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AUTOMORPHISM GROUPS OF HYPERELLIPTIC RIEMANN SURFACES

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Hurwitz [3] proved that a Riemann surface of genus g has at most $84(g-1)$ automorphisms. If the surface is hyperelliptic this bound may be very much shortened.

In this paper we obtain all surfaces of genus $g>3$ with more than $8(g-1)$ automorphisms, and their corresponding automorphism groups. As a consequence of these results, all hyperelliptic Riemann surfaces with more than $8(g-1)$ automorphisms appear to be symmetric.

The surfaces of low genus were studied by Wiman [12] and A. and I. Kuribayashi [4, 5].

The methods of our study of Riemann surfaces involve the representation of compact Riemann surfaces as quotient spaces of Fuchsian groups [6]. A Fuchsian group is a discrete subgroup of the group of orientable isometries of the hyperbolic plane D. If the quotient space D/Γ, Γ being a Fuchsian group, is compact, then Γ has the following presentation:

$$\langle a_1, b_1, \cdots, a_g, b_g, x_1, \cdots, x_r \mid a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}x_1 \cdots x_r = x_i^{m_i} = 1\rangle.$$

Then we call $(g, [m_1, \cdots, m_r])$ the signature of Γ and g is the genus of D/Γ. The numbers m_i are called proper periods. When X is a surface of genus g, it may be expressed as D/Γ, Γ having signature $(g, [-])$.

If G is a group of automorphisms of the Riemann surface D/Γ, then G may be written as Γ'/Γ, where Γ' is another Fuchsian group. A Fuchsian group K with signature $(g, [m_1, \cdots, m_r])$ has associated an area

$$\text{Area}\,[K] = 2\pi\Big(2g-2+\sum_{i=1}^{r}\Big(1-\frac{1}{m_i}\Big)\Big) = 2\pi|K|,$$

and the order of $G=\Gamma'/\Gamma$ is $|G|=|\Gamma|/|\Gamma'|$.

Let now D/Γ be a Riemann surface of genus g, and $G=\Gamma'/\Gamma$ its group of automorphisms. If $|G|>8(g-1)$, then $|\Gamma'|<\frac{2(g-1)}{8(g-1)}=\frac{1}{4}$. We list the Fuchsian groups Γ' with $|\Gamma'|<\frac{1}{4}$.

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i) $(0, [2, 3, m])$ $m \geqq 7$
ii) $(0, [2, 4, m])$ $m \geqq 5$
iii) $(0, [2, 5, m])$ $5 \leqq m \leqq 19$
iv) $(0, [2, 6, m])$ $6 \leqq m \leqq 11$
v) $(0, [2, 7, m])$ $7 \leqq m \leqq 9$
vi) $(0, [3, 3, m])$ $4 \leqq m \leqq 11$
vii) $(0, [3, 4, 4])$
viii) $(0, [3, 4, 5])$
ix) $(0, [2, 2, 2, 3])$.

A Riemann surface X is hyperelliptic if it admits an involution ϕ such that X/ϕ has genus 0. In terms of Fuchsian groups the surface D/Γ is hyperelliptic if and only if there exists a unique Fuchsian group Γ_1 such that $|\Gamma_1 : \Gamma| = 2$, and the signature of Γ_1 is $(0, [2, \overset{2g+2}{\cdots\cdots}, 2])$ [9]. Furthermore, the element Γ_1/Γ is central in the automorphism group of D/Γ, and so when $G = \Gamma'/\Gamma$ is the automorphism group of the surface we have the following relations: $\Gamma \lhd \Gamma_1 \lhd \Gamma'$.

THEOREM. *Let X be a hyperelliptic Riemann surface of genus g and G its automorphism group. If $g > 3$ and $|G| > 8(g-1)$, then one of the following holds:*

Genus of X	Presentation of G	Order of G
	$\langle a, b \mid a^2 = b^4 = (ab)^m = 1,\ ab^2 = b^2 a\rangle$	$4m = 8(g+1)$
	$\langle a, b \mid a^2 = b^4 = (ab)^m = 1,\ b^2 = (ab)^{m/2}\rangle$	$2m = 8g$
5	$\langle a, b \mid a^2 = b^3 = (ab)^{10} = 1,\ (ab)^5 = (ba)^5\rangle$	120
9	$\langle a, b \mid a^2 = b^5 = (ab)^6 = 1,\ (ab)^3 = (ba)^3\rangle$	120
6	$\langle a, b \mid a^2 = b^6 = (ab)^8 = 1,\ b^3 = (ab)^4\rangle$	48
15	$\langle a, b \mid a^2 = b^6 = (ab)^{10} = 1,\ b^3 = (ab)^5\rangle$	120
5	$\langle a, b \mid a^3 = b^4 = (ab)^4 = 1,\ ab^2 = b^2 a\rangle$	48
14	$\langle a, b \mid a^3 = b^4 = (ab)^5 = 1,\ ab^2 = b^2 a\rangle$	120

Besides, in each case the surface is unique up to conformal equivalence.

Proof. Let $X = D/\Gamma$, with genus g. Since X is hyperelliptic, there exists an automorphism ϕ of X, of order 2, and hence a Fuchsian group Γ_1 with signature $(0, [2, \overset{2g+2}{\cdots\cdots}, 2])$ such that $|\Gamma_1 : \Gamma| = 2$. If $G = \Gamma'/\Gamma$ is the automorphism group of D/Γ, then $\Gamma \lhd \Gamma_1 \lhd \Gamma'$. So there exists an epimorphism $\theta_1 : \Gamma' \to G_1 = G/\langle\phi\rangle$ whose kernel is isomorphic to Γ_1.

We look for the existence of this epimorphism θ_1, according to the signature of Γ', i)—ix) above.

1) Let Γ' have signature i). Let m be odd. We have $\theta_1 : \Gamma' \to G_1$ with

kernel Γ_1. The order of G_1 is $|\Gamma_1|/|\Gamma'|=\frac{g-1}{(m-6)/6m}=\frac{6m(g-1)}{m-6}$. From [8, Cor. 2] since m is odd, the proper periods of the kernel of θ_1 are produced by the image of x_1. Hence $\theta_1(x_1)$ is 1, and the number of proper periods in $\ker\theta_1$ is $|G_1|=\frac{6m(g-1)}{m-6}$. We have $\frac{6m(g-1)}{m-6}=2g+2$, and $g=\frac{2m-3}{m+3}$, impossible.

Now, if m is even, the proper periods of $\ker\theta_1$ may be obtained from x_1 and x_3. Then its number is

$$\alpha\frac{6m(g-1)}{m-6}+\beta\frac{6m(g-1)}{(m-6)(m/2)},$$

α (resp., β) being 1 when $\theta_1(x_1)$ (resp., $\theta_1(x_3)$) is 1 (resp., an element of order $m/2$) and being 0 when $\theta_1(x_1)$ (resp., $\theta_1(x_3)$) is an element of order 2 (resp., order m) [8, Cor. 2]. So

$$\alpha\frac{6m(g-1)}{m-6}+\beta\frac{6m(g-1)}{(m-6)(m/2)}=2g+2$$

and this holds only for $\alpha=0$, $\beta=1$, $g=2$ or 5. Thus the unique case in our scope is $\alpha=0$, $\beta=1$, $m=10$, $g=5$.

2) Let Γ' have signature ii). Now the order of G_1 is $\frac{4m(g-1)}{m-4}$ and we distinguish again m odd and m even. By the same argument above, we consider the number of proper periods of $\ker\theta_1$:

Let m be odd. Then

$$\alpha\frac{4m(g-1)}{m-4}+\beta\frac{2m(g-1)}{m-4}=2g+2$$

and in no case we obtain an integer value of g.

When m is even.

$$\alpha\frac{4m(g-1)}{m-4}+\beta\frac{2m(g-1)}{m-4}+\gamma\frac{8(g-1)}{m-4}=2g+2.$$

The possibilities that appear now are:

$$(*)\quad \alpha=0,\quad \beta=1,\quad \gamma=0,\quad g=\frac{m-2}{2},$$

$$\alpha=1,\quad \beta=0,\quad \gamma=0,\quad m=12,\quad g=2,$$

$$\alpha=0,\quad \beta=0,\quad \gamma=1,\quad m=6,\quad g=3,$$

$$(*)\quad \alpha=0,\quad \beta=1,\quad \gamma=1,\quad g=\frac{m}{4}.$$

For our purposes we will deal only with the starred cases.

3) Let Γ' have signature iii). The order of G_1 is $\dfrac{10m(g-1)}{3m-10}$.

If m is odd,

$$\frac{10m(g-1)}{3m-10}=2g+2, \qquad \text{impossible.}$$

If m is even,

$$\alpha\frac{10m(g-1)}{3m-10}+\beta\frac{20(g-1)}{3m-10}=2g+2$$

and it holds just for $\alpha=0$, $\beta=1$, $m=6$, $g=9$.

4) Let Γ' have signature iv). Then the order of G_1 is $\dfrac{3m(g-1)}{m-3}$ and the number of proper periods of $\ker\theta_1$ is, when m is odd,

$$\alpha\frac{3m(g-1)}{m-3}+\beta\frac{m(g-1)}{m-3}=2g+2$$

and then $\alpha=1$, $\beta=1$, $m=9$, $g=2$, what is out of our scope.

If m is even,

$$\alpha\frac{3m(g-1)}{m-3}+\beta\frac{m(g-1)}{m-3}+\gamma\frac{6(g-1)}{m-3}=2g+2$$

and then we have

$$\begin{array}{llllll} & \alpha=1, & \beta=0, & \gamma=1, & m=8, & g=2, \\ & \alpha=0, & \beta=1, & \gamma=1, & m=6, & g=3, \\ (*) & \alpha=0, & \beta=1, & \gamma=1, & m=8, & g=6, \\ (*) & \alpha=0, & \beta=1, & \gamma=1, & m=10, & g=15, \\ & \alpha=1, & \beta=1, & \gamma=1, & m=12, & g=2, \end{array}$$

from which the starred cases are the interesting ones.

5) Let Γ' have signature v). The order of G_1 is $\dfrac{14m(g-1)}{5m-14}$. If $m=7$ or 9 we have

$$\frac{14m(g-1)}{5m-14}=2g+2, \qquad \text{impossible.}$$

If $m=8$,

$$\alpha\frac{112(g-1)}{26}+\beta\frac{28(g-1)}{26}=2g+2, \qquad \text{impossible.}$$

6) When Γ' has signature vi), the order of G_1 is $\dfrac{3m(g-1)}{m-3}$. The kernel of

θ_1 has proper periods only when m is even, and their number is

$$\frac{6m(g-1)}{m-3}=2g+2$$

holding only when $m=4$, $g=2$, out of our hypothesis.

7) Let Γ' have signature vii). The order of G_1 is $6(g-1)$ and the number of proper periods of $\ker\theta_1$ is $\alpha 3(g-1)$, with $\alpha=1$ or 2; if $\alpha=1$, $g=5$, and if $\alpha=2$, $g=2$.

8) If Γ' has signature viii), the order of G_1 is $\frac{60(g-1)}{13}$ and the number of proper periods is

$$\frac{30(g-1)}{13}=2g+2$$

holding for $g=14$.

9) If Γ' has signature ix), the order of G_1 is $6(g-1)$ and there are $\alpha 6(g-1)$ proper periods, $1\leqq\alpha\leqq 3$. When $\alpha=1$, it is $g=2$ and the other cases are impossible.

Now we check that all these cases actually hold. Since G_1 is a group of automorphisms of $X/\langle\phi\rangle$, a surface of genus 0, by [1, §§ 4.3, 4.4], this group must be one of the following: C_q, D_q, A_4, S_4 or A_5. (In what follows we use for finite groups the notation of [1]). In each of the cases we construct the epimorphism θ satisfying $\pi\theta=\theta_1$, where $\pi: G\to G_1=G/\langle\phi\rangle$ is the canonical projection. Since $\ker\theta$ has no proper periods, $\theta(x_i)$ must have order m_i. We study separately the eight possibilities.

I. Γ': (0, [2, 3, 10]), $g=5$, $|G_1|=60$, $\alpha=0$, $\beta=1$. Then the signature of Γ_1 is (0, $[2^{(12)}]$) (where $2^{(p)}$ means that there exist p proper periods equal to two), and $\theta_1: \Gamma'\to G_1$ is defined [9] by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1}.$$

satisfying $x^2=y^3=(xy)^5=1$. Hence $G_1=A_5$. Now the epimorphism $\theta: \Gamma'\to G$ with kernel Γ is given by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x_3)=(ab)^{-1},$$

with $a^2=b^3=(ab)^{10}=1$. Since $\pi\theta=\theta_1$, $(ab)^5$ is a central element and so $(ab)^5=(ba)^5$. This group is just the group

$$\langle R, S\mid R^{10}=S^3=(RS)^2=(R^{-4}S)^2=1\rangle$$

described in [11, p. 46]. For, call $a=RS$, $b=S$. Then $S^3=b^3=1$; $(RS)^2=a^2=1$; since $ab=S^{-1}R^{-1}S$, the order of R is the one of ab, i.e., 10; finally $(R^{-4}S)^2=((ba)^4b)^2=(ab)^{10}=1$. By the definition of the epimorphism, $D/\ker\theta$ is a Riemann surface of genus g, and its hyperellipticity comes from the automorphism

$(ab)^5$.

II. $\Gamma' : (0, [2, 4, m])$, m even, $g=(m-2)/2$, $|G_1|=4(g+1)=2m$, $\alpha=0$, $\beta=1$, $\gamma=0$. The signature of Γ_1 is $(0, [2^{(2g+2)}])$ and $\theta_1 : \Gamma' \to G_1$ is defined by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1},$$

$x^2=y^2=(xy)^m=1$, and obviously $G_1=D_m$. The epimorphism $\theta : \Gamma' \to G$ with kernel Γ is given by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x^3)=(ab)^{-1},$$

with $a^2=b^4=(ab)^m=1$. Since $\pi\theta=\theta_1$, b^2 is a central element and so $ab^2=b^2a$. This group has order $8(g+1)$ and has presentation

$$\langle T, U \mid T^4=U^{2(g+1)}=(TU)^2=(T^{-1}U)^2=1\rangle$$

(see [7, Th. 4]). Effectively, we call $b=T$, $a=TU$, and the presentation follows at once. Now $D/\ker\theta$ is a Riemann surface of genus g and b^2 gives the hyperellipticity.

III. $\Gamma' : (0, [2, 4, m])$, m a multiple of 4, $g=m/4$, $|G_1|=4g=m$, $\alpha=0$, $\beta=1$, $\gamma=1$. Γ_1 is again $(0, [2^{(2g+2)}])$ and θ_1 is given by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1}$$

satisfying $x^2=y^2=(xy)^{m/2}=1$, and G_1 is $D_{m/2}$. Now $\theta : \Gamma' \to G$ is defined by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x_3)=(ab)^{-1},$$

$a^2=b^4=(ab)^m=1$, and by unicity of the central element, $b^2=(ab)^{m/2}$. Consider the group $C_m=\langle y \mid y^m=1\rangle$, $C_2=\langle x \mid x^2=1\rangle$, and $C_m \times_\phi C_2$ with $\phi : C_2 \to \mathrm{Aut}\,(C_m)$, $\phi(1)=Id$, $\phi(x)=\psi$, $\psi(y)=y^{(m/2)-1}$. In this situation $a=(x, 1)$, $ab=(1, y)$ and G is so $C_m \rtimes C_2$. $D/\ker\theta$ is a Riemann surface of genus g and $b^2=(ab)^{m/2}$ gives the hyperellipticity.

IV. $\Gamma' : (0, [2, 5, 6])$, $g=9$, $|G_1|=60$, $\alpha=0$, $\beta=1$. The signature of Γ_1 is $(0, [2^{(20)}])$ and θ_1 is defined by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1}.$$

satisfying $x^2=y^5=(xy)^3=1$, G_1 being so A_5. Now θ is given by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x_3)=(ab)^{-1},$$

with $a^2=b^5=(ab)^6=1$, and $(ab)^3=(ba)^3$. By [1, p. 76] G is the quotient of the group 2[6]5 by adding the relation "the central element has order two". X is a hyperelliptic surface and $\phi=(ab)^3$.

V. $\Gamma' : (0, [2, 6, 8])$, $g=6$, $|G_1|=24$, $\alpha=0$, $\beta=1$, $\gamma=1$. The signature of Γ_1 is $(0, [2^{(14)}])$ and θ_1 is defined by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1}.$$

with $x^2=y^3=(xy)^4=1$. So G_1 is S_4. Now the epimorphism θ is given by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x_3)=(ab)^{-1}.$$

satisfying $a^2=b^6=(ab)^8=1$, and, as above, $b^3=(ab)^4$. G is the group $\langle 3, 4 | 2 ; 2\rangle \cong S_4\times C_2$ [1, p. 72], X is a hyperelliptic surface and $\phi=b^3=(ab)^4$.

VI. Γ': (0, [2, 6, 10]), $g=15$, $|G_1|=60$, $\alpha=0$, $\beta=1$, $\gamma=1$. Γ_1 has signature (0, $[2^{(32)}]$) and the epimorphism θ_1 is defined by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1},$$

with $x^2=y^3=(xy)^5=1$, and G_1 is A_5. The epimorphism θ is given by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x_3)=(ab)^{-1}$$

with $a^2=b^6=(ab)^{10}=1$ and $b^3=(ab)^5$. G is the group $\langle 3.5 | 2 ; 2\rangle \cong A_5\times C_2$[1, p. 72]. X is a hyperelliptic surface and $\phi=b^3=(ab)^5$.

VII. Γ': (0, [3, 4, 4]), $g=5$, $|G_1|=24$, $\alpha=1$. In this case the signature of Γ_1 is (0, $[2^{(12)}]$). The epimorphism θ_1 is defined by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1},$$

with $x^3=y^2=(xy)^4=1$, and G_1 is S_4. The epimorphism θ is given by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x_3)=(ab)^{-1},$$

with $a^3=b^4=(ab)^4=1$, and $ab^2=b^2a$. G is the quotient of the group 3[4]4 ([1, p. 76]) by adding the relation "the central element has order two". X is a hyperelliptic surface and $\phi=b^2$.

VIII. Γ': (0, [3, 4, 5]), $g=14$, $|G_1|=60$. Γ_1 has signature (0, $[2^{(30)}]$) and θ_1 is given by

$$\theta_1(x_1)=x, \qquad \theta_1(x_2)=y, \qquad \theta_1(x_3)=(xy)^{-1},$$

satisfying $x^3=y^2=(xy)^5=1$. We have $G_1=A_5$ and θ is defined by

$$\theta(x_1)=a, \qquad \theta(x_2)=b, \qquad \theta(x_3)=(ab)^{-1},$$

with $a^3=b^4=(ab)^5=1$ and $ab^2=b^2a$. G is the quotient of the group 3[4]5 ([1, p. 76]) by adding the relation "the central element has order two". X is a hyperelliptic surface and $\phi=b^2$.

Observe that in all cases I-VIII Γ' is a triangle group and so its Teichmüller space has dimension 0. Besides, the epimorphism θ is unique modulo Aut(Γ') and Aut(G). Hence, using [2, § 3], the surface $D/\ker\theta$ is unique in each case, up to conformal equivalence.

COROLLARY. *If X is a hyperelliptic Riemann surface of genus $g>3$ with $|\mathrm{Aut}(X)|>8(g-1)$, then X is symmetric.*

Proof.

Cases I and III come from [10, note in p. 24 and th. 4, resp.], and case II from [11, p. 46]. In cases IV-VIII, since $a \to a^{-1}$, $b \to b^{-1}$ is an automorphism of the group, it follows from [10, th. 2] that X is symmetric.

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