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ON COEFFICIENT INEQUALITIES IN THE CLASS Σ

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§ 0. Introduction.

Let Σ denote the class of functions

$$f(z)=z+\sum_{n=1}^{\infty} b_n z^{-n},$$

analytic and univalent in $|z|>1$. In this paper we shall prove the following theorems.

THEOREM 1. (i) *If* $-1/3\leqq\lambda<1$, *then*

$$\mathrm{Re}\{b_5+b_1b_3+b_2^2+\lambda(2b_3+b_1^2)\}\leqq\frac{1}{3}(1+3\lambda^2-\lambda^3).$$

Equality occurs if and only if

$$f(z)=z\{(1+3\lambda z^{-2}+3b_2z^{-3}+3\lambda z^{-4}+z^{-6})^{2/3}+2(b_1-\lambda)z^{-2}\}^{1/2}$$

where b_2 *is real and* $|b_2|\leqq(1/3)\min\{2+6\lambda, 2(1-\lambda)^{3/2}\}$, *with* $\arg(b_1-\lambda)=\pi \pmod{2\pi}$ *and* $0\leqq|b_1-\lambda|\leqq(1/2)\min\{[(2+6\lambda)+3b_2]^{2/3}, [(2+6\lambda)-3b_2]^{2/3}\}$, *or with* $\arg(b_1-\lambda)=\pi/3, -\pi/3 \pmod{2\pi}$ *and* $0\leqq|b_1-\lambda|\leqq(1/2)\min\{[2(1-\lambda)^{3/2}+3b_2]^{2/3}, [2(1-\lambda)^{3/2}-3b_2]^{2/3}\}$.

(ii) *If* $1\leqq\lambda$, *then*

$$\mathrm{Re}\{b_5+b_1b_3+b_2^2+\lambda(2b_3+b_1^2)\}\leqq\lambda.$$

Equality occurs if and only if

$$f(z)=z\{1+2b_1z^{-2}+z^{-4}\}^{1/2}$$

where b_1 *is real and* $-1\leqq b_1\leqq 1$.

THEOREM 2.

$$\mathrm{Re}\{b_4+b_1b_2\}\leqq 2/5.$$

Extremal functions must satisfy

$$f(z^2)^{5/2}-\frac{5}{2}b_1f(z^2)^{1/2}=z^5+e^{i\theta}z^{-5}.$$

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Duren [2] proved the inequality in Theorem 1 (i) when $\lambda=0$. And Kubota [4] proved the inequality in Theorem 2 when b_1 is real.

Our proofs depend on Schiffer's variational method [6], and our key lemmas are that the extremal functions omit the value 0 in $|z|>1$, which are proved by making use of Bombieri's theorem [1].

§1. Proof of Theorem 1.

1.1. SOME LEMMAS. Firstly we assume that $-1/3\leqq\lambda<1$.

LEMMA 1.1. *Every extremal function $f(z)$ omits the value 0 in $|z|>1$.*

Proof. Applying Schiffer's variational method to this problem, we find the associated quadratic differential

$$Q(w)dw^2=w^2(w^2-2(b_1-\lambda))dw^2\,.$$

Assume first that $b_1-\lambda\neq0$. Put $b_1-\lambda=|b_1-\lambda|e^{i\beta}$. It is clear that the trajectories of $Q(w)dw^2$ are symmetric with respect to the origin. Let Δ denote the critical trajectories of $Q(w)dw^2$. Assume that $\beta\neq n\pi/3$. On the ray $J_1=\{ite^{i\beta/2}:0<t<\infty\}$,

$$\operatorname{Im}\{Q(ite^{i\beta/2})(d(ite^{i\beta/2}))^2\}=-t^2(t^2+2|b_1-\lambda|)\sin3\beta(dt)^2\neq0\,.$$

Hence by Bombieri's theorem [1] $\bar{J}_1$ meets a component of Δ which goes through the origin only at the origin. The same fact is true for $-J_1=\{ite^{i\beta/2}:-\infty<t<0\}$. On $J_2=\{te^{i\beta/2}:0<t<(2|b_1-\lambda|)^{1/2}\}$,

$$\operatorname{Im}\{Q(te^{i\beta/2})(d(te^{i\beta/2}))^2\}=t^2(t^2-2|b_1-\lambda|)\sin3\beta(dt)^2\neq0\,.$$

Hence $\bar{J}_2$ meets one component of Δ at most at the origin and another component at most at the critical point $(2(b_1-\lambda))^{1/2}$. The similar fact holds for $-J_2$. On $J_3=\{te^{i\beta/2}:(2|b_1-\lambda|)^{1/2}<t<\infty\}$,

$$\operatorname{Im}\{Q(te^{i\beta/2})(d(te^{i\beta/2}))^2\}=t^2(t^2-2|b_1-\lambda|)\sin3\beta(dt)^2\neq0\,.$$

Hence $\bar{J}_3$ meets Δ at most at the critical point $(2(b_1-\lambda))^{1/2}$, because three critical trajectories must meet at $(2(b_1-\lambda))^{1/2}$ with equal angles. The similar fact holds for $-J_3$. Furthermore, on the short segments $J_\alpha=\{te^{i(\beta+\alpha)/2}:0<t<\varepsilon\}$,

$$\begin{aligned}&\operatorname{Im}\{Q(te^{i(\beta+\alpha)/2})(d(te^{i(\beta+\alpha)/2}))^2\}\\&\qquad=t^2(t^2\sin(3\beta+3\alpha)-2|b_1-\lambda|\sin(3\beta+2\alpha))(dt)^2\neq0\end{aligned}$$

for all sufficiently small α and ε. Thus none of the four trajectories which tend to the origin can be tangential along J_2. The same fact holds along $-J_2$. Hence by the fact that the four trajectories meet at the origin with equal angles, we know that each of them remains in each of the quadrants divided by J_2+J_3, J_1, $(-J_2)+(-J_3)$ and $-J_1$.

Assume that $\beta=0$. Then

$$Q(w)dw^2=w^2(w^2-2|b_1-\lambda|)dw^2.$$

In this case Δ has three components and is symmetric with respect to the real axis. In the case of $\beta=2n\pi/3$, the shape of Δ is the rotated one of Δ in the case of $\beta=0$.

Assume that $\beta=\pi$. Then

$$Q(w)dw^2=w^2(w^2+2|b_1-\lambda|)dw^2.$$

In this case Δ has only one component and is symmetric with respect to the real axis. In the case of $\beta=(2n+1)\pi/3$, the shape of Δ is the rotated one of Δ in the case of $\beta=\pi$.

Now assume that $b_1-\lambda=0$. Then

$$Q(w)dw^2=w^4dw^2.$$

In this case Δ consists of six rays meeting at the origin with equal angles.

We denote the image of $|z|=1$ by the extremal function $f(z)$ by Γ. Γ is on the trajectories of $Q(w)dw^2$. So Γ must be on Δ and go through the origin, because the conformal centre

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta=0.$$

This completes the proof.

We prepare two more lemmas. The next one is a special case of Jenkins' general coefficient theorem.

LEMMA 1.2. (Jenkins, e.g. [5, Theorem 8.12.]) *Let*

$$\psi(w)=w+a_2w^{-2}+a_3w^{-3}+a_4w^{-4}+a_5w^{-5}+\cdots$$

be univalent and admissible for the quadratic differential

$$Q(w)dw^2=(A_0w^4+A_1w^3+A_2w^2+A_3w)dw^2.$$

Then

$$\mathrm{Re}(A_0a_5+A_1a_4+A_2a_3+A_3a_2+A_0a_2^2)\leqq 0.$$

If equality holds, then

$$\frac{Q(\psi(w))}{Q(w)}\psi'(w)^2\equiv 1.$$

LEMMA 1.3. *If* $-1/3\leqq\lambda\leqq 1$, *then*

$$\mathrm{Re}\{b_5+b_1b_3+\lambda(2b_3+b_1^2)\}\leqq\frac{1}{3}(1+3\lambda^2-\lambda^3)$$

for all odd functions $f(z)$ in Σ. Equality occurs if and only if

$$f(z)=z\{(1+3\lambda z^{-2}+3\lambda z^{-4}+z^{-6})^{2/3}+2(b_1-\lambda)z^{-2}\}^{1/2}$$

with $\arg(b_1-\lambda)=\pi$ (mod. 2π) *and* $0\leqq|b_1-\lambda|\leqq(1/2)(2+6\lambda)^{2/3}$, *or with* $\arg(b_1-\lambda)=\pi/3,\ -\pi/3$ (mod. 2π) *and* $0\leqq|b_1-\lambda|\leqq2^{-1/3}(1-\lambda)$.

Proof. From any odd function $f(z)=z+b_1z^{-1}+b_3z^{-3}+b_5z^{-5}+\cdots$ in Σ, we obtain the univalent function

$$f(z^{1/2})^2=z+c_0+c_1z^{-1}+c_2z^{-2}+\cdots$$

where $c_1=2b_3+b_1^2$ and $c_2=2(b_5+b_1b_3)$. Hence

$$b_5+b_1b_3+\lambda(2b_3+b_1^2)=\frac{1}{2}(c_2+2\lambda c_1).$$

By this relation and Jenkins' results ([3], Lemma 3 and Corollary 10), we obtain the desired result.

1.2. SCHIFFER'S DIFFERENTIAL EQUATION. We denote the extremal function by

$$f(z)=z+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+b_4z^{-4}+b_5+z^{-5}+\cdots.$$

Put

$$\psi(w)=-f(-f^{-1}(w))=w-2b_2w^{-2}-(2b_4+4b_1b_2)w^{-4}-4b_2^2w^{-5}+\cdots.$$

Because $\Gamma=f(|z|=1)$ is on the critical trajectories of the quadratic differential $Q(w)dw^2=w^2(w^2-2(b_1-\lambda))dw^2$, we have

$$\psi(w)^2(\psi(w)^2-2(b_1-\lambda))\psi'(w)^2=w^2(w^2-2(b_1-\lambda))$$

by making use of Lemma 1.2. Expanding the left hand side and comparing the coefficients, we have

$$b_4+b_1b_2+\lambda b_2=0.$$

Hence the extremal function $f(z)$ satisfies Schiffer's differential equation

$$\begin{aligned}&f(z)^2(f(z)^2-2(b_1-\lambda))z^2f'(z)^2\\ (1)\qquad &=z^6+2\lambda z^4-(2b_3+b_1^2)z^2-6\Big(b_5+b_1b_3+b_2^2+\frac{2}{3}\lambda(2b_3+b_1^2)\Big)-(2\bar{b}_3+\bar{b}_1^2)z^{-2}+2\lambda z^{-4}+z^{-6}.\end{aligned}$$

We denote the right hand side by $q(z)$. Then

$$q(z)\leqq0\qquad\text{on } |z|=1.$$

1.3. *Proof of Theorem* 1(i). Suppose that $b_1-\lambda=0$. Then Schiffer's differential equation (1) becomes

$$f(z)^4z^8f'(z)^2=z^6q(z)$$
$$=\left[\prod_{j=1}^{6}(z-e^{i\alpha_j})\right]^2=[z^6+Az^5+Bz^4+Cz^3+Dz^2+Ez+F]^2,$$

by Lemma 1.1. Comparing coefficients, we have $A=C=E=0$, $B=\lambda$, $F^2=1$, $DF=\lambda$, $D=-b_3-\lambda^2$ and $-3(b_5+b_1b_3+b_2^2+(2/3)\lambda(2b_3+b_1^2))=F+\lambda D$. It follows that $F=-1$, $D=-\lambda$ and $b_3=\lambda-\lambda^2$ by the fact that $q(z)\leqq 0$ on $|z|=1$. Hence we have

$$b_5+b_1b_3+b_2^2+\lambda(2b_3+b_1^2)=\frac{1}{3}(1+3\lambda^2-\lambda^3).$$

And the extremal function $w=f(z)$ must satisfy

$$w^4z^8(dw/dz)^2=[z^6+\lambda z^4-\lambda z^2-1]^2.$$

By solving this differential equation we have

$$w=z(1+3\lambda z^{-2}+3b_2z^{-3}+3\lambda z^{-4}+z^{-6})^{1/3}$$

where b_2 is real and $|b_2|\leqq(1/3)\min\{2+6\lambda,\ 2(1-\lambda)^{3/2}\}$.

Now assume that $b_1-\lambda\neq 0$ and put $b_1-\lambda=|b_1-\lambda|e^{i\beta}$ as before.

Case a). Suppose that $\beta=(2n+1)\pi/3$ and that $\Gamma=f(|z|=1)$ contains both of critical points $\pm(2(b_1-\lambda))^{1/2}$. Then we know as above that

$$b_5+b_1b_3+b_2^2+\lambda(2b_3+b_1^2)=\frac{1}{3}(1+3\lambda^2-\lambda^3)$$

and that the extremal function must satisfy

$$w^2(w^2-2(b_1-\lambda))z^8(dw/dz)^2=[z^6+\lambda z^4-\lambda z^2-1]^2.$$

By solving this differential equation we have

$$w=z\{(1+3\lambda z^{-2}+3b_2z^{-3}+3\lambda z^{-4}+z^{-6})^{2/3}+2(b_1-\lambda)z^{-2}\}^{1/2}$$

where b_2 is real and $|b_2|\leqq(1/3)\min\{2+6\lambda,\ 2(1-\lambda)^{3/2}\}$, with $\arg(b_1-\lambda)=\pi$ (mod. 2π) and $0<|b_1-\lambda|\leqq(1/2)\min\{[(2+6\lambda)+3b_2]^{2/3},\ [(2+6\lambda)-3b_2]^{2/3}\}$, or with $\arg(b_1-\lambda)=\pi/3,\ -\pi/3$ (mod. 2π) and $0<|b_1-\lambda|\leqq(1/2)\min\{[2(1-\lambda)^{3/2}+3b_2]^{2/3},\ [2(1-\lambda)^{3/2}-3b_2]^{2/3}\}$.

Case b). Suppose that $\beta=(2n+1)\pi/3$ and Γ contains exactly one of the critical points $\pm(2(b_1-\lambda))^{1/2}$. Because $q(-z)=q(z)$ we have

$$f(-z)^2(f(-z)^2-2(b_1-\lambda))f'(-z)^2=f(z)^2(f(z)^2-2(b_1-\lambda))f'(z)^2$$

in $|z|>1$. If $f(z_0)=(2(b_1-\lambda))^{1/2}$ for some z_0, $|z_0|>1$, then

$$f(-z_0)=0 \quad\text{or}\quad f(-z_0)=-(2(b_1-\lambda))^{1/2}.$$

By Lemma 1.1 we have $f(-z_0)=-(2(b_1-\lambda))^{1/2}$. But this contradicts the assumption. Hence this case cannot occur.

Case c). Now suppose that Γ does not contain the critical points $\pm(2(b_1-\lambda))^{1/2}$. By Lemma 1.1 and the fact that $q(-z)=q(z)$ we can put

$$f(z)^2(f(z)^2-2(b_1-\lambda))z^8f'(z)^2=z^6q(z)$$
$$=[(z^2-e^{2i\alpha_1})(z^2-e^{2i\alpha_2})]^2(z^2-r^2e^{2i\alpha_3})(z^2-r^{-2}e^{2i\alpha_3})$$

for some real α_j $(j=1, 2, 3)$ and $r>1$. Putting $w=f(z)$, we have

$$w(w^2-2(b_1-\lambda))^{1/2}dw$$
$$=z^{-4}(z^2-e^{2i\alpha_1})(z^2-e^{2i\alpha_2})((z^2-r^2e^{2i\alpha_3})(z^2-r^{-2}e^{2i\alpha_3}))^{1/2}dz\,.$$

Hence

$$\int_{(2(b_1-\lambda))^{1/2}}^{f(e^{i\alpha_1})}w(w^2-2(b_1-\lambda))^{1/2}dw$$
$$=\int_{re^{i\alpha_3}}^{e^{i\alpha_1}}z^{-4}(z^2-e^{2i\alpha_1})(z^2-e^{2i\alpha_2})((z^2-r^2e^{2i\alpha_3})(z^2-r^{-2}e^{2i\alpha_3}))^{1/2}dz$$

and

$$\int_{-(2(b_1-\lambda))^{1/2}}^{f(-e^{i\alpha_1})}w(w^2-2(b_1-\lambda))^{1/2}dw$$
$$=\int_{-re^{i\alpha_3}}^{-e^{i\alpha_1}}z^{-4}(z^2-e^{2i\alpha_1})(z^2-e^{2i\alpha_2})((z^2-r^2e^{2i\alpha_3})(z^2-r^{-2}e^{2i\alpha_3}))^{1/2}dz\,.$$

The integrand in the right hand side is a single-valued odd function on the domain D, the complement of the slits $(\infty, -re^{i\alpha_3})$, $(-e^{i\alpha_3}/r, e^{i\alpha_3}/r)$ and $(re^{i\alpha_3}, \infty)$. Taking the integral path γ in D from $re^{i\alpha_3}$ to $e^{i\alpha_1}$ and $-\gamma$ from $-re^{i\alpha_3}$ to $-e^{i\alpha_1}$, it follows that

$$\int_{(2(b_1-\lambda))^{1/2}}^{f(e^{i\alpha_1})}w(w^2-2(b_1-\lambda))^{1/2}dw=\int_{-(2(b_1-\lambda))^{1/2}}^{f(-e^{i\alpha_1})}w(w^2-2(b_1-\lambda))^{1/2}dw\,.$$

Thus we have

$$\frac{1}{3}(f(e^{i\alpha_1})^2-2(b_1-\lambda))^{3/2}=\frac{1}{3}(f(-e^{i\alpha_1})^2-2(b_1-\lambda))^{3/2}\,.$$

Hence

$$f(e^{i\alpha_1})=-f(-e^{i\alpha_1})\,.$$

By a similar calculation we also have

$$f(e^{i\alpha_2})=-f(-e^{i\alpha_2})\,.$$

Hence Γ is symmetric with respect to the origin. So the extremal function

$f(z)$ must be an odd function. This case is contained in Lemma 1.3. This completes the proof of Theorem 1 (i).

1.4. *Proof of Theorem* 1 (ii). It follows from Theorem 1 (i) that

$$\mathrm{Re}\,(b_5+b_1b_3+b_2^2+2b_3+b_1^2)\leqq 1\,.$$

By making use of the inequality

$$\mathrm{Re}\,(2b_3+b_1^2)\leqq 1$$

which is one of Grunsky's inequalities, we have

$$\begin{aligned}&\mathrm{Re}\{b_5+b_1b_3+b_2^2+\lambda(2b_3+b_1^2)\}\\&\quad=\mathrm{Re}\,(b_5+b_1b_3+b_2^2+2b_3+b_1^2)+(\lambda-1)\,\mathrm{Re}\,(2b_3+b_1^2)\\&\quad\leqq 1+(\lambda-1)=\lambda\end{aligned}$$

for all $\lambda\geqq 1$. Equality occurs only for the functions which satisfy $\mathrm{Re}\,(2b_3+b_1^2)=1$. These are

$$f(z)=z(1+2b_1z^{-2}+z^{-4})^{1/2}$$

where b_1 is real and $-1\leqq b_1\leqq 1$. In fact these functions satisfy $\mathrm{Re}\{b_5+b_1b_3+b_2^2+\lambda(2b_3+b_1^2)\}=\lambda$. Hence we obtain the desired result.

§ 2. Proof of Theorem 2.

2.1. A LEMMA. We start again the following

LEMMA 2.1. *Every extremal function* $f(z)$ *omits the value* 0 *in* $|z|>1$.

Proof. The associated quadratic differential of this problem is

$$Q(w)dw^2=w(w^2-b_1)dw^2\,.$$

Take an extremal function $f(z)=z+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+b_4z^{-4}+\cdots$. We can put $b_1=|b_1|e^{i\alpha}$ with $0\leqq\alpha<4\pi/5$ by rotation. The local structure of critical trajectories of $Q(w)dw^2$ around the critical points 0, $(b_1)^{1/2}$, $-(b_1)^{1/2}$ and ∞ is well known. Let us denote the critical trajectories of $Q(w)dw^2$ by Δ. If $b_1=0$ then Δ consists of five rays joining at the origin with equal angles. We suppose that $b_1\neq 0$.

Case a). $\alpha=0$. Δ is symmetric with respect to the real axis. Let J be the imaginary axis $\{it:-\infty<t<\infty\}$. Along J

$$\mathrm{Im}\{Q(w)dw^2\}=t(t^2+b_1)(dt)^2\neq 0$$

for $t\neq 0$. Hence by Bombieri's theorem $\bar{J}$ meets the component of Δ which goes through the origin only at the origin. So we can conclude that $\Gamma=f(|z|=1)$ contains the origin because the conformal centre

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta=0.$$

Case b). $\alpha=\pi/5$. Let J_1 be $\{te^{\pi i/10}: 0<t<\infty\}$. Along J_1

$$\operatorname{Im}\{Q(w)dw^2\}=t(t^2-|b_1|)(dt)^2\neq 0$$

for $t\neq|b_1|^{1/2}$. Hence $\bar{J}_1$ meets one component of Δ at most at the origin and the other component at most at $(b_1)^{1/2}$. The similar fact holds along $-J_1$. Let J_2 be $\{ite^{\pi i/10}: 0<t<\infty\}$. Along J_2

$$Q(w)dw^2=-t(t^2+|b_1|)(dt)^2.$$

Hence J_2 is an orthogonal trajectory and $-J_2$ is a critical trajectory. Thus Γ passes through the origin by

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta=0.$$

Case c). $0<\alpha<\pi/5$ or $\pi/5<\alpha<2\pi/5$. Let J_1 be $\{te^{i\alpha/2}: 0<t<\infty\}$. Along J_1

$$\operatorname{Im}\{Q(w)dw^2\}=t(t^2-|b_1|)\sin(5\alpha/2)(dt)^2\neq 0$$

for $t\neq|b_1|^{1/2}$. Hence $\bar{J}_1$ meets two components of Δ at most at the origin and at $(b_1)^{1/2}$ respectively. The similar fact holds along $-J_1$. On $J_2=\{te^{-\pi i/5}: 0<t<\infty\}$,

$$\operatorname{Im}\{Q(w)dw^2\}=t|b_1|\sin(\alpha+2\pi/5)(dt)^2\neq 0.$$

Hence $\bar{J}_2$ meets the component of Δ which passes through the origin only at the origin. The same is true for $-J_2$. The similar considerations can be applyed to the lines $\{te^{\pi i/5}: -\infty<t<\infty\}$ and $\{it: -\infty<t<\infty\}$. Now we can readily prove that Γ goes through the origin by the above facts, the local structure of critical trajectories of $Q(w)dw^2$ around the critical points and

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta=0.$$

Case d). $2\pi/5\leqq\alpha<4\pi/5$. By the rotation $w=e^{-4\pi i/5}\zeta$, $w(w^2-b_1)dw^2=\zeta(\zeta^2-b_1e^{-2\pi i/5})d\zeta^2$. It means that this case is essentially included in the above three cases. This completes the proof.

2.2. *Proof of Theorem* 2. Let $f(z)=z+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+b_4z^{-4}+\cdots$ be an extremal function. We can take its square-root transformation $g(z)=f(z^2)^{1/2}$ by Lemma 2.1. Let $F_5(w)$ be the fifth Faber polynomial of $g(z)$ defined by

$$F_5(g(z))=z^5+\sum_{n=1}^{\infty}a_{5n}z^{-n}.$$

Then Grunsky's inequality says that

$$\sum_{n=1}^{\infty}n|a=a_{5n}|^2\leqq 5$$

and especially

$$|a_{55}|\leqq 1.$$

Because $a_{55}=(5/2)(b_4+b_1b_2)$ in this case, we obtain

$$|b_4+b_1b_2|\leqq 2/5.$$

If equality holds then $|a_{55}|=1$ and therefore $a_{5n}=0$ for $n\neq 5$. Hence we have $F_5(g(z))=z^5+e^{i\theta}z^{-5}$. Since $F_5(w)=w^5-(5/2)b_1w$ in this case, we obtain the desired relation

$$f(z^2)^{5/2}-\frac{5}{2}b_1f(z^2)^{1/2}=z^5+e^{i\theta}z^{-5}.$$

Expanding the left hand side of this relation, we have $b_2=0$. It means that $b_2=0$ for each extremal function. Thus we can deduce that $|b_4+b_1b_2|<2/5$ if $b_2\neq 0$. Moreover, it follows directly from Theorem 2 that if $b_2=0$ then $|b_4|\leqq 2/5$.

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