S. UDAGAWA
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BI-ORDER REAL HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

By Seiichi Udagawa

§ 0. Introduction.

Real hypersurfaces in a complex projective space have been studied by many differential geometers (for example, see [2], [4], [5], [6], [10] and [11]).

Typical examples of real hypersurfaces in a complex projective space are homogeneous ones ([10]). They are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 2 (see [10]). Homogeneous real hypersurfaces in a complex projective space have been characterized by many differential geometers, who have used the multiplicities of the principal curvatures of them.

In [7], some of homogeneous real hypersurfaces are characterized from a different point of view. In this paper, we give some generalizations of the results in [7].

Let $f: M \to E^N$ be an isometric immersion from an n-dimensional compact Riemannian manifold into an N-dimensional Euclidean space, and let Δ and $\mathrm{Spec}(M)=\{0<\lambda_1<\lambda_2<\cdots\}$ be the Laplacian of M and the spectrum of M, respectively. Then, it is known that f can be decomposed as $f=f_0+\sum_{k\in N} f_k$, where $\Delta f_k=\lambda_k f_k$ and f_0 is a constant mapping, and the addition is convergent componentwise for the L^2-topology on $c^\infty(M)$. We say that the immersion is of *order* $\{k\}$ (or *mono-order*) if $f=f_0+f_k$, $k\in \boldsymbol{N}$, $f_k\neq 0$, and of *order* $\{k, l\}$ (or *bi-order*) if $f=f_0+f_k+f_l$, $k, l\in \boldsymbol{N}$, $1>k$, $f_k, f_l\neq 0$. Moreover, we say that M is mono-order (resp. bi-order) if f is mono-order (resp. bi-order). From a well known result of Takahashi [12], we know that M is mono-order if and only if M is a minimal submanifold of some sphere of E^N. Minimal submanifolds of a sphere have been studied by many differential geometers. Then, we consider the following problems:

i) Investigate the necessary and sufficient conditions for f to be bi-order.

ii) Classify the bi-order immersions.

These problems are very interesting, but very difficult. Hence, we consider the restricted problems as follows: Let $F: \boldsymbol{C}P^m(4)\to E^N$ be the first standard imbedding of an m-dimensional complex projective space of constant holomorphic sectional curvature 4 into an N-dimensional Euclidean space (for details, see

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§1), and let $\phi: M^n \to CP^m(4)$ be an isometric immersion of an n-dimensional Riemannian manifold. We consider $f = F \circ \phi: M^n \to E^N$. Note that f is mono-order (resp. bi-order) if and only if the restriction of an arbitrary first eigenfunction of $CP^m(4)$ to M^n is also an eigenfunction of M^n (resp. sum of two eigenfunctions of M^n corresponding to distinct eigenvalues). It is interesting to study the problems: To what extent, does the analytic condition (such as the order of the immersion) determine the geometry of submanifolds? If ϕ's are Kaehler immersions, mono-order and bi-order immersions are completely determined ([8], [13]). In fact, they are totally geodesic Kaehler submanifolds (order $\{1\}$) or Einstein parallel Kaehler submanifolds (order $\{1, 2\}$). Moreover, every totally real submanifold M^m of CP^m is mono-order. In this paper, we consider the case that M is a real hypersurface of CP^m.

Then, Martinez and Ros [7] obtained the following

THEOREM A. *Let M be a real hypersurface of $CP^m(4)$ $(m \geqq 2)$. Then, M is minimal in some sphere of E^N if and only if M is locally congruent to the geodesic hypersphere*

$$\pi(S^1(\sqrt{1/2(m+1)}) \times S^{2m-1}(\sqrt{(2m+1)/2(m+1)})),$$

where $\pi: S^{2m+1}(1) \to CP^m(4)$ is the usual fibration.

THEOREM B. *Let M be a compact minimal real hypersurface of $CP^m(4)$. Then, M is bi-order in E^N if and only if M is congruent to one of the following:*

i) *a geodesic hypersphere $\pi(S^1(\sqrt{1/2m} \times S^{2m-1}(\sqrt{(2m-1)/2m}))$, $m \geqq 2$,*

ii) *$\pi(S^m)(\sqrt{1/2}) \times S^m(\sqrt{1/2}))$, m: odd, $m \geqq 3$.*

Remark 1. In fact, $CP^m(4)$ is minimally imbedded in a sphere $S^{(m+1)^2-2}(\sqrt{m/2(m+1)})$ of radius $\sqrt{m/2(m+1)}$ and $\pi(S^1(\sqrt{1/2(m+1)}) \times S^{2m-1}(\sqrt{(2m+1)/2(m+1)}))$ is minimal in a small hypersphere $S^{(m+1)^2-3}(\sqrt{(4m^2-1)/8m(m+1)})$ of $S^{(m+1)^2-2}(\sqrt{m/2(m+1)})$.

Our results are the following

THEOREM 1. *Let M be a compact real hypersurface with constant mean curvature in $CP^m(4)$. Then, M is bi-order in E^N if and only if M is congruent to one of the following*

i) *a geodesic hypersphere $\pi(S^1(\sqrt{r_1}) \times S^{2m-1}(\sqrt{r_2}))$ with $r_1 + r_2 = 1$, $r_1 \neq 1/2(m+1)$.*

ii) *$\pi(S^p(\sqrt{p/2(m+1)}) \times S^q(\sqrt{(q+2)/2(m+1)}))$,*

iii) *$\pi(S^p(\sqrt{(p+1)/2(m+1)}) \times S^q(\sqrt{(q+1)/2(m+1)}))$,*

where p, $q > 1$, and p and q are odd with $p + q = 2m$.

A submanifold M of a sphere S^{N-1} in E^N is called *mass-symmetric* in S^{N-1} if the center of gravity of M coincides with the center of S^{N-1} in E^N. Then, we have

THEOREM 2. *Let M be a compact real hypersurface of $CP^m(4)$. Then, M is mass-symmetric and bi-order in E^N if and only if M is congruent to the case* iii) *in Theorem* 1.

Remark 2. Since $CP^m(4)$ is mass-symmetric in $S^{(m+1)^2-2}(\sqrt{m/2(m+1)})$, the condition that M is mass-symmetric is equivalent to the one that the center of gravity of M coincides with the center of gravity of $CP^m(4)$.

For Kaehler submanifolds in CP^m, some kind of spectral inequality involving λ_1 and λ_2 of M is obtained ([14]). For real hypersurfaces as well as Koehler submanifolds, the same kind of spectral inequality will be proved in § 5. In fact, Theorem 1 and Theorem 2 are used to prove the spectral inequality.

All manifolds are assumed to be connected and of real dimension $\geqq 2$ unless otherwise stated.

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§ 1. Preliminaries.

Let CP^m be a complex projective space obtained as a quotient space of the unit sphere $S^{2m+1}(1)=\{z\in C^{m+1} | zz^*=z\bar{z}^t=1\}$ by identifying z with λz, $\lambda\in C$ and $|\lambda|=1$. Let g_0 be the canonical metric on CP^m, which is the invariant metric such that the fibration $\pi: S^{2m+1}(1)\to CP^m$ is a Riemannian submersion. It is known that (CP^m, g_0) has constant holomorphic sectional curvature 4. Moreover, its Riemannian curvature tensor $\tilde{R}$ is given by

$$(1.1)\qquad \tilde{R}(X, Y)Z=g_0(Y, Z)X-g_0(X, Z)Y+g_0(\tilde{J}Y, Z)\tilde{J}X -g_0(\tilde{J}X, Z)\tilde{J}Y+2g_0(X, \tilde{J}Y)\tilde{J}Z \quad \text{for any } X, Y, Z\in T(CP^m),$$

where $\tilde{J}$ is the complex structure tensor of CP^m. Let $HM(m+1)=\{P\in gl(m+1, C) | \bar{P}=P^t\}$ be the set of all Hermitian matrices of degree $(m+1)$ with the metric g such that

$$(1.2)\qquad g(P, Q)=(1/2)tr(PQ) \qquad \text{for any } P, Q\in HM(m+1).$$

Sakamoto [9] proved that the mapping $\tilde{F}: S^{2m+1}(1)\to HM(m+1)$ given by

$$(1.3)\qquad \tilde{F}(z)=z^*z=\bar{z}^t z, \qquad z\in S^{2m+1}(1)$$

induces an immersion $F: CP^m\to HM(m+1)$ such that the following hold:

i) $F(CP^m)=\{P\in HM(m+1) \mid P^2=P \text{ and } tr(P)=1\}$,

ii) F is an equivariant full isometric imbedding into

$$H_1M(m+1)=\{P\in HM(m+1) \mid tr(P)=1\}.$$

In the following, we identify CP^m with $F(CP^m)$. We denote by D the Riemannian connection of $HM(m+1)$, by $\tilde{\nabla}$ the induced connection of CP^m and denote by $\tilde{\nabla}^{\perp}$, $\tilde{\sigma}$, $\tilde{A}$ and $\tilde{H}$, the normal connection, the second fundamental form, the corresponding shape operator and the mean curvature vector of CP^m in $HM(m+1)$, respectively.

LEMMA 1. *The tangent and normal speces at each point $B\in CP^m$ are given respectively by*

$$T_BCP^m=\{X\in HM(m+1) \mid XB+BX=X\}$$
$$T_B^{\perp}CP^m=\{Z\in HM(m+1) \mid ZB=BZ\}. \tag{1.4}$$

Moreover, we have

$$\tilde{J}X=\sqrt{-1}(I-2B)X, \tag{1.5}$$

$$\tilde{\sigma}(X, Y)=(XY+YX)(I-2B), \qquad \tilde{A}_ZX=(XZ-ZX)(I-2B), \tag{1.6}$$

$$\tilde{H}_B=(2/m)(I-(m+1)B), \tag{1.7}$$

$$\tilde{\sigma}(\tilde{J}X, \tilde{J}Y)=\tilde{\sigma}(X, Y), \tag{1.8}$$

$$\tilde{\nabla}\tilde{\sigma}=0, \tag{1.9}$$

$$g(\tilde{\sigma}(X, Y), \tilde{\sigma}(V, W))=2g(X, Y)g(V, W)+g(X, V)g(Y, W)$$
$$+g(X, W)g(Y, V)+g(\tilde{J}X, V)g(\tilde{J}Y, W)+g(\tilde{J}X, W)g(\tilde{J}Y, V), \tag{1.10}$$

$$g(\tilde{\sigma}(X, Y), I)=0, \qquad g(\tilde{\sigma}(X, Y), B)=-g(X, Y), \tag{1.11}$$

where I is the identity matrix of $HM(m+1)$ and $X, Y, V, W\in T_B(CP^m)$, $Z\in T_B^{\perp}(CP^m)$.

§2. Real hypersurfaces satisfying a certain equation.

Let M be a real hypersurface in CP^m. We put

$$\tilde{J}X=JX+fN, \tag{2.1}$$

where X is a unit tangent vector of M and N is the unit normal vector of M and f is the 1-form on M. The pair (J, f) is called the almost contact structure of M. Let $e_1, \cdots, e_{2m-1}, N$ be a local field of orthonormal frames of CP^m such that, restricted to M, $e_1, \cdots, e_{2m-1}$ are tangent to M. With reference to the local field of orthonormal frames, we can verify the following relations

$$\textstyle\sum_k J_{ik}J_{kj}=f_if_j-\delta_{ij}, \qquad \sum_j f_jJ_{ji}=0, \tag{2.2}$$

$$\sum_i f_i^2=1\,, \qquad J_{ij}+J_{ji}=0\,,$$

where $J=(J_{ij})$, $f=(f_i)$ and $i, j, k=1, \cdots, 2m-1$. We denote by ∇ the induced connection of M, and denote by σ and A, the second fundamental form of M in CP^m and the corresponding shape operator, respectively. We may put

$$\sigma(e_i, e_j)=h_{ij}N\,, \qquad h_{ij}=h_{ji}\,. \tag{2.3}$$

We may define the covariant derivative of σ by

$$(\nabla'_{e_i}\sigma)(e_j, e_k)=e_i(h_{jk})N-\sigma(\nabla_{e_i}e_j, e_k)-\sigma(e_j, \nabla_{e_i}e_k)\,, \tag{2.4}$$

and we put

$$(\nabla'_{e_i}\sigma)(e_j, e_k)=h_{jki}N\,. \tag{2.5}$$

Then, we have the Gauss equation and the Codazzi equation, respectively

$$R_{ijkl}=\delta_{jk}\delta_{il}-\delta_{ik}\delta_{jl}+J_{jk}J_{il}-J_{ik}J_{jl}+2J_{ji}J_{kl}+hJ_{jk}h_{il}-h_{ik}h_{jl}\,, \tag{2.6}$$

$$h_{kji}-h_{kij}=f_iJ_{jk}-f_jJ_{ik}+2f_kJ_{ji}\,, \tag{2.7}$$

where $R_{ijkl}=g(R(e_i, e_j)e_k, e_l)$ and R is the curvature tensor of M.

Let H and $\hat{H}$ be the mean curvature vector of M in CP^m and the mean curvature vector of M in $H_1M(m+1)$, respectively. Then, we see that

$$H=(1/(2m-1))\sum_i\sigma(e_i, e_i)=(1/(2m-1))\sum_i h_{ii}N\,, \tag{2.8}$$

$$\begin{aligned}\hat{H}_B&=(1/(2m-1))\{\sum_i\sigma(e_i, e_i)+\tilde{\sigma}(e_i, e_i)\}\\ &=H_B+(1/(2m-1))\{4(I-(m+1)B)-\tilde{\sigma}(N, N)\}\\ &\qquad\qquad \text{for any point } B\in M\,,\end{aligned} \tag{2.9}$$

where we have used (1.7). The following relations are verified:

$$\Delta B=-(2m-1)\hat{H}_B=\sum_{k\in N}\lambda_k B_k\,, \tag{2.10}$$

$$\Delta^2 B=-(2m-1)\Delta\hat{H}_B=\sum_{k\in N}\lambda_k^2 B_k\,, \tag{2.11}$$

where $\Delta B_k=\lambda_k B_k$. Then, it follows from the definition of bi-order immersion that if M is compact, M is bi-order if and only if

$$\Delta\hat{H}_B=a\hat{H}_B+b(B-B_0)\,, \tag{2.12}$$

where a and b are positive constants and B_0 is the constant part of B. If M is of order $\{k, l\}$, then we have

$$a=\lambda_k+\lambda_l\,, \qquad b=\lambda_k\lambda_l/(2m-1)\,. \tag{2.13}$$

Therefore, computing $\Delta\hat{H}$, we want to derive the conditions that M satisfies the equation (2.12).

Remark 3. $\Delta\hat{H}$ is computed in [7] under the assumption that M is minimal in CP^m.

PROPOSITION 1. *Let M be a real hypersurface of CP^m. Then, we have*

$$D_{e_i}\hat{H}=-A_H e_i+(1/(2m-1))\{-2(2m+1)e_i+2f_i\textstyle\sum_j f_j e_j \tag{2.14}$$
$$+\textstyle\sum_j e_i(h_{jj})N\}+\tilde{\sigma}(e_i, H)+(2/(2m-1))\tilde{\sigma}(Ae_i, N),$$

$$\Delta\hat{H}_B=(1/(2m-1))\textstyle\sum_{i,j,k}\{2h_{jji}h_{ik}+h_{iik}h_{jj}\}e_k \tag{2.15}$$
$$-(1/2m-1))\textstyle\sum_{i,j}h_{jjii}N+(4/(2m-1))\tilde{J}A\tilde{J}N+(6m+2+\|\sigma\|^2)H$$
$$+(2/(2m-1))\textstyle\sum_i\tilde{\sigma}(Ae_i, Ae_i)+2\sum_i\tilde{\sigma}(e_i, A_He_i)$$
$$+(8(2m+1)/(2m-1))(I-(m+1)B)-(4/(2m-1))\textstyle\sum_{i,j}h_{jji}\tilde{\sigma}(e_i, N)$$
$$-\{2(2m+2+\|\sigma\|^2)/(2m-1)+(2m-1)\|H\|^2\}\tilde{\sigma}(N, N).$$

Proof. From (2.8), (2.9) and (1.9), we obtain

$$D_{e_i}\hat{H}=-A_He_i+(1/(2m-1))\textstyle\sum_j e_i(h_{jj})N+\tilde{\sigma}(e_i, H) \tag{2.16}$$
$$-(4(m+1)/(2m-1))e_i+(1/(2m-1))\{\tilde{A}_{\tilde{\sigma}(N,N)}e_i+2\tilde{\sigma}(Ae_i, N)\}.$$

Using (1.10), we get

$$\tilde{A}_{\tilde{\sigma}(N,N)}e_i=2e_i+2g(\hat{J}N, e_i)\hat{J}N. \tag{2.17}$$

Since (2.1) implies $\tilde{J}N=-\sum_j f_j e_j$, (2.16) and (2.17) yield (2.14). In the following, we assume that $(\nabla_{e_i}e_j)_B=0$ for any i and j at $B\in M$. Then, we obtain

$$\Delta\hat{H}_B=-\textstyle\sum_i D_{e_i}D_{e_i}\hat{H}_B \tag{2.18}$$
$$=\textstyle\sum_i\{\nabla_{e_i}(A_He_i)+\sigma(e_i, A_He_i)+\tilde{\sigma}(e_i, A_He_i)\}$$
$$+(2(2m+1)/(2m-1))\textstyle\sum_i\{\sigma(e_i, e_i)+\tilde{\sigma}(e_i, e_i)\}$$
$$-(2/(2m-1))\textstyle\sum_i\{-g(JAe_i, e_i)\tilde{J}N+g(\tilde{J}N, e_i)(-\tilde{J}Ae_i+\tilde{\sigma}(e_i, \tilde{J}N))\}$$
$$-(1/(2m-1))\textstyle\sum_{i,j}\{e_ie_i(h_{jj})N+e_i(h_{jj})(-Ae_i+\tilde{\sigma}(e_i, N))\}$$
$$-\textstyle\sum_i\{-\tilde{A}_{\tilde{\sigma}(e_i,H)}e_i+\tilde{\sigma}(\sigma(e_i, e_i), H)$$
$$+(1/(2m-1))\tilde{\sigma}(e_i, e_i(h_{jj})N)-\tilde{\sigma}(e_i, A_He_i)\}$$
$$-(2/(2m-1))\textstyle\sum_i\{-\tilde{A}_{\tilde{\sigma}(Ae_i,N)}e_i+\tilde{\sigma}(\nabla_{e_i}(Ae_i), N)$$
$$+\tilde{\sigma}(\sigma(e_i, Ae_i), N)-\tilde{\sigma}(Ae_i, Ae_i)\}.$$

Moreover, we see that

$$\sum_i \nabla_{e_i}(A_H e_i)=(1/(2m-1))\sum_{i,j,k}\{h_{jji}h_{ik}+h_{iik}h_{jj}\}e_k$$

(by (2.2), (2.4) and (2.7)),

$$\sum_i \nabla_{e_i}(Ae_i)=\sum_{i,k}h_{iik}e_k \qquad \text{(by (2.2), (2.4) and (2.7))},$$

$$\sum_i g(JAe_i, e_i)\tilde{J}N=0 \qquad \text{(by (2.2) and (2.3))},$$

$$\sum_i g(\tilde{J}N, e_i)\tilde{J}Ae_i=\tilde{J}A\tilde{J}N,$$

$$\sum_i g(\tilde{J}N, e_i)\tilde{\sigma}(e_i, \tilde{J}N)=\tilde{\sigma}(N, N) \qquad \text{(by (1.8))},$$

$$\sum_i \tilde{A}_{\tilde{\sigma}(e_i, H)}e_i=2(m-1)H \qquad \text{(by (1.10) and (2.2))},$$

$$\sum_i \tilde{A}_{\tilde{\sigma}(Ae_i, N)}e_i=(2m-1)H+\tilde{J}A\tilde{J}N \qquad \text{(by (1.10) and (2.2))}.$$

Therefore, (2.18) is reduced to (2.15). Q.E.D.

LEMMA 2. *Let M be a real hypersurface of CP^m. Then, the following relations are verified*:

(2.19) $$g(B, B)=1/2\,,$$

(2.20) $$g(B, \hat{H})=-1\,,$$

(2.21) $$g(\hat{H}, \hat{H})=\|H\|^2+4(2m^2-1)/(2m-1)^2\,,$$

(2.22) $$g(\Delta\hat{H}, B)=-(2m-1)\|H\|^2-4(2m^2-1)/(2m-1)\,,$$

(2.23) $$g(\Delta\hat{H}_B, \hat{H}_B)=(1/2)\Delta\|H\|^2+\|dH\|^2+(8(m+1)(4m^2-2m-1))/(2m-1)^2 +(4/(2m-1)^2)\|\sigma\|^2+(10m+6+\|\sigma\|^2)\|H\|^2 -(4/(2m-1)^2)\|A\tilde{J}N\|^2-(8/(2m-1))g(A\tilde{J}N, \tilde{J}H)\,.$$

Lemma 2 can be obtained by using (1.2), (1.4), (1.10), (1.11) and (2.15). Now, we prepare the following

LEMMA 3. *Let M be a real hypersurface of CP^m. Assume that M is mass-symmetric and satisfies the equation* (2.12). *Then,*

$$\|H\|=constant.$$

Proof. If M is mass-symmetric, it follows from (1.7) that

$$B_0=(1/(m+1))I\,.$$

Then, using (2.12), Proposition 1 and Lemma 2, we have

$$\|H\|^2=(1/(2m-1))\{a-b/2+b/2(m+1)-4(2m^2-1)/(2m-1)\}.$$

Therefore, we see that $\|H\|^2=$constant. Q.E.D.

§3. Proofs of Theorems.

In consideration of Lemma 3, we assume in the following that M satisfies the equation (2.12) and has constant mean curvature in CP^m. First, (2.12) and Lemma 2 imply that

$$(3.1)\qquad g(B, B_0)=(1/b)\{-a+b/2+(2m-1)\|H\|^2+4(2m^2-1)/(2m-1)\}$$
$$=\text{constant}\qquad \text{for any } B\in M.$$

Then, we see that

$$(3.2)\qquad g(B-B_0, B-B_0)=1/2-2g(B, B_0)+g(B_0, B_0)$$
$$=\text{constant}.$$

(3.2) implies that M is contained in some sphere of radius $\sqrt{1/2-2g(B, B_0)+g(B_0, B_0)}$ with center B_0. Since CP^m is imbedded in a sphere $S^{(m+1)^2-2}_{(1/(m+1))I}(\sqrt{m/2(m+1)})$ of radius $\sqrt{m/2(m+1)}$ with center $(1/(m+1))I$, we consider the following two cases.

Case 1. $B_0\neq(1/(m+1))I$.

In this case, M is fully immersed in a small hypersphere of $S^{(m+1)^2-2}_{(1/(m+1))I}(\sqrt{m/2(m+1)})$ because CP^m is fully imbedded in $H_1M(m+1)$ and M is a real hypersurface of CP^m. We define a hyperplane $H_1'M(m+1)$ of $H_1M(m+1)$ as follows

$$H_1'M(m+1)=\left\{Q\in H_1M(m+1)\mid Q=\left[\begin{array}{c|c} S_1 & S_3\\ \hline \bar{S}_3^t & S_2\end{array}\right],\right.$$
$$S_1\in HM(u+1),\quad S_2\in HM(v+1),\quad u+v+1=m,$$
$$\left.tr(S_1)=r_1,\quad tr(S_2)=r_2,\quad r_1+r_2=1\right\}.$$

Then, we may assume that M is contained in $H_1'M(m+1)$. Since

$$Q_0=\left[\begin{array}{c|c} (r_1/(u+1))I_{u+1,u+1} & 0\\ \hline 0 & (r_2/(v+1))I_{v+1,v+1}\end{array}\right]$$

with $r_1+r_2=1$ is contained in $H_1'M(m+1)$ and

$$g(B-Q_0, B-Q_0)=1/2-(1/2)(r_1^2/(u+1)+r_2^2/(v+1))$$
$$=\text{constant}\qquad \text{for any } B\in M\subset H_1'M(m+1),$$

we see that

$$B_0=\left[\begin{array}{c|c}(r_1/(u+1))I_{u+1,u+1} & 0 \\ \hline 0 & (r_2/(v+1))I_{v+1,v+1}\end{array}\right].$$

On the other hand, for any $B\in M$, we see that

$$B=z^*z=\begin{bmatrix}|z_1|^2 & \bar{z}_1z_2 \cdots\cdots & \bar{z}_1z_{m+1} \\ \bar{z}_2z_1 & |z_2|^2 \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots \\ \bar{z}_{m+1}z_1 & \cdots\cdots & |z_{m+1}|^2\end{bmatrix} \tag{3.3}$$

with $|z_1|^2+\cdots+|z_{u+1}|^2=r_1$, $|z_{u+2}|^2+\cdots+|z_{m+1}|^2=r_2$. Therefore, M is locally congruent to $\pi(S^q(\sqrt{r_1})\times S^q(\sqrt{r_2}))$ with $r_1+r_2=1$, $p=2u+1$, $q=2v+1$ and $p+q=2m$. For any $(z, w)\in S^p(\sqrt{r_1})\times S^q(\sqrt{r_2})\subset S^{2m+1}(1)$, we see that

$$\begin{aligned}\tilde{F}((z, w))&=\begin{bmatrix}z^*\\ w^*\end{bmatrix}(z, w)\\ &=\left[\begin{array}{c|c}\bar{z}_iz_j & \bar{z}_iw_j \\ \hline \bar{w}_jz_i & \bar{w}_iw_j\end{array}\right]\in H_1M(m+1).\end{aligned}$$

Using the properties of Δ for a Riemannian product manifold and the fact that π is a Riemannian submersion with the totally geodesic fibres (see [1]), we obtain

$$\Delta B=-(2m-1)\hat{H}=\left[\begin{array}{c|c}0 & (1/r_1+(2m-1)/r_2)\bar{z}_1w_i \\ \hline (1/r_1+(2m-1)/r_2)\bar{w}_iz_1 & (4m/r_2)\bar{w}_iw_j-4I\end{array}\right], \tag{3.4}$$

$$\Delta^2B=-(2m-1)\Delta\hat{H}=\left[\begin{array}{c|c}0 & (1/r_1+(2m-1)/r_2)^2\bar{z}_1w_i \\ \hline (1/r_1+(2m-1)/r_2)^2\bar{w}_iz_1 & (16m^2/r_2^2)\bar{w}_iw_j-(16m/r_2)I\end{array}\right],$$

if $p=1$,

$$\Delta B=-(2m-1)\hat{H}=\left[\begin{array}{c|c}(2(p+1)/r_1)\bar{z}_iz_j-4I & (p/r_1+q/r_2)\bar{z}_iw_j \\ \hline (p/r_1+q/r_2)\bar{w}_jz_i & (2(q+1)/r_2)\bar{w}_iw_j-4I\end{array}\right] \tag{3.5}$$

$$\begin{aligned}\Delta^2B&=-(2m-1)\Delta\hat{H}\\ &=\left[\begin{array}{c|c}(4(p+1)^2/r_1^2)\bar{z}_iz_j-(8(p+1)/r_1)I & (p/r_1+q/r_2)^2\bar{z}_iw_j \\ \hline (p/r_1+q/r_2)^2\bar{w}_jz_i & (4(q+1)^2/r_2^2)\bar{w}_iw_j-(8(q+1)/r_2)I\end{array}\right],\end{aligned}$$

if $p, q>1$.

For the case of $p=1$, M satisfies the equation (2.12) if and only if the following equations hold

$$\begin{aligned}&(1/r_1+(2m-1)/r_2)^2=(1/r_1+(2m-1)/r_2)a-(2m-1)b\\ &16m^2/r_2^2=4ma/r_2-(2m-1)b,\end{aligned} \tag{3.6}$$

where a and b are positive constants. Then, we see that every $\pi(S^1(\sqrt{r_1})\times S^{2m-1}(\sqrt{r_2}))$ with $r_1\neq 1/(m+1)$ (i.e., $B_0\neq(1/(m+1))I$) and $r_1\neq 1/2(m+1)$ satisfies the equation (2.12). For the case of $p, q>1$, M satisfies the equation (2.12) if and only if the following equations hold

$$
\begin{aligned}
&4(p+1)^2/r_1^2=2(p+1)a/r_1-(2m-1)b\\
&4(q+1)^2/r_2^2=2(q+1)a/r_2-(2m-1)b \qquad (3.7)\\
&(p/r_1+q/r_2)^2=(p/r_1+q/r_2)a-(2m-1)b\,.
\end{aligned}
$$

In consideration of the condition $r_2(p+1)\neq r_1(q+1)$ (i.e., $B_0\neq(1/(m+1))I$), from (3.7) we can see that every $\pi(S^p(\sqrt{p/2(m+1)})\times S^q(\sqrt{(q+2)/2(m+1)}))$ satisfies the equation (2.12).

Case 2. $B_0=(1/(m+1))I$

If M has constant mean curvature, (2.15) can be reduced to

$$
\begin{aligned}
(3.8)\qquad \Delta\hat{H}_B=&(4/(2m-1))(\tilde{J}A\tilde{J}N+(6m+2+\|\sigma\|^2)H+(2/(2m-1))\textstyle\sum_i\tilde{\sigma}(Ae_i, Ae_i)\\
&+2\textstyle\sum_i\tilde{\sigma}(e_i, A_He_i)+(8(2m+1)/(2m-1))(I-(m+1)B)\\
&-\{2(2m+2+\|\sigma\|^2)/(2m-1)+(2m-1)\|H\|^2\}\tilde{\sigma}(N, N)\,.
\end{aligned}
$$

Since I and B are normal to $\boldsymbol{CP}^m$ (see (1.4)), if M satisfies the equation (2.12), it follows from (2.9) and (3.8) that

$$(3.9)\qquad A\tilde{J}N=\mu\tilde{J}N\,,$$

where $\mu=(6m+2+\|\sigma\|^2-a)\alpha/4$ and $\alpha=\sum_i h_{ii}$. Now, we prepare the following lemmas,

LEMMA 4 ([6]). *If $A\tilde{J}N=\mu\tilde{J}N$ for some real function μ on M, then, μ is locally constant.*

We denote by V_r the eigenspace of A corresponding to the eigenvalue r.

LEMMA 5 ([6]). *Assume that $A\tilde{J}N=\mu\tilde{J}N$. If X belongs to V_r and is orthogonal to $\tilde{J}N$, then JX belongs to $V_{(\mu r+2)/(2r-\mu)}$.*

Using (1.10), (1.11) and (2.9), we obtain

$$
\begin{aligned}
(3.10)\qquad &g(\tilde{\sigma}(X, Y), a\hat{H}_B+b(B-(1/(m+1))I))\\
&=(2(2m+1)a/(2m-1)-b)g(X, Y)-(2a/(2m-1))g(X, \tilde{J}N)g(Y, \tilde{J}N)\\
&\qquad\qquad\qquad\text{for any } X, Y\in T_BM\,.
\end{aligned}
$$

On the other hand, it follows from (1.10), (1.11) and (3.8) that

(3.11) $g(\tilde{\sigma}(X, Y), \Delta\hat{H}_B)$

$$=\{(16m^2+16m+2\alpha^2)/(2m-1)\}g(X, Y)$$
$$+(4/(2m-1))\{\alpha g(AX, Y)+g(AX, AY)+\alpha g(AJX, JY)+g(AJX, AJY)\}$$
$$-2\{(4m+4+2\|\sigma\|^2+\alpha^2)/(2m-1)\}g(\tilde{J}N, X)g(\tilde{J}N, Y),$$

where $\alpha=\sum_i h_{ii}$ and $X, Y\in T_BM$. We define the subspace $\tilde{V}$ of TM as follows

$$\tilde{V}=\{X\in TM \mid g(X, \tilde{J}N)=0\}.$$

Then, for any $X\in\tilde{V}$, (3.10) and (3.11) imply that

$$A^2X-JA^2JX+\alpha AX-\alpha JAJX+\beta X=0, \tag{3.12}$$

where $\beta=4m(m+1)-(2m+1)a/2+(2m-1)b/4+\alpha^2/2$. Since $\tilde{J}X\in TM$, we can see that

$$J^2X=J(\tilde{J}X)=\tilde{J}^2X=-X. \tag{3.13}$$

Therefore, if X belongs to V_Λ, then it follows from (3.12), (3.13) and lemma 5 that

$$\{4\Lambda^4+4(\alpha-\mu)\Lambda^3+2(\mu^2-\alpha\mu+2\beta)\Lambda^2+4(\mu+\alpha-\beta\mu)\Lambda+4-2\alpha\mu+\beta\mu^2\}X=0 \quad \text{for any } X\in V_\Lambda\cap\tilde{V}. \tag{3.14}$$

Consequently, (3.9), Lemma 4 and (3.14) imply that $\tilde{J}N$ is a principal vector and M has at most five distinct constant principal curvatures. Then, we need the following

LEMMA 6 ([4]). *Let M be a real hypersurface of CP^m. Assume that $\tilde{J}N$ is a principal vector and M has constant principal curvatures. Then, M is a homogeneous one.*

It follows from Lemma 6 that M is a homogeneous real hypersurface of CP^m.

§4. Mss-symmetric homogeneous real hypersurfaces.

Homogeneous real hypersurfaces can be divided into six types, which are the types of A_1, A_2, B, C, D and E (see [11]). They are listed in Table 1. In Table 1, κ is the number of the distinct principal curvatures and p and q are odd number with $p, q>1$ and $p+q=2m$. First, we consider the type A_1.

TYPE A_1. In this case, M has two distinct constant principal curvatures. Therefore, a principal curvature λ must satisfy the equation

$$(\Lambda-\lambda)^4=0.$$

Table 1.

types	dimension	κ	principal curvatures	multiplicities
A_1	$2m-1$ $(m\geqq 2)$	2	$\lambda=\cot r$ $\mu=2\cot 2r$	$m(\lambda)=2(m-1)$ $m(\mu)=1$
A_2	$2m-1$ $(m\geqq 3)$	3	$\lambda_1=\cot r$ $\lambda_2=-\tan r$ $\mu=2\cot 2r$	$m(\lambda_1)=p-1$ $m(\lambda_2)=q-1$ $m(\mu)=1$
B	$2m-1$ $(m\geqq 2)$	3	$\lambda_1=\cot(r-\pi/4)$ $\lambda_2=-\tan(r-\pi/4)$ $\mu=2\cot 2r$	$m(\lambda_1)=m=1$ $m(\lambda_2)=m-1$ $m(\mu)=1$
C	$4n-3$ $(n\geqq 3)$	5	$\lambda_i=\cot(r-\pi i/4)$ $(i=1, 2, 3, 4)$ $\mu=2\cot 2r$	$m(\lambda_i)=2(n-2)$ $(i=2, 4)$ $m(\lambda_i)=2$ $(i=1, 3)$ $m(\mu)=1$
D	17	5	$\lambda_i=\cot(r-\pi i/4)$ $(i=1, 2, 3, 4)$ $\mu=2\cot 2r$	$m(\lambda_i)=4$ $(i=1, 2, 3, 4)$ $m(\mu)=1$
E	29	5	$\lambda_i=\cot(r-\pi i/4)$ $(i=1, 2, 3, 4)$ $\mu=2\cot 2r$	$m(\lambda_i)=8$ $(i=2, 4)$ $m(\lambda_i)=6$ $(i=1, 3)$ $m(\mu)=1$

Comparing this with (3.14), we obtain $\lambda=0$. This is a contradiction.

If M has three distinct constant principal curvatures, the following cases can be considered:

Case i) $(A-\lambda_1)^2(A-\lambda_2)^2=0$.

Case ii) $(A-\lambda_1)(A-\lambda_2)^3=0$.

Case iii) $(A-\lambda_1)^3(A-\lambda_2)=0$.

TYPE A_2. For the case i), the following equation must be satisfied:

$$A^4-2(\lambda_1+\lambda_2)A^3+(\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2)A^2-2\lambda_1\lambda_2(\lambda_1+\lambda_2)A+\lambda_1^2\lambda_2^2=0.$$

Comparing this with (3.14), we obtain

$$\begin{aligned} &\alpha-\mu=-2(\lambda_1+\lambda_2), \quad \mu^2-\alpha\mu+2\beta=2(\lambda_1^2+4\lambda_1\lambda_2+\lambda_2^2), \\ &\mu+\alpha-\beta\mu=-2\lambda_1\lambda_2(\lambda_1+\lambda_2), \quad 4-2\alpha\mu+\beta\mu^2=4\lambda_1^2\lambda_2^2. \end{aligned} \tag{4.1}$$

This, together with Table 1, implies

$$\sin r=\sqrt{(p+1)/2(m+1)}\,, \qquad \cos r=\sqrt{(q+1)/2(m+1)}\,.$$

Therefore, M is locally congruent to

$$\pi(S^p(\sqrt{(p+1)/2(m+1)}\times S^q(\sqrt{(q+1)/2(m+1)})\,.$$

In the same way, we can verify that the cases ii) and iii) never occur.

TYPE B. In this type, we can easily see that all the cases i), ii) and iii) never occur.

Finally, we consider the types C, D and E.

TYPES C, D and E. In these types, the following equation must be satisfied

$$(\Lambda-\lambda_1)(\Lambda-\lambda_2)(\Lambda-\lambda_3)(\Lambda-\lambda_4)=0\,.$$

Comparing this with (3.14), we obtain

$$\begin{aligned} &\alpha-\mu=-\textstyle\sum_{i=1}^4\lambda_i\,, \qquad \mu^2-\alpha\mu+2\beta=2\textstyle\sum_{i<j}\lambda_i\lambda_j\,, \\ &\mu+\alpha-\beta\mu=-\textstyle\sum_{i<j<k}\lambda_i\lambda_j\lambda_k\,, \qquad 4-2\alpha\mu+\beta\mu^2=4\lambda_1\lambda_2\lambda_3\lambda_4\,. \end{aligned} \tag{4.2}$$

Since $\lambda_1\lambda_2\lambda_3\lambda_4=1$ and $\mu\neq 0$ for these types, the last equation in (4.2) implies that $\beta=2\alpha/\mu$. This, together with (4.2), yields

$$4\mu+(\mu^2-4)\textstyle\sum_i\lambda_i=2\mu\textstyle\sum_{i<j}\lambda_i\lambda_j\,.$$

Then, from Table 1, we can easily see that this is a contradiction. Thus, Theorem 1 is proved. Theorem 2 is a immediate consequence of Theorem 1 and Lemma 3.

§ 5. Spectral inequality.

The following spectral inequality is proved in [3] and [8].

THEOREM C. *Let $x: M\to E^N$ be an isometric immersion of an n-dimensional compact Riemannian manifold into an N-dimensional Euclidean space. Then,*

$$\int_M\{n^2g(\Delta\hat{H},\hat{H})-n^2(\lambda_1+\lambda_2)g(\hat{H},\hat{H})-n\lambda_1\lambda_2 g(x,\hat{H})\}\geqq 0\,. \tag{5.1}$$

The equality holds if and only if M is of order $\{1\}$ *or order* $\{2\}$ *or order* $\{1, 2\}$.

Using Theorem C, we prove the following

THEOREM 3. *Let M be a compact real hypersurface of $\boldsymbol{CP}^m$. Then, we have*

$$(5.2)\qquad \{8(m+1)(4m^2-2m-1)-4(2m^2-1)(\lambda_1+\lambda_2)+(2m-1)\lambda_1\lambda_2\}\mathrm{vol}(M)$$
$$+(2m-1)^2\int_M\{(10m+6+\|\sigma\|^2-\lambda_1-\lambda_2)\|H\|^2+\|dH\|^2\}$$
$$\geqq 4\int_M\{\|A\tilde{J}N\|^2-\|\sigma\|^2+2(2m-1)g(A\tilde{J}N, \tilde{J}H)\}.$$

A) *If M has constant mean curvature H, then the equality in* (5.2) *holds if and only if M is congruent to one of the following*

1) $\pi(S^1(\sqrt{1/2(m+1)})\times S^{2m-1}(\sqrt{(2m+1)/2(m+1)}))$ (*order* $\{1\}$),

2) $\pi(S^1(\sqrt{r_1})\times S^{2m-1}(\sqrt{r_2}))$ *with* $r_1<1/(m+1)$, $r_1\neq 1/2(m+1)$ *and* $r_1+r_2=1$ (*order* $\{1, 2\}$),

3) $\pi(S^p(\sqrt{(p+1)/2(m+1)})\times S^q(\sqrt{(q+1)/2(m+1)}))$ (*order* $\{1, 2\}$), *with* $p, q>1$, $p+q=2m$ *and* p, q: *odd.*

B) *If M is mass-symmetric, then the equality in* (5.2) *holds if and only if M is congruent to the case* 3) *in* A).

Remark 4. For the case of M being minimal in CP^m, we have Corollary 4.6 in [7].

Proof. (5.2) can be obtained by Lemma 2 and Theorem C. The equality in (5.2) holds if and only if M is of order $\{1\}$ or order $\{2\}$ or order $\{1, 2\}$. Then. if M has constant mean curvature H and M is of order $\{1, 2\}$, it follows from Theorem 1 that M is congruent to one of the cases i), ii) and iii) in Theorem 1. Therefore, it is enouph to investigate the order of the immersions of the cases i), ii) and iii),

First, note that (see [1])

$$\mathrm{Spec}(\pi(S_1\times S_2))=\{\lambda\in\mathrm{Spec}(S_1\times S_2)\mid \Delta(f_k\times f_l)=\lambda(f_k\times f_l).$$
$$\text{with } k+l=\text{even},\ f_j:\text{ a homogeneous polynomial of degree } j\},$$

where S_1 and S_2 are the spheres with the natural metrics.

Case i). $M=\pi(S^1(\sqrt{r_1})\times S^{2m-1}(\sqrt{r_2}))$, $r_1\neq 1/(m+1)$, $1/2(m+1)$. In this case, from (2.13) and (3.6) we have $\lambda_k=4m/r_2$, $\lambda_l=1/r_1+(2m-1)/r_2$. If $r_1>1/2(m+1)$ (i.e., $\lambda_k>\lambda_l$), M is of order $\{1, 2\}$ if and only if $r_1<1/(m+1)$. If $r_1<1/2(m+1)$, M is of order $\{1, 2\}$ for any r_1. If $r_1=1/2(m+1)$, M is of order $\{1\}$.

Case ii). $M=\pi(S^p(\sqrt{p/2(m+1)})\times S^q(\sqrt{(q+2)/2(m+1)}))$, $p, q>1$, $p+q=2m$ and p, q: odd. In this case, from (2.13) and (3.7) we have $\lambda_k=4(p+1)(m+1)/p$, $\lambda_l=4(q+1)(m+1)/(q+2)$. Therefore, M is of order $\{1, 2\}$ if and only if

$$2(m+1)+2q(m+1)/(q+2)\geqq 4(p+1)(m+1)/p.$$

But, this is impossible.

Case iii). $M=\pi(S^p(\sqrt{(p+1)/2(m+1)})\times S^q(\sqrt{(q+1)/2(m+1)}))$, $p, q>1$, $p+q=2m$ and p, q: odd. In this case, from (2.13), (3.12) and (4.1) we have

$$\lambda_k=2(m+1)(p(q+1)+q(p+1))/(p+1)(q+1),\ \lambda_l=4(m+1).$$

Therefore, M is of order {1, 2}, If M is mass-symmetric, Theorem 2, together with the above arguments, gives the result. Q. E. D.

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NIHON UNIVERSITY
DEPARTMENT OF MATHEMATICS
SCHOOL OF MEDICINE
ITABASHI, TOKYO, 173 JAPAN