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AN EXAMPLE OF AN OPEN RIEMANN SURFACE NOT UNIFORMLY LARGE WITH RESPECT TO GREEN'S FUNCTIONS

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§ 1. Introduction and the main result.

Let R be an open Riemann surface. To assure certain uniformness of R, we may impose the following conditions on R.

(G) Assuming that R admits Green's functions $g(\cdot, q; R)$ with the pole $q \in R$, there is a positive constant M such that $\{p \in R: g(p, q; R) > M\}$ is simply connected for every $q \in R$.

(H) Letting $d_R(\cdot, \cdot)$ be Poincaré's hyperbolic distance on R, there is a positive ε such that $\{p \in R: d_R(p, q) < \varepsilon\}$ is simply connected for every $q \in R$.

A surface satisfying (H) is called one with a positive injectivity radius and has several nice properties (cf. [4]). The condition (G) is recently considered in [3] and [7].

Remark 1. The condition (G) implies (H). In fact, let F_p be a Fuchsian group acting on $\{|z|<1\}$ and corresponding to a universal covering map π_p of $\{|z|<1\}$ to a Riemann surface R satisfying (G) such that $\pi_p(0)=p$ for arbitrarily given $p \in R$. Then since $g(\pi_p(z), p; R)=\sum_{f \in F_p} \log|1/f(z)|$, π_p is injective on the disk $\{z: \log|1/z|>M\}$, which has some hyperbolic radius depending only on M.

Remark 2. In case of finite surfaces, (G) is equivalent to (H). In fact, in this case each of (G) and (H) is equivalent to the condition for non-existence of punctures.

But in general, (H) does not necessarily implies (G). Actually, the purpose of this note is to show the following.

THEOREM. *There is a regular Riemann surface of Parreau-Widom type which satisfies* (*H*) *but not* (*G*).

Here for regular Riemann surfaces of Parreau-Widom type, see for example [5]. We will construct a family of Riemann surfaces satisfying (H) but not (G) in § 2, and give a proof of Theorem in § 3.

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§ 2. Construction.

Let S_1 be a compact bordered Riemann surface of genus one whose border consists of a single geodesic d_1. Namely, we have a Riemann surface R_1 of genus one with one hole and a bordered subsurface S_1(called the Nielsen kernel, cf. [2]) of R_1 such that R_1-S_1 is a doubly connected region and the border of S_1 is a compact simple geodesic in R_1(with respect to Poincaré's hyperbolic metric on R_1). Let c be the length of d_1 and S_0 be a compact bordered Riemann surface of genus one whose border consists of two closed geodesics d_0 and d_0' with the same length c. Also we set $W_1=R_1-(S_1-d_1)$.

For every $n\geqq 2$, we can construct inductively a compact bordered Riemann surface S_n from S_{n-1} and S_0 by gluing the border d_{n-1} of S_{n-1} to d_0 isometrically so that the hyperbolic metrics on S_{n-1} and S_0 coincide with that on S_n. And for every n, let R_n be a Riemann surface (called the Nielsen extension of S_n, cf. [2]) obtained from S_n and W_1 by gluing d_n and d_1 considered as the border of W_1. In the sequel, we denote by $S_{1,n}$ and $W_{1,n}$ the parts of R_n corresponsing to S_1 and W_1 in R_n, respectively.

Also fix a point p_1 in S_1 and, denote by p_n the point on $S_{1,n}$ corresponding to p_1 for every n.

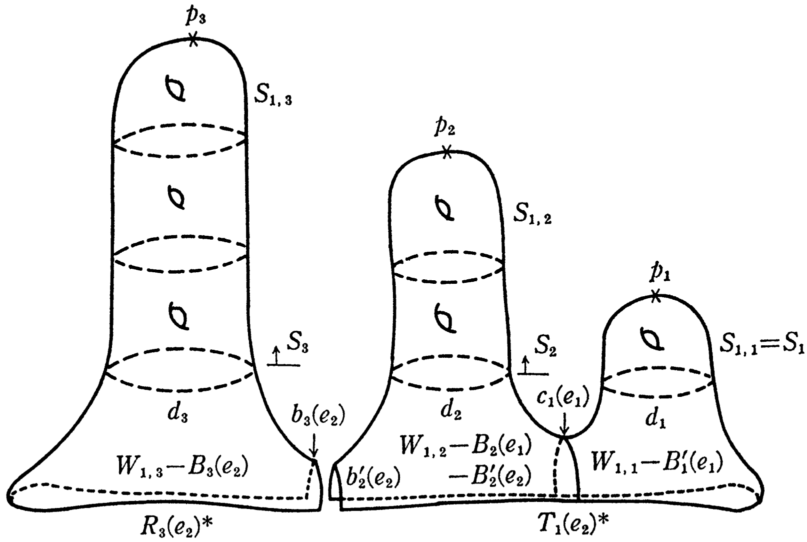
Here recall that W_1 can be represented as the quotient bordered surface of $\Delta=\{z: \operatorname{Im} z>0, \operatorname{Re} z\geqq 0\}$ by the elementary group generated by $A(z)=az$, where $a>1$ and then is determined by c. For every n and positive $e<(a^{1/2}-1)/2$, we denote by $B_n(e)$ and $B_n'(e)$ the subregions of $W_{1,n}$ corresponding to $\{z\in\Delta: |z-1|<e\}$ and $\{z\in\Delta: |z-a^{1/2}|<e\cdot a^{1/2}\}$, respectively. Note that $B_n(e)$ and $B_n'(e)$ are mutually disjoint. Let $b_n(e)$ and $b_n'(e)$ be the relative boundaries of $B_n(e)$ and $B_n'(e)$ in $W_{1,n}$, respectively.

Then for any given sequence $\{e_n\}_{n=1}^{\infty}$ of positive numbers e_n with $e_n<(a^{1/2}-1)/2$, we can construct a sequence $\{T_n(e_n)\}_{n=0}^{\infty}$ of Riemann surfaces, where we set $T_0(e_0)=R_1$, inductively as follows.

Set $T_0(e_1)^*=R_1-B_1'(e_1)$ and $R_2(e_1)^*=R_2-B_2(e_1)$, and glue $b_1'(e_1)$ to $b_2(e_1)$ by the mapping corresponding to $f(z)=-e_1^2\cdot a^{1/2}/(z-a^{1/2})+1$. Then as before we have a Riemann surface $T_1(e_1)$ having the curve $c_1(e_1)$ resulting from $b_1'(e_1)$ and $b_2(e_1)$ as a geodesic (with respect to the hyperbolic metric on $T_1(e_1)$). Next suppose that we have constructed $\{T_n(e_n)\}_{n=1}^{k-1}$. Then set $T_{k-1}(e_k)^*=T_{k-1}(e_{k-1})-B_k'(e_k)$ and $R_{k+1}(e_k)^*=R_{k+1}-B_{k+1}(e_k)$, and glue $b_k'(e_k)$ to $b_{k+1}(e_k)$ similarly as above. We denote by $T_k(e_k)$ the resulting Riemann surface. See Figure below.

Now since $T_n(e_{n+1})^*$ can be considered as a bordered subsurface of $T_{n+1}(e_{n+2})^*$ for every n, we can consider a Riemann surface $R=\bigcup_{n=1}^{\infty}T_n(e_{n+1})^*$ as the inductive limit of $\{T_n(e_{n+1})^*\}_{n=1}^{\infty}$. And we have the following.

PROPOSITION 1. *For every sequence $\{e_n\}_{n=1}^{\infty}$ of positive numbers e_n such that $e_n<(a^{1/2}-1)/2$, the surface $R=\bigcup_{n=1}^{\infty}T_n(e_{n+1})^*$ satisfies (H) but not (G).*



[Figure] The parts of the surface $T_2(e_2)$.

Proof. It is clear from the above construction that R satisfies (H) and that R admits Green's functions.

Next let $S=\bigcup_{n=1}^{\infty} S_n$ be the inductive limit of $\{S_n\}_{n=1}^{\infty}$. Then $\{S_n-d_n\}_{n=1}^{\infty}$ gives a canonical exhaustion of S. Also from the construction we can see by Nevanlinna's modular test (cf. [1, IV. 15D]) that S admits no Green's functions, or equivalently, that $g(\cdot, p_1; S_n-d_n)$ tends to $+\infty$ locally uniformly on S as n tends to $+\infty$, where we regard p_1 as a point on S.

Since $g(\cdot, p_n; R)>g(\cdot, p_1; S_n-d_n)$ on S_n-d_n considered as a subsurface of R(where p_1 is identified with p_n) for every n, we can find, for any given M, an N such that $\{p\in R: g(p, p_N; R)>M\}$ contains $S_{1,N}$, hence is not simply connected.

Thus we conclude that R does not satisfy (G). q. e. d.

§ 3. Proof of Theorem.

First we will show the following

PROPOSITION 2. *Let $\{t_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ be two sequences of positive numbers which satisfy the following conditions; for every n, it holds that*

1) $t_n<\min\{t_{n-1}, 1/n^3\}$ *and* $e_n<(a^{1/2}-1)/2$,

2) $D_n=\{p\in T_{n-1}(e_{n-1}): g_{1,n-1}>4t_n\}$ *is homeomorphic to* $T_{n-1}(e_{n-1})$ *and contains* D_{n-1},

3) $g_{1,n-1}<t_n/2$ *on* $\partial T_{n-1}(e_n)^*=b_n'(e_n)$, *and*

4) $|g_{1,n}-g_{1,n-1}|<t_n/2^{n+1}$ *on* $T_{n-1}(e_n)^*$,

where we set $t_0=1$, $D_0=\emptyset$ *and* $g_{1,n}=g(\cdot, p_1; T_n(e_n))$ *for every* n.

Then $R=\bigcup_{n=1}^{\infty} T_n(e_{n+1})^*$ *is a regular Riemann surface of Parreau-Widom type.*

Proof. To show that R is regular, first not that, by 4), $g_{1,n}$ converges locally uniformly on $R-\{p_1\}$ to a positive harmonic function, say h, as n tends to $+\infty$. Moreover, 1), 3) and 4) imply that

5) $|h-g_{1,n-1}|\leqq\sum_{m=n}^{\infty}|g_{1,m}-g_{1,m-1}|<\sum_{m=n}^{\infty}t_m/2^{m+1}<t_n/2^n$ on $T_{n-1}(e_n)^*$, and

6) $h\leqq\sum_{m=n}^{\infty}|g_{1,m}-g_{1,m-1}|+g_{1,n-1}<t_n/2^n+t_n/2\leqq t_n$ on $\partial T_{n-1}(e_n)^*$,

for every n.

Hence by the maximal principle, for every $\varepsilon>0$, we can find a compact set F in R such that $h<\varepsilon$ on $R-F$, which implies that $h=g(\cdot, p_1; R)$ and that R is a regular Riemann surface.

Next recall that a regular Riemann surface R is of Parreau-Widom type if and only if

$$\sum_{q\in Z} g(q, p_1; R)<+\infty,$$

where Z is the set of all critical points of $g(\cdot, p_1; R)$ including multiplicity (cf. [5, V. 1C Theorem]).

Fix n arbitrarily. Since $L_n=\{p\in R: g(p, p_1; R)=2t_n\}$ is contained in $T_{n-1}(e_n)^*-\bar{D}_n$ by 4), 5) and 6), and since $T_{n-1}(e_n)^*-\bar{D}_n$ is a doubly connected region by 2), we see that L_n is a simple closed analytic curve and $D'_n=\{p\in R: g(p, p_1; R)>2t_n\}$ is homeomorphic to $T_{n-1}(e_{n-1})$.

Here it is well known (as a corollary of Riemann-Roch theorem, cf. [1, V. 27A]) that D'_n contains exactly $2\cdot\sum_{k=1}^{n} k=n(n+1)$ critical points of $g(\cdot, p_1; R)$. Hence $D'_n-D'_{n-1}$ contains $2n$ such points and it holds that

$$\sum_{q\in Z} g(q, p_1; R)\leqq\sum_{q\in Z\cap D'_1} g(q, p_1; R)+\sum_{n=1}^{\infty}4(n+1)t_n,$$

which is finite by 1). Thus we conclude that R is of Parreau-Widom type.

q. e. d.

Thus to complete the proof of Theorem, it remins to give such sequences $\{t_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ as in Proposition 2.

For this purpose, fix n arbitrarily, and suppose that $\{t_k\}_{k=1}^{n-1}$ and $\{e_k\}_{k=1}^{n-1}$ are determined. Then take t_n so small that 1) and 2) in Proposition 2 hold, and then take $e_{n,0}$ so small that $g_{1,n-1}<t_n/2^{n+2}$ on $b'_n(e_{n,0})$. Next set $E=\{p\in T_{n-1}(e_{n-1}): g_{1,n-1}(p)\geqq t_n/2^{n+2}\}$, then E is compact in $T_{n-1}(e_{n-1})-\overline{B'_n(e_{n,0})}$. And if we find an $e'_n<e_{n,0}$ such that

$$|g(\cdot, p_1; T_n(e'_n))-g_{1,n-1}|<t_n/2^{n+2} \quad \text{on} \quad E,$$

we can conclude by the maximal principle that 3) and 4) hold with $e_n=e'_n$.

Here the existence of such an e'_n follows by the fact that $g(\cdot, p_1; T_n(e'_n))$ converges to $g_{1,n-1}$ locally uniformly on $T_{n-1}(e_{n-1})$ as e'_n tends to 0. This fact seems to be essentially well-known. But the author can not find any adequate reference, so we include a proof.

For every $e'_n < e_{n,0}$, let $\hat{T}_n(e'_n)$ be the double of $T_n(e'_n)$ with respect to two ideal boundary arcs of $T_n(e'_n)$ between $b'_n(e_{n,0})$ and $b_{n+1}(e_{n,0})$. Then clearly, $\hat{T}_n(e'_n)$ admits Green's functions, and it holds that

$$g(p,\ p_1;\ T_n(e'_n)) = g(p,\ p_1;\ \hat{T}_n(e'_n)) - g(p,\ p_1^*;\ \hat{T}_n(e'_n))$$

on $T_n(e'_n)$ for every e'_n, where p_1^* is the mirror image of p_1. Since $\hat{T}_n(e'_n)$ converges to a Riemann surface with one node corresponding to $e'_n = 0$ in the sence of the conformal topology as e'_n tends to 0, we can see the assertion by [6, Corollary 1].

Now we have obtained t_n and e_n satisfying 1)-4) in Proposition 2. And by induction, we can show the existence of desired sequences, and finish the proof of Theorem. q. e. d.

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