant terms. By this relation, (2.4), (2.5) and (2.7) we obtain the desired estimate for $b_3+(1/2)b_1^2+\lambda b_2$. Expanding the left hand side of (2.1), we know that all of the coefficients of $f(z)=z+\sum_{n=1}^{\infty}b_nz^{-n}$ are represented in terms of λ. Thus the extremal function is unique.

REFERENCES

[1] GARABEDIAN, P.R. AND M. SCHIFFER, A coefficient inequality for schlicht functions, Ann. of Math. **61** (1955), 116-136.

[2] JENKINS, J.A., On certain extremal problems for the coefficients of univalent functions, J. Analyse Math. **18** (1967), 173-184.

[3] KIRWAN, W.E., AND G. SCHOBER, New inequalities from old ones, Math. Z. **180** (1982) 19-40.

[4] LANDAU, E., Über die Blochsche Konstante und zwei verwandte Weltkonstanten, Math. Z. **30** (1929), 608-634.

[5] LEUNG, Y.J. AND G. SCHOBER, Low order coefficient estimates in the class Σ, Ann. Acad. Sci. Fenn. AI Math. **11** (1986), 39-61.

[6] OZAWA, M., Coefficient estimates for the class Σ, Kodai Math. J. **9** (1986), 123-133.

[7] POMMERENKE, Ch., *Univalent Functions* (With a chapter on quadratic differentials by G. Jensen), Vandenhoeck & Ruprecht in Göttingen, 1975.

[8] SCHOBER, G., *Univalent Functions—Selected Topics*. Lecture Notes in Mathematics 478, Springer Verlag, Berlin·Heidelberg·New York, 1975.

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