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KODAI MATH. J.
10 (1987), 231—241

# CALCULATION OF DENSITY FOR THE MULTIVARIATE POISSON DISTRIBUTION

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## §1. Summary.

On the bivariate case, each Poisson density has five different recurrence relations, in the followin lines, we denote them as $r.r$'s. and conversely, if we need to calculate the density, we can omit one of the two main relations. On the trivariate case, there are 13 different $r.r$'s. Let us put the $r.r$'s in the concrete, on the bivariate case.

It is known that a density function of Poisson distribution $p(k, l)$ $(k, l=0, 1, 2, \cdots)$ satisfies two main recurrence relations:

$$k\ p(k, l)=\lambda_{10}p(k, 1-l)+\lambda_{11}p(k-1, l-1)\,,$$

$$l\ p(k, l)=\lambda_{01}p(k-1, l)+\lambda_{11}p(k-1, l-1)\,,$$

Conversely, if we need to calculate the density using the $r.r$'s., not using the direct calculation, one of the relations would be meaning less except in the usage of its reduced relations:

$$k\ p(k, 0)=\lambda_{10}p(k-1, 0)\,, \qquad l\ p(0, l)=\lambda_{01}p(0, l-1)$$

$$\text{and} \quad p(0, 0)=\exp(-\lambda_{10}-\lambda_{01}-\lambda_{11})\,.$$

On the trivariate case, we have three main recurrence relations ($r.r$'s.):

$$k\ p(k, l, m)=\lambda_{100}p(k-1, l, m)+\lambda_{110}p(k-1, l-1, m)$$
$$+\lambda_{101}p(k-1, l, m-1)+\lambda_{111}p(k-1, l-1, m-1)\,,$$

$$l\ p(k, l, m)=\lambda_{010}p(k, l-1, m)+\lambda_{110}p(k-1, l-1, m)$$
$$+\lambda_{011}p(k, l-1, m-1)+\lambda_{111}p(k-1, l-1, m-1)\,,$$

$$m\ p(k, l, m)=\lambda_{001}p(k, l, m-1)+\lambda_{101}p(k-1, l, m-1)$$
$$+\lambda_{011}p(k, l-1, m-1)+\lambda_{111}p(k-1, l-1, m-1)\,.$$

To calculate the density $p(k, l, m)$ from the $r.r$'s. we need one of the rela-

Received December 12, 1986

tion and their reduced relations.

In this paper we will express the $r.r$'s. for the multivarivate case and it will be shown that if we need to calculate the density we use only one of the $n$ main relations and their reduced relations. Because they include the other main relations as shown in the last section of this paper.

## §2. Notations and Definitions.

$p(k\,;\lambda)$: univariate Poisson density with parameter $\lambda$

$\boldsymbol{i}=(i_1, i_2, \cdots, i_n)$ $i_j=0$ or $1$ $(j=1, 2, \cdots, n)$

$\boldsymbol{k}=(k_1, k_2, \cdots, k_n)$ $k_j\geqq 0$ nonnegative integer for $j=1, 2, \cdots, n$

$\boldsymbol{O}=(0, 0, \cdots, 0)$

$\boldsymbol{E}_0=\{0, 1\}^n$, $\boldsymbol{E}=\{0, 1\}^n-0$

$p(\boldsymbol{k})$: $n$-variate Poisson density

$\alpha_i$, $\beta_i$: nonnegative integers

$\alpha=\{\alpha_i, \boldsymbol{i}\in E\}$ vector with nonnegative components $\alpha_i$ for some ordering of $i$, say, the linary scale.

$[C(\boldsymbol{k})]$: a restriction of $\alpha$ depending only on $k$ a set of $\alpha$

$\boldsymbol{p}=\{p_i\,; \boldsymbol{i}\in E_0\}$ vector with $2^n$ components of probabilities such that $\sum_{i\in E_0} p_i=1$ (*)

$\lambda=\{\lambda_i\,; \boldsymbol{i}\in E\}$ vector with $2^n-1$ nonnegative parameters.

$\sharp\boldsymbol{k}$: the number of positive components of a vector $\boldsymbol{k}$

—calculable, $p_0$-calculable—

$p(\boldsymbol{k})$ is calculable on $S$: If we can calculate all the values $p(\boldsymbol{k})$ for $\boldsymbol{k}\in S$ from some prescribed relations then we call $p(\boldsymbol{k})$ to be "calculable on $S$" or simply "calculable". In the same way, if we can calculate all the values $p(\boldsymbol{k})$ for $\boldsymbol{k}\in S-(0, 0)$ from some prescribed relations except the boundary condition indicating the value of origin $p_0$, we call $p(\boldsymbol{k})$ for $\boldsymbol{k}\in S-(0, 0)$ to be "$p_0$-calculable on $S$" or simply "$p_0$-calculable".

$[C(\boldsymbol{k})]$: a set of vectors $\alpha=\{\alpha_i, i\in E\}$ with a restriction listed as below, and we will shorten it as $[C]$

$$[C]=[C(\boldsymbol{k})]=\{\alpha\,;\ \sum_{i\neq 0} i\alpha_i=\boldsymbol{k}\}$$

where $\boldsymbol{i}\neq 0$ means $\boldsymbol{i}\in E$ and $0\neq\boldsymbol{i}\leqq\boldsymbol{k}$ and $\sum_{[C]} u_\alpha$ means the sum of $u_\alpha$ where $\alpha$ varies on the set $[C]$.

$n$-variate Poisson density:

$$p(\boldsymbol{k})=\sum_{[C]}\prod_{\boldsymbol{i}\in E} p(\alpha_i\,;\lambda_i)\,,$$

we denote this distribution as $P(\lambda)$. If $X$ is a $n$-variate random vector with the distribution $P(\lambda)$ then we can express $X$ as

$$X=\sum_{\boldsymbol{i}\in E}\boldsymbol{i}X_{\boldsymbol{i}}$$

where $X_{\boldsymbol{i}}$ is an univariate Poisson distribution $P(\lambda_{\boldsymbol{i}})$ and $X_{\boldsymbol{i}}$ are mutually independent for $\boldsymbol{i}\in E$, see Kawamura [4].

$\boldsymbol{k}\geqq\boldsymbol{i}$: A relation of two vectors of both sides, which means any two corresponding components $k_j$ and $i_j$ satisfy $\boldsymbol{k}_j\geqq\boldsymbol{i}_j$. We use the relation as $0\leqq\boldsymbol{i}\leqq\boldsymbol{k}$.

## § 3. Introduction.

On the bivariate case, the density function of $P(\lambda)$ is expressed as

$$p(\boldsymbol{k},\boldsymbol{l})=\sum_{\delta=0}^{\boldsymbol{k}\triangle\boldsymbol{l}}P(\boldsymbol{k}-\delta;\lambda_{10})P(\boldsymbol{l}-\delta;\lambda_{01})P(\delta;\lambda_{11})$$

for any nonnegative integers $k$, $l$ and we used $\boldsymbol{k}\triangle\boldsymbol{l}=\min(\boldsymbol{k},\boldsymbol{l})$, see Kawamura [2] and Polak [7].

The density function $p(\boldsymbol{k},\boldsymbol{l})$: a nonnegative function defined on the lattice points of the first quadrant including two axes, satisfies following recurrence relations, see Johnson and Kotz [1].

$$\boldsymbol{k}\ p(\boldsymbol{k},\boldsymbol{l})=\lambda_{10}p(\boldsymbol{k}-1,\boldsymbol{l})+\lambda_{11}p(\boldsymbol{k}-1,\boldsymbol{l}-1)$$

$$\boldsymbol{l}\ p(\boldsymbol{k},\boldsymbol{l})=\lambda_{01}p(\boldsymbol{k},\boldsymbol{l}-1)+\lambda_{11}p(\boldsymbol{k}-1,\boldsymbol{l}-1)$$

It is better to express the relations as followings, because of they don't include the reduced forms of at least one of $\boldsymbol{k}$ and $l$ equals zero.

$$\boldsymbol{k}=\boldsymbol{l}=0 \qquad p(0,0)=\exp(-\lambda_{10}-\lambda_{01}-\lambda_{11}) \tag{1}$$

$$\boldsymbol{k}\geqq1,\ \boldsymbol{l}=0 \qquad \boldsymbol{k}\ p(\boldsymbol{k},0)=\lambda_{10}p(\boldsymbol{k}-1,0) \tag{2}$$

$$\boldsymbol{k}=0,\ \boldsymbol{l}\geqq1 \qquad \boldsymbol{l}\ p(0,\boldsymbol{l})=\lambda_{01}p(0,\boldsymbol{l}-1) \tag{3}$$

$$\boldsymbol{k},\ \boldsymbol{l}\geqq1 \qquad \boldsymbol{k}\ p(\boldsymbol{k},\boldsymbol{l})=\lambda_{10}p(\boldsymbol{k}-1,\boldsymbol{l})+\lambda_{11}p(\boldsymbol{k}-1,\boldsymbol{l}-1) \tag{4}$$

$$\boldsymbol{k},\ \boldsymbol{l}\geqq1 \qquad \boldsymbol{l}\ p(k,\boldsymbol{l})=\lambda_{01}p(\boldsymbol{k},\boldsymbol{l}-1)+\lambda_{11}p(\boldsymbol{k}-1,\boldsymbol{l}-1). \tag{5}$$

The first equality (1) is not a relation but explains the density. If we need to calculate the density $p(\boldsymbol{k},\boldsymbol{l})$ $(\boldsymbol{k},l\geqq0)$, we can calculate $p(0,0)$ by (1) and $p(\boldsymbol{k},0)$ $(\boldsymbol{k}\geqq1)$ are calculable from (2) and the induction for $k$ and also $p(0,l)$ $(\boldsymbol{l}\geqq1)$ are calculable from the same argument. For $p(\boldsymbol{k},\boldsymbol{l})$ $(\boldsymbol{k},\boldsymbol{l}\geqq1)$ we need one of the main relations (4) and (5). That is, we don't need one of the other relation to prove the calculability of the density $p(\boldsymbol{k},\boldsymbol{l})$ $(\boldsymbol{k},\boldsymbol{l}\geqq0)$, see Kawamura [5].

In the case of trivariate Poisson distribution $P(\lambda)$. The density is written as

$$p(\boldsymbol{k})=p(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=\sum_{[C]}\prod_{\boldsymbol{i}\neq0}p(\alpha_{\boldsymbol{i}},\lambda_{\boldsymbol{i}}) \qquad (\boldsymbol{k}\geqq0)$$

for any pair of nonnegative integers $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, see for Kawamura [3] and Liu [6], where $\boldsymbol{k}=(k_1, k_2, k_3)$ and $\boldsymbol{k}\geqq 0$ means every component $k_j$ $(j=1, 2, 3)$ is nonnegative. And for given $\boldsymbol{k}\geqq 0$, $[C]$ means a set of pair $\alpha=\{\alpha_i\,;\,\boldsymbol{i}\in E\}$ having a restriction listed as follows

$$[C]=\{\alpha\,;\ \sum_{\boldsymbol{i}\neq 0}\boldsymbol{i}\alpha_{\boldsymbol{i}}=\boldsymbol{k}\}$$

and $\sum_{[C]} u_\alpha$ means the sum of $u_\alpha$ where $\alpha$ varies on the set $[C]$. Another expression for $[C]$:

$$\sum_{\boldsymbol{i}\neq 0}\boldsymbol{i}\alpha_{\boldsymbol{i}}=k \;\langle=\rangle\; \sum_{\boldsymbol{i}_1=1}\boldsymbol{i}_1\alpha_{\boldsymbol{i}}=\boldsymbol{k}_1,\quad \sum_{\boldsymbol{i}_2=1}\boldsymbol{i}_2\alpha_{\boldsymbol{i}}=\boldsymbol{k}_2,\quad \sum_{\boldsymbol{i}_3=1}\boldsymbol{i}_3\alpha_{\boldsymbol{i}}=\boldsymbol{k}_3$$

$$\langle=\rangle\; \sum_{\boldsymbol{i}_j=1}\alpha_{\boldsymbol{i}}=\boldsymbol{k}_j \quad (j=1, 2, 3)\,.$$

THEOREM 3-1. *For given trivariate Poisson density $p(\boldsymbol{k})$ $(\boldsymbol{k}\geqq 0)$, we have an expression of the recurrence relations*

$$\boldsymbol{k}=0\,;\quad p(0)=\prod_{\boldsymbol{i}\in E}p(0\,;\,\lambda_{\boldsymbol{i}})=\exp\big(-\sum_{\boldsymbol{i}\in E}\lambda_{\boldsymbol{i}}\big) \qquad (1),$$

$$\boldsymbol{k}\neq 0\,;\quad \boldsymbol{k}_j\ p(\boldsymbol{k})=\sum_{\boldsymbol{i}_j=1,\,\boldsymbol{i}\leq\boldsymbol{k}}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i})$$

*for $j$ such that $\boldsymbol{k}j\geqq 1$ $(j=1, 2, \cdots, n)$.*

We can combine the last equation as

$$\boldsymbol{k}\ p(\boldsymbol{k})=\sum_{\boldsymbol{i}\in E,\,\boldsymbol{i}\leq\boldsymbol{k}}\boldsymbol{i}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-i) \qquad (2)\sim(13).$$

We have a concrete expression of the recurrence relations as given in the next corollary.

COROLLARY 3-1. *For given trivariate Poisson density $p(\boldsymbol{k})$ $(\boldsymbol{k}\geqq 0)$, we have a concrete expression of the r.r's:*

$$p(0, 0, 0)=\exp(-\lambda_{100}-\lambda_{010}-\lambda_{001}-\lambda_{110}-\lambda_{101}-\lambda_{011}-\lambda_{111}) \qquad (1)$$

$$\boldsymbol{k}_1\geqq 1 \qquad \boldsymbol{k}_1p(k_1, 0, 0)=\lambda_{100}p(k_1-1, 0, 0) \qquad (2)$$

$$\boldsymbol{k}_2\geqq 1 \qquad \boldsymbol{k}_2p(0, \boldsymbol{k}_2, 0)=\lambda_{010}p(0, \boldsymbol{k}_2-1, 0) \qquad (3)$$

$$\boldsymbol{k}_3\geqq 1 \qquad \boldsymbol{k}_3p(0, 0, \boldsymbol{k}_3)=\lambda_{001}p(0, 0, \boldsymbol{k}_3-1) \qquad (4)$$

$$\boldsymbol{k}_2,\ \boldsymbol{k}_3\geqq 1 \quad \boldsymbol{k}_2p(0, \boldsymbol{k}_2, \boldsymbol{k}_3)=\lambda_{010}p(0, \boldsymbol{k}_2-1, \boldsymbol{k}_3)+\lambda_{011}p(0, \boldsymbol{k}_2-1, \boldsymbol{k}_3-1) \qquad (5)$$

$$\boldsymbol{k}_3p(0, \boldsymbol{k}_2, \boldsymbol{k}_3)=\lambda_{001}p(0, \boldsymbol{k}_2, \boldsymbol{k}_3-1)+\lambda_{011}p(0, \boldsymbol{k}_2-1, \boldsymbol{k}_3-1) \qquad (6)$$

$$\boldsymbol{k}_3,\ \boldsymbol{k}_1\geqq 1 \quad \boldsymbol{k}_3p(\boldsymbol{k}_1, 0, \boldsymbol{k}_3)=\lambda_{001}p(\boldsymbol{k}_1, 0, \boldsymbol{k}_3-1)+\lambda_{101}p(\boldsymbol{k}_1-1, 0, \boldsymbol{k}_3-1) \qquad (7)$$

$$k_1 p(k_1, 0, k_3)=\lambda_{100} p(k_1-1, 0, k_3)+\lambda_{101} p(k_1-1, 0, k_3-1) \quad (8)$$

$$k_1, k_2 \geqq 1 \quad k_1 p(k_1, k_2, 0)=\lambda_{100} p(k_1-1, k_2, 0)+\lambda_{110} p(k_1-1, k_2-1, 0) \quad (9)$$

$$k_2 p(k_1, k_2, 0)=\lambda_{010} p(k_1, k_2-1, 0)+\lambda_{110} p(k_1-1, k_2-1, 0) \quad (10)$$

$$k_1, k_2, k_3 \geqq 1 \quad k_1 p(k_1, k_2, k_3)=\lambda_{100} p(k_1-1, k_2, k_3)+\lambda_{110} p(k_1-1, k_2-1, k_3)$$
$$+\lambda_{101} p(k_1-1, k_2, k_3-1)+\lambda_{111} p(k_1-1, k_2-1, k_3-1) \quad (11)$$

$$k_2 p(k_1, k_2, k_3)=\lambda_{010} p(k_1, k_2-1, k_3)+\lambda_{110} p(k_1-1, k_2-1, k_3)$$
$$+\lambda_{011} p(k_1, k_2-1, k_3-1)+\lambda_{111} p(k_1-1, k_2-1, k_3-1) \quad (12)$$

$$k_3 p(k_1, k_2, k_3)=\lambda_{001} p(k_1, k_2, k_3-1)+\lambda_{101} p(k_1-1, k_2, k_3-1)$$
$$+\lambda_{011} p(k_1, k_2-1, k_3-1)+\lambda_{111} p(k_1-1, k_2-1, k_3-1). \quad (13)$$

Conversely, if we calculate the density function $p(\boldsymbol{k})$ $(\boldsymbol{k} \geqq 0)$ from the relations, we need only one of 3 main relations (11)$\sim$(13) and their reduced relations. If we search deligently we have 13 relations and we need only specified 8 from the 13 relations to calculate the density conversely.

THEOREM 3-2 (8 relations). *If nonnegative function* $q(\boldsymbol{k})$ *defined on* $\boldsymbol{k} \geqq 0$ *satisfy* (1), (2), (3), (4), (5) *or* (6), (7) *or* (8), (9) *or* (10) *and* (11) *or* (12) *or* (13) *then* $q(\boldsymbol{k})$ *consists with the density of the Poisson distribution* $p(\boldsymbol{k})$ $(\boldsymbol{k} \geqq 0)$.

This theorem means the minimal $r.r$'s. are (1)$\sim$(4) and one of (5) and (6), and one of (7) and (8), and one of (9) and (10), and one of (11), (12) and (13) to get the calculablity.

COROLLARY 3-2 (7 relations). *If a density functions* $q(\boldsymbol{k})$ *on* $k \geqq 0$ *satisfy* (2) $\sim$(4), (5) *or* (6), (7) *or* (8), (9) *or* (10) *and* (11) *or* (12) *or* (13) *then* $q(\boldsymbol{k})$ *consists with one of the density* $p(\boldsymbol{k})$ *on* $\boldsymbol{k} \geqq 0$.

## § 4. Multivariate Poisson distribution and its recurrence relations.

We assume $X_0$ to be $n$-variate bivariate distribution $B(N, p)$, $X_0$ is explained as the sum of $N$ independent distributions $B(1, p)$. We can derive $n$-variate Poisson distribution $P(\lambda)$ by the limiting distribution of $X_0$ with the restriction $Np_i \to \lambda_i$ as $N \to \infty$ for every $\boldsymbol{i} \in E$. Denote $X$ as the $n$ variate Poisson distribution $P(\lambda)$ then we can represent the density

$$p(X=\boldsymbol{k})=\sum_{[C]} \prod_{\boldsymbol{i} \neq 0} p(\alpha_i, \lambda_i)$$

where $p(\alpha_i, \lambda_i)$ $(\boldsymbol{i} \in E)$ are usual univariate Poisson density

$$p(\alpha_i, \lambda_1)=(\lambda_1/\alpha_i!)\exp(-\lambda_i) \qquad (\boldsymbol{i}\in E)$$

and $[C]$ should be generalized to $n$-variate $\alpha$ set

$$[C]=\{\alpha\,;\ \sum_{\boldsymbol{i}\neq 0}\boldsymbol{i}\alpha_{\boldsymbol{i}}=\boldsymbol{k}\},$$

see Kawamura [4]. Then we have a recurrence relation of $P(\lambda)$

$$\boldsymbol{k}\ p(\boldsymbol{k})=\sum_{0\leqq \boldsymbol{i}\leqq \boldsymbol{k}}\boldsymbol{i}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i}) \qquad (*)$$

and if $\boldsymbol{k}=0$ we have $p(0)=\prod_{\boldsymbol{i}\in E}p(0, \lambda_{\boldsymbol{i}})=\exp(-\sum_{\boldsymbol{i}\in E}\lambda_{\boldsymbol{i}})$ or we can express $(*)$ as

$$\boldsymbol{k}_j p(\boldsymbol{k})=\sum_{\boldsymbol{i}_j=1,\, \boldsymbol{i}\leqq \boldsymbol{k}}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i})$$

where $\boldsymbol{k}_j\geqq 1$ for some $j=1, 2, \cdots, n$. We can summarize these relations as following.

THEOREM 4-1. *The density $p(\boldsymbol{k})$ of $n$-variate Poisson distribution $P(\lambda)$ satisfies the recurrence relations*

$$p(0)=\exp(-\sum_{\boldsymbol{i}\in E}\lambda_{\boldsymbol{i}})$$

*and*

$$\boldsymbol{k}\ p(\boldsymbol{k})=\sum_{0\leqq \boldsymbol{i}\leqq \boldsymbol{k}}\boldsymbol{i}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i})\,.$$

*Proof.* If $\boldsymbol{k}_j\geqq 1$ then

$$\boldsymbol{k}_j=[\sum_{\boldsymbol{i}\in E}\boldsymbol{i}\alpha_{\boldsymbol{i}}]_j=\sum_{\boldsymbol{i}\in E}\boldsymbol{i}_j\alpha_{\boldsymbol{i}}=\sum_{\boldsymbol{i}_j=1}\alpha_{\boldsymbol{i}}\,,$$

$$\begin{aligned}\boldsymbol{k}_j p(\boldsymbol{k})&=\boldsymbol{k}_j\sum_{[C]}\prod_{\boldsymbol{i}\neq 0}p(\alpha_{\boldsymbol{i}}, \lambda_{\boldsymbol{i}})\\ &=\sum_{[C]}(\sum_{\boldsymbol{i}_j=1}\alpha_{\boldsymbol{i}})\prod_{\boldsymbol{i}\neq 0}p(\alpha_{\boldsymbol{i}}, \lambda_{\boldsymbol{i}})\\ &=\sum_{[C]}\sum_{\boldsymbol{i}_j=1}\lambda_{\boldsymbol{i}}\prod_{\boldsymbol{i}\neq 0}p(\beta_{\boldsymbol{i}}, \lambda_{\boldsymbol{i}}) \qquad (j=1, 2, \cdots, n)\,,\end{aligned}$$

where $\beta_{\boldsymbol{i}}=\alpha_{\boldsymbol{i}}-1$ with $\boldsymbol{i}$ satisfying $\boldsymbol{i}_j=1$ and otherwise $\beta_{\boldsymbol{i}}=\alpha_{\boldsymbol{i}}$.

$$\begin{aligned}\boldsymbol{k}_j p(\boldsymbol{k})&=\sum_{\boldsymbol{i}_j=1}\lambda_{\boldsymbol{i}}\sum_{[C]}\prod_{\boldsymbol{i}\neq 0}p(\beta_{\boldsymbol{i}}, \lambda_{\boldsymbol{i}})\\ &=\sum_{\boldsymbol{i}_j=1}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i}) \qquad (j=1, 2, \cdots, n)\,,\end{aligned}$$

where $p(\beta_{\boldsymbol{i}}, \lambda_{\boldsymbol{i}})=0$ for $\beta_{\boldsymbol{i}}=-1$. We can summarize these relations as

$$\boldsymbol{k}\ p(\boldsymbol{k})=\sum_{0\leqq \boldsymbol{i}\leqq \boldsymbol{k}}\boldsymbol{i}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i}) \qquad \boldsymbol{k}\neq 0\,. \qquad \blacksquare$$

The recurrence relations just proved involves $n\ 2^{n-1}$ different relations which are caused on the boundary conditions regarding $k$. And if $\boldsymbol{k}=0$ then we have directly from the density

$$p(0)=\prod_{i\in E} p(0\,;\lambda_i)=\prod_{i\neq 0}\exp(-\lambda_i)=\exp(-\sum_{i\neq 0}\lambda_i)\,.$$

In the case if $\boldsymbol{k}\neq 0$ we could classify $\boldsymbol{k}$ by the number of positive components of the vector as followings.

If $\sharp\boldsymbol{k}=1$, that is, only one of the components $\boldsymbol{k}_j\geqq 1$ then we have

$$\boldsymbol{k}_j p(\boldsymbol{k})=\lambda_i p(\boldsymbol{k}-\boldsymbol{i}) \qquad (0\neq\boldsymbol{i}\leqq\boldsymbol{k})$$

where $\boldsymbol{i}\leqq\boldsymbol{k}$ and $\boldsymbol{i}\neq 0$ include only one $\boldsymbol{i}$ such that $\boldsymbol{i}_j=1$ otherwise $\boldsymbol{i}_k=0$ for $k\neq j$. Practically we have for $\sharp\boldsymbol{k}=1$,

$$\boldsymbol{k}_j p(0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)=\lambda_{0\cdots 010\cdots 0}\,p(0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0)\,.$$

We can vary $j$ from 1 to $n$, so we have $n$ relations for $\sharp\boldsymbol{k}=1$. The number of relations would be $\binom{n}{1}$, having one term in the right side.

Also, in the case $\sharp\boldsymbol{k}=2$ ($\boldsymbol{k}_i$, $\boldsymbol{k}_j\geqq 1$, $\boldsymbol{i}<j$), we have practically

$$\boldsymbol{k}_j p(0\cdots 0\,\boldsymbol{k}_i\,0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)=\lambda_{0\cdots 010\cdots 000\cdots 0}p(0\cdots 0\,\boldsymbol{k}_i-1\,0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)\,.$$
$$+\lambda_{0\cdots 010\cdots 010\cdots 0}\,p(0\cdots 0\,\boldsymbol{k}_1-1\,0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0)\,.$$
$$\boldsymbol{k}_j p(0\cdots 0\,\boldsymbol{k}_i\,0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)=\lambda_{0\cdots 000\cdots 010\cdots 0}p(0\cdots 0\,k_i\,0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0)\,.$$
$$+\lambda_{0\cdots 010\cdots 010\cdots 0}p(0\cdots 0\,\boldsymbol{k}_i-1\,0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0)\,.$$

The number of relations would be $\binom{n}{2}$, and each has two terms in the right side.

By the induction for $\sharp\boldsymbol{k}$ we have totally

$$1\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n}=n2^{n-1}$$

relations. We can conclude as the theorem.

THEOREM 4-2. *The density function $p(\boldsymbol{k})$ on $\boldsymbol{k}\geqq 0$ of n-variate Poisson distribution $P(\lambda)$ has totally*

$$1+1\binom{n}{1}+2\binom{n}{2}+\cdots+n\binom{n}{n}=n2^{n-1}+1$$

*mutually different recurrence relations depending on the boundary condition for $\boldsymbol{k}$, including the density*

$$p(0)=\exp(-\sum_{i\neq 0}\lambda_i)\,.$$

## § 5. Minimal relations for getting a multivariate Poisson density.

Let's consider the main theme of calculation of the density. Primarily we will try to calculate directly from the density function but we will risk summing up a lot of accidental error, really we can calculate the density by computer if we make the software being careful not to summarize the error. It is, generally, a very difficalt way to calculate the density and we will be able to improve the situation by these recurrence relations.

To calculate the density $p(\boldsymbol{k})$ of $n$-variate Poisson distribution $P(\lambda)$ we need only prespecified $2^n$ relations from the $1+n2^{n-1}$ relations given in the theorem 4-2. We can express the assertion in the following theorem.

As denoted and defined in the preceding section, we use the notation $\sharp\boldsymbol{k}$ as the number of positive components of $k=(\boldsymbol{k}_1, \boldsymbol{k}_2, \cdots, \boldsymbol{k}_n)$, if we can calculate the density $p(\boldsymbol{k})$ for $\boldsymbol{k}\in S$ by some prescribed relations, then we call $p(\boldsymbol{k})$ to be "calculable on $S$" or simply "calculable".

THEOREM 5-1. *The density of n-variate Poisson distribution $p(\boldsymbol{k})$ for $\boldsymbol{k}\geqq 0$ is "calculable" from the prescribed $2^n$ relations including the relation*

$$p(0)=\exp(-\sum_{\boldsymbol{i}\neq 0}\lambda_{\boldsymbol{i}}).$$

In the case of $\sharp\boldsymbol{k}=0$ ($\boldsymbol{k}=0$), we have one relation $p(0)=\exp(-\sum_{\boldsymbol{i}\neq 0}\lambda_{\boldsymbol{i}})$ we need it to calculate the density.

In the case of $\sharp\boldsymbol{k}=1$. If $\boldsymbol{k}_j\geqq 1$ for fixed $j=1, 2, \cdots, n$ we have one relation

$$\boldsymbol{k}_j p(0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)=\lambda_{0\cdots 010\cdots 0}\,p(0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0).$$

and we need it to do this. The selection of $j$ may be $\binom{n}{1}$ cases then we have totally $\binom{n}{1}$ relations and we need all of them to do this.

In the case of $\sharp\boldsymbol{k}=2$, if $\boldsymbol{k}_i, \boldsymbol{k}_j\geqq 1$ for fixed $\boldsymbol{i}, j$; $1\leqq\boldsymbol{i}<j\leqq n$ then we have two relations

$$\boldsymbol{k}_i p(0\cdots 0\,\boldsymbol{k}_i\,0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)=\lambda_{0\cdots 010\cdots 000\cdots 0}\,p(0\cdots 0\,\boldsymbol{k}_i-1\,0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)$$
$$+\lambda_{0\cdots 010\cdots 010\cdots 0}\,p(0\cdots 0\,\boldsymbol{k}_i-1\,0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0),$$

and

$$\boldsymbol{k}_j p(0\cdots 0\,\boldsymbol{k}_i\,0\cdots 0\,\boldsymbol{k}_j\,0\cdots 0)=\lambda_{0\cdots 000\cdots 010\cdots 0}\,p(0\cdots 0\,\boldsymbol{k}_i\,0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0)$$
$$+\lambda_{0\cdots 010\cdots 010\cdots 0}\,p(0\cdots 0\,\boldsymbol{k}_i-1\,0\cdots 0\,\boldsymbol{k}_j-1\,0\cdots 0).$$

The selection of $\boldsymbol{i}$ and $j$ may be $\binom{n}{2}$ cases, each case has two relations then we have totally $2\binom{n}{2}$ relations and we need one of the relation in each case to do this so that we need $\binom{n}{2}$ relations to do this.

In the case of $\# \boldsymbol{k}=t$ $(0<t\leqq n)$ if $\boldsymbol{k}_{j_1}, \boldsymbol{k}_{j_2}, \cdots, \boldsymbol{k}_{j_t}\geqq 1$ for fixed $1\leqq j_1<j_2<\cdots<j_t\leqq n$, we have $t$ different relations and we need one of the relations to calculate the density. The selection of $j_1<j_2<\cdots<j_t$ may be in $\binom{n}{t}$ cases, so we have totally $t\binom{n}{t}$ relations and we need $\binom{n}{t}$ relations to do this. Additionary we can confirm each of the $t$ relations has $2^{t-1}$ terms in maximum in the right side of the relation; if $\lambda_i>0$ for every $\boldsymbol{i}$; $\boldsymbol{i}\in E$, $\boldsymbol{i}\leqq\boldsymbol{k}$ then our relation

$$\boldsymbol{k}_j p(\boldsymbol{k})=\sum_{\boldsymbol{i}\in E,\, \boldsymbol{i}\leqq\boldsymbol{k}} \lambda_i p(\boldsymbol{k}-\boldsymbol{i})$$

has maximum number of terms for $\boldsymbol{k}_j\geqq 1$. See DISCUSSION.

In the case of $\# \boldsymbol{k}=n$, $\boldsymbol{k}=(\boldsymbol{k}_1, \boldsymbol{k}_2, \cdots, \boldsymbol{k}_n)$ if any component of $\boldsymbol{k}$ is a positive integer then we have $n$ different relations and we need one of the $n$ relations to calculate the density. The selection of such a case may be $\binom{n}{n}$ case, so we have totally $n$ relations and we need one of the relations to calculate the density.

*Proof of the theorem.* Put $\# \boldsymbol{k}=1$ and $\boldsymbol{k}=(0\cdots 0\, \boldsymbol{k}_j\, 0\cdots 0)$ where $\boldsymbol{k}_j>0$ and assume $\boldsymbol{i}\leqq\boldsymbol{k}$ then $\boldsymbol{i}=(0\cdots 010\cdots 0)$. In the case we can use specified recurrence relation

$$\boldsymbol{k}_j p(\boldsymbol{k})=\lambda_i p(k-\boldsymbol{i}), \qquad \boldsymbol{i}=(0\cdots 010\cdots 0)$$

then we have

$$p(\boldsymbol{k})=(\lambda_i/\boldsymbol{k}_j)p(\boldsymbol{k}-\boldsymbol{i}). \tag{1}$$

If $\boldsymbol{k}-\boldsymbol{i}=(0\cdots 0\, \boldsymbol{k}_j-1\, 0\cdots 0)\neq 0$ in the right side, then we use the relation in iteration,

$$\begin{aligned}\boldsymbol{k}_j(\boldsymbol{k}_j-1)p(\boldsymbol{k})&=\lambda_i(\boldsymbol{k}_j-1)p(\boldsymbol{k}-\boldsymbol{i})\\ &=\lambda_i 2p(\boldsymbol{k}-2\boldsymbol{i}).\end{aligned}$$

By the iteration and $\boldsymbol{k}-\boldsymbol{k}_j\boldsymbol{i}=0$, we conclude

$$\boldsymbol{k}_j!\, p(\boldsymbol{k})=\lambda_i\boldsymbol{k}_j p(0). \tag{2}$$

We can calculate all the density $p(\boldsymbol{k})$ for every $\boldsymbol{k}$ satisfying $\# \boldsymbol{k}=1$ and $\boldsymbol{k}_j\geqq 1$ for fiexed $j$ from $j$-th $r.r.$ of type(1). From the arbitrariness of $j$, we can calculate all the density $p(\boldsymbol{k})$ for every $\boldsymbol{k}$ satisfying $\# \boldsymbol{k}\leqq 1$, from our $n$ recurrence relations of type(1) $(j=1, 2, \cdots, n)$. If we put $\# \boldsymbol{k}=t$ then in the case $t=0$ $(\boldsymbol{k}=0)$ we know that $p(\boldsymbol{k})$ is calculable from the relation. In the case $t=1$ $(\# \boldsymbol{k}=1)$ all the density $p(\boldsymbol{k})$ for $\# \boldsymbol{k}=1$ are calculable which is proved by using the recurrence relations of type(1) and the fact of $p(\boldsymbol{k})$ is calculable on $\boldsymbol{k}=0$. That is, the fact means all the density $p(\boldsymbol{k})$ are calculable on $\# \boldsymbol{k}$.

To proceed, in the proof by the induction for $t$. If $t=0$ then $p(\boldsymbol{k})$ is calculable, and if $t=1$ then $p(\boldsymbol{k})$ is calculable. The proof is made using the fact $p(0)$ $(t=0)$ is calculable. Therefore if we assume $p(\boldsymbol{k})$ is calculable for every $\boldsymbol{k}$ such that $\# \boldsymbol{k}<t$ and let prove that $p(\boldsymbol{k})$ is calculable on $\# \boldsymbol{k}=t$ using prescribed type $t$ recurrence relations. In the case $\# \boldsymbol{k}=t$, we use our notations $\boldsymbol{i}\in E$, $E=\{0, 1\}^n-0$ and $(\boldsymbol{k})_j=\boldsymbol{k}_j$. For every $\boldsymbol{k}_j>0$ $(1\leqq j\leqq n)$ we can express the recurrence relation of type $t$

$$\boldsymbol{k}_j p(\boldsymbol{k})=\sum_{\{\boldsymbol{i};\boldsymbol{i}_j=1,\,\boldsymbol{i}\leqq\boldsymbol{k}\}}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i}) \qquad (*)$$

where $\{\ \}$ means $\{i\,;\,\boldsymbol{i}_j=1,\ \boldsymbol{i}\leqq\boldsymbol{k},\ \boldsymbol{i}\in E\}=\{\boldsymbol{i}\,;\,(\boldsymbol{i})_j=1,\ 0\leqq\boldsymbol{k}-\boldsymbol{i},\ \boldsymbol{i}\in E\}$, and put $\boldsymbol{i}_1$ the number of $\{\ \}$ then we have $\#(\boldsymbol{k}-\boldsymbol{i}_1)\leqq t$. It $\#(\boldsymbol{k}-\boldsymbol{i}_1)<t$ then $p(\boldsymbol{k}-\boldsymbol{i}_1)$ in the right side of the relation $(*)$ is calculable from the assumption of the induction and otherwise if $\#(\boldsymbol{k}-\boldsymbol{i}_1)=t$ then $(\boldsymbol{k}-\boldsymbol{i}_1)_j=\boldsymbol{k}_j-1\geqq 1$ and we can use the relation $(*)$ of type $t$ for $p(\boldsymbol{k}-\boldsymbol{i}_1)$ one more time then we have

$$(\boldsymbol{k}-\boldsymbol{i}_1)_j p(\boldsymbol{k}-\boldsymbol{i}_1)=\sum_{\{\boldsymbol{i};\boldsymbol{i}_j=1,\,1\leqq\boldsymbol{k}-\boldsymbol{i}_1,\,\boldsymbol{i}\in E\}}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i})\,.$$

Put $\boldsymbol{i}_2$ the member of $\{\ \}$ then we have $\#(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)\leqq\#(\boldsymbol{k}-\boldsymbol{i}_1)\leqq t$. If $\#(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)<t$ then $p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)$ in the right side of the relation is calculable from the assumption of the induction and otherwise if $\#(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)=t$ then $(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)_j$ $=\boldsymbol{k}_j-2\geqq 1$ and we can use the relation $(*)$ of type $t$ for $p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)$ one more time then we have

$$(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)_j p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)=\sum_{\{\boldsymbol{i};\boldsymbol{i}_j=1,\,\boldsymbol{i}\leqq\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2,\,\boldsymbol{i}\in E\}}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2)\,.$$

Put $\boldsymbol{i}_3$ the member of $\{\ \}$ then we have $\#(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3)\leqq t$. If $\#(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3)$ $<t$ then $p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3)$ in the right side of the relation is calculate from the assumption (of the induction) and otherwise if $\#(k-\boldsymbol{i}_1-\boldsymbol{i}_2-\boldsymbol{i}_3)=t$ we have to continue this process so as to satisfy all the terms of the right side of the relation $(*)$ to be calculable. The process to get calculability require at most $\boldsymbol{k}_j-1$ steps and we put $s=\boldsymbol{k}_j-1$ then we have

$$(\boldsymbol{k}_j-s)p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\cdots-\boldsymbol{i}_s)=\sum_{\{\boldsymbol{i};\boldsymbol{i}_j=1,\,\boldsymbol{i}\leqq\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\cdots-\boldsymbol{i}_s,\,\boldsymbol{i}\in E\}}\lambda_{\boldsymbol{i}}p(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\cdots-\boldsymbol{i}_s-\boldsymbol{i}).$$

where $\#(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\cdots-\boldsymbol{i}_s)\leqq t$ and if $\#(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\cdots-\boldsymbol{i}_s)<t$ then the left side is already calculable by the assumption and otherwise, in the right side, $\#(k-\boldsymbol{i}_1-\boldsymbol{i}_2-\cdots-\boldsymbol{i}_s-\boldsymbol{i})<t$ because $(\boldsymbol{k}-\boldsymbol{i}_1-\boldsymbol{i}_2-\cdots-\boldsymbol{i}_s-\boldsymbol{i})_j=\boldsymbol{k}_j(s+1)=0$ for every member of $\{\ \}$. Therefore we have a result that all the terms of the right side of the relation $(*)$ are calculable where we used only the relation $(*)$ of type $t$ repeatedly and the assumption for the induction.

We recognize that the proof is made only on using the relation of type $t$ and disregard the other $t-1$ relations in the case $\#\boldsymbol{k}=t$. And arbitrariness of $t$ (on $0\leqq t\leqq n$), we have the conclusion of the theorem.

Theorem 5-1. The density of $n$-variate Poisson distribution $p(\boldsymbol{k})$ for $\boldsymbol{k}\geqq 0$ is calculable from precribed $2^n$ relations including the relation

$$p(0)=\exp(-\sum_{i\in E}\lambda_i).$$

If we omit the relation $p(0)=\exp(-\sum_{i\in E}\lambda_i)$ and put $p(0)=p_0$ and using the other prescribed $2^{n-1}$ relations, we can calculate all the values $p(\boldsymbol{k})$ for $\boldsymbol{k}\geqq 0$ as the function of $p_0$ and if we introduce the assumption of the function to be a density, that is, summing up to unity then all the function $p(\boldsymbol{k})$ $\boldsymbol{k}\geqq 0$ consists with the multivariate Poisson density.

THEOREM 5-2. *The function $p(\boldsymbol{k})$ for $\boldsymbol{k}\geqq 0$ is calculable from prescribed $2^n-1$ relations if we assume $p(\boldsymbol{k})$ for $\boldsymbol{k}\geqq 0$ to be a density.*

The proof may be completed by putting $p(0)=p_0$ and using prescribed $2^n-1$ relations all the density is calculable as the multiple of $p_0$ in the same discussion of theorem 5-1 and $\sum_{k\geqq 0}p(\boldsymbol{k})=1$. That is, the function is proved as $p_0$-calculable.

## DISCUSSION

Through this paper, we have counted the number of relations and the number of terms in a relation including $\lambda_i=0$ for some of $\boldsymbol{i}\in E$.

If we consider multivariate Poisson distribution usually we need to treat $\lambda_i\geqq 0$ for $\boldsymbol{i}\in E$ then occasionally some parameters $\lambda_{i_1}, \cdots, \lambda_{i_t}$ $(\boldsymbol{i}_1, \cdots, \boldsymbol{i}_t\in E)$ are equal to zero.

So we have to understand the numbers in these theorems to be counted at most!

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