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A NOTE ON CONTINUITY OF GREEN'S FUNCTIONS ON RIEMANN SURFACES

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§1. Introduction and main results.

Let R be an arbitrary Riemann surface admitting Green's functions, and denote by $g(\cdot, r)$ Green's function with the pole $r\in R$. Also set $U(r, M)=\{s\in R : g(s, r)>M\}$ for every $r\in R$ and positive M. Then we have the following

THEOREM 1. *Let q be a point on R. Take a positive constant M so large that $U(q, M)$ is simply connected. Then it holds that*

$$\|dg(\cdot, q')-dg(\cdot, q)\|_{R-U(q,M)}<6\cdot e^{M+4}\cdot\exp(-g(q', q))$$

for every $q'\in U(q, M+4)$.

Theorem 1 is a corollary of Lemma 2 in §2, which also gives the following

THEOREM 2. *Under the same assumptions as in Theorem* 1, *it holds that*

$$\left|\int_d {}^*dg(\cdot, q')-\int_d {}^*dg(\cdot, q)\right|\leqq 9(\lambda_d)^{1/2}\cdot e^{M+4}\cdot\exp(-g(q', q))$$

for every 1-*cycle d on $R-U(q, M+4)$ and $q'\in U(q, M+4)$. where λ_d is the extremal length of the homology class of d on R.*

THEOREM 3. *Let p and q be two distinct points on R. Take a positive M so large that $U(q, M)$ is contained in $R-\{p\}$ and simply connected. Then it holds that*

$$|g(q', p)-g(q, p)|\leq 5(g(q, p))^{1/2}\cdot e^{M+4}\cdot\exp(-g(q', q))$$
$$\leq 5M^{1/2}\cdot e^{M+4}\cdot\exp(-g(q', q))$$

for every $q'\in U(q, M+4)$.

The proof of Theorems 1 and 2, 3 are given in §1 and §2, respectively. Here we note the following corollary of Theorem 3.

COROLLARY. *Let R be a Riemann surface satisfying the following condition;*

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(*) *there is a positive constant M such that $U(q, M)$ is simply connected for every $q \in R$.*

Then Green's functions are locally uniformly Lipschitz-like continuous, i.e. it holds that

$$|g(q_1, p)-g(q_2, p)| \leq 5M^{1/2} \cdot e^{M+4} \cdot \exp(-g(q_1, q_2))$$

for every q_1 and q_2 in $R-\overline{U(p, M)}$ such that $g(q_1, q_2)>M+4$.

Next combining Theorem 3 with the comparison theorem in [4], we can show, in general, the following local Lipschitz-like continuity of Green's functions.

THEOREM 4. *Under the same assumption as in Theorem 3, it holds that*

$$|g(q_1, p)-g(q_2, p)| \leq C_1 \cdot (M+1)^{1/2} \cdot e^M \cdot \exp(-g(q_1, q_2))$$

for every q_1 and q_2 in $U(q, M+C_0)$, where C_0 and C_1 are suitable absolute constants.

In case that $g(q, p)$ is sufficeintly large, or equivalently q is sufficiently near to q, we can show the following

THEOREM 5. *Let p and q be distinct points on R such that $M=g(q, p)>C_2$ and $U(q, M-C_2)$ is simply connected with a suitable absolute constant C_2. Then it holds that*

$$|g(q_1, p)-g(q_2, p)| \leq C_3 \cdot e^M \cdot \exp(-g(q_1, q_2))$$

for every q_1 and q_2 in $U(q, M+C_2)$ with a suitable absolute constant C_3.

The proofs of Theorems 4 and 5 are given in §4. And finally as an application of Theorem 1, we will include in §5 a remark to the remainder terms of variational formulas in [5].

§2. The proof of Theorem 1.

Let $g^*(s)$ be a conjugate harmonic function of $g(s, q)$ on $U(q, M)$, and set $z=Z(s)=e^{M+2} \cdot \exp(-g(s, q)-i \cdot g^*(s))$. Then $Z(U(q, M+n))=U(e^{2-n})$ for every non-negative n, where we set $U(\rho)=\{z: |z|<\rho\}$ for every positive ρ. For every a in $U(1/e)$, define a mapping f_a of R onto itself by setting

$$Z \circ f_a \circ Z^{-1}(z)=z+a \qquad \text{on } U(1), \text{ and}$$

$$=(a|z|+z) \cdot (1+a(|z|/z))^{-\log|z|}$$

on $U(e)-\overline{U(1)}$, and by letting f_a be the identical mapping on $R-U(q, M+1)$, where we choose the branch of $\log(1+a(|z|/z))$ so that $\log 1=0$.

Note that f_a is conformal outside of $W=\overline{U(q, M+1)}-U(q, M+2)$, and we can show the following

LEMMA 1. *If* $|a|<(e-2)/e$ $(<1/e)$, *then* f_a *is* $(1+k_a)/(1-k_a)$-*quasiconformal on* R *with* $k_a \leq e\cdot|a|/(e-2)$.

Proof. Set $F(z)=Z\circ f_a\circ Z^{-1}(z)$ on $U(e)-\overline{U(1)}$, then by a simple computation we have

$$\frac{F_z}{F}=\frac{1}{2z}\Big(2-(1-\log|z|)\cdot\frac{a|z|}{z+a|z|}-\log\Big(1+a\frac{|z|}{z}\Big)\Big), \quad \text{and}$$

$$\frac{F_{\bar z}}{F}=\frac{1}{2\bar z}\Big((1-\log|z|)\cdot\frac{a|z|}{z+a|z|}-\log\Big(1+a\frac{|z|}{z}\Big)\Big)$$

Since $|a|<1/e$, it holds that

$$\Big|\frac{a|z|}{z+a|z|}\Big|\leq\frac{|a|}{1-|a|}\leq e|a|/(e-1)<1/(e-1)\,, \quad \text{and}$$

$$|\log(1+a(|z|/z))|\leq|a|\sum_{n=0}^{\infty}|a|^n<e|a|/(e-1)<1/(e-1)\,.$$

Hence we have

$$|F_{\bar z}/F_z|\leq\Big(\frac{2e}{e-1}\cdot|a|\Big)/(2-2/(e-1))\leq e|a|/(e-2) \quad (<1)\,.$$

q. e. d.

Now fix $q'\in U(q, M+4)$ and set $a=Z(q')$. Then $|a|=e^{M+2}\cdot\exp(-g(q', q))<1/e^2<(e-2)/e$. Writting $\varphi_r=dg(\cdot, r)+i{}^*dg(\cdot, r)$ for every $r\in R$, we set $\omega=\varphi_{q'}\circ f_a-\varphi_q$, where $\varphi_{q'}\circ f_a$ is the pull-back of $\varphi_{q'}$ by f_a. Then we know the following lemma, which implies the assertion of Theorem 1 (cf. [1, Theorem 5], [3, Proposition 5]).

LEMMA 2. *It holds that*

$$\|\omega\|_R\leq\frac{\sqrt{2}\cdot k_a}{1-k_a}\|\varphi_q\|_W<6\cdot e^2\cdot|a|\,,$$

where $\|\alpha\|_E$ *is the Dirichlet norm of* α *on a Borel set* E.

Proof. For the sake of convenience, we include the proof. Since $\mathrm{Re}\,\omega\in\Gamma_{e0}(R)$ and $\mathrm{Im}\,\omega\in\Gamma_c(R)$, we have

$$(*) \qquad \iint_R \omega\wedge\bar\omega=2i\cdot(\mathrm{Re}\,\omega, {}^*\mathrm{Im}\,\omega)_R=0\,,$$

where and in the sequel, $\Gamma(R)$ is the Hilbert space of real square integrable differentials on R, $\Gamma_c(R)$ and $\Gamma_h(R)$ are subspaces of $\Gamma(R)$ consisting of closed and harmonic differentials, respectively, $\Gamma_{e0}(R)$ is the orthogonal complement of $\Gamma_h(R)$ in $\Gamma_c(R)$, and we set $(\alpha, \beta)_E=\iint_E \alpha\wedge{}^*\beta$ for every α and β in $\Gamma(R)$ and E as above.

Writting $\varphi_{q'}=g(w)dw$ with a generic local parameter w on R, we have by (*)

$$\|g\circ f_a\cdot(f_a)_w dw-\varphi_q\|_W\leq\|g\circ f_a\cdot(f_a)_w dw-\varphi_q\|_R$$
$$=\|g\circ f_a\cdot(f_a)_{\bar{w}}d\bar{w}\|_R\leq k_a\cdot\|g\circ f_a\cdot(f_a)_w dw\|_W\,,$$

which implies that

$$\|g\circ f_a\cdot(f_a)_w dw\|_W\leq\frac{1}{1-k_a}\|\varphi_q\|_W\,.$$

Thus we have

$$\|\omega\|_R^2=\|g\circ f_a\cdot(f_a)_w dw-\varphi_q\|_R^2+\|g\circ f_a\cdot(f_a)_{\bar{w}}d\bar{w}\|_R^2$$
$$\leq 2(k_a)^2\cdot\|g\circ f_a\cdot(f_a)_w dw\|_W^2\leq 2(k_a/(1-k_a))^2\cdot\|\varphi_q\|_W^2\,,$$

which shows the first inequality. Next, since $\|\varphi_q\|_W^2=4\pi$ and $|k_a|\leq e\cdot|a|/(e-2)$ $<1/e(e-2)$ by Lemma 1, we can see the second inequality. q. e. d.

§ 3. The proofs of Theorems 2 and 3.

Theorems 2 and 3 follows from Lemma 2 by recalling the following facts (cf. [2, § 3]). Again for the sake of convenience, we include their proofs.

LEMMA 3. *Under the same assumptions as in Theorem* 2, *it holds that*

$$\int_d {}^*dg(\cdot q')-\int_d {}^*dg(\cdot,\, q)=-\mathrm{Re}\iint_R \omega\wedge\theta_d\,,$$

where, letting σ_d *be the differential in* $\Gamma_h(R)$ *such that* $(\alpha, \sigma_d)_R=\int_d \alpha$ *for every* $\alpha\in\Gamma_h(\boldsymbol{R})$, *we set* $\theta_d=\sigma_d+i^*\sigma_d$.

Proof. Since $\mathrm{Re}\,\omega\in\Gamma_{e0}(R)$ and $\mathrm{Im}\,\omega\in\Gamma_c(R)$, we have $(\mathrm{Re}\,\omega, {}^*\sigma_d)_R=0$ and $(\mathrm{Im}\,\omega, \sigma_d)_R=\int_d \omega$, which implies the assertion. q. e. d.

LEMMA 4. *Under the same assumptions as in Theorem* 3, *it holds that*

$$g(q', p)-g(q, p)=\frac{1}{2\pi}\cdot\mathrm{Re}\iint_R \omega\wedge{}^*\varphi_p\,.$$

Proof. Since $U(q, M)\not\ni p$, we can find a positive N so large that $U(p, N-1)$ is simply connected and disjoint from $U(q, M+1)$. Fix such an N, and let $J(s)$

be a smooth function on R such that $J(s)\equiv 1$ on $U(p, N-1/2)$ and $J(s)\equiv 0$ on $R-U(p, N-1)$. Set

$$\omega_1=d((1-J(\cdot))\cdot(g(f_a(\cdot), q')-g(\cdot, q))),$$

then we have $\omega_1\in\Gamma_{e0}(R_1)$ with $R_1=R-\overline{U(p, N)}$. Hence by Green's formula, we have

$$(\mathrm{Re}\,\omega, \mathrm{Re}\,\overline{\varphi_p})_{R_1}=(\mathrm{Re}\,\omega-\omega_1, dg(\cdot, p))_{R_1}$$

$$=\int_{-\partial U(p,N)}(g(\cdot, q')-g(\cdot, q))\cdot{}^*dg(\cdot, p)=2\pi(g(p, q')-g(p, q)).$$

Similarly we can see that

$$(\mathrm{Im}\,\omega, \mathrm{Im}\,\overline{\varphi_p})_{R_1}=\int_{-\partial U(p,N)}-g(\cdot, p)\cdot\mathrm{Im}\,\omega$$

$$=N\int_{-\partial U(p,N)}-\mathrm{Im}\,\omega=0.$$

Finally since ω is holomorphic on $U(p, N)$, it holds that $\iint_{U(p,N)}\omega\wedge{}^*\varphi_p=0$, and we have the desired equation. q. e. d.

Proof of Theorem 2. By Lemma 3, we have

$$\left|\int_d {}^*dg(\cdot, q')-\int_d {}^*dg(\cdot, q)\right|\leq\left|\iint_R\omega\wedge\theta_d\right|\leq\|\omega\|_R\cdot\|\theta_d\|_R.$$

Since $\|\theta_d\|_R^2=2\lambda_d$ by Accola's theorem, we conclude the assertion by Lemma 2. q. e. d.

Proof of Theorem 3. Since $\omega\wedge{}^*\varphi\equiv 0$ on $R-\overline{W}\cup\{p\}$, we have

$$|g(q, p)-g(q', p)|\leq(1/2\pi)\cdot\left|\iint_R\omega\wedge{}^*\varphi_p\right|$$

$$=(1/2\pi)\cdot\left|\iint_W\omega\wedge{}^*\varphi_p\right|\leq(1/2\pi)\cdot\|\omega\|_R\cdot\|\varphi_p\|_W<e^2\cdot\|\varphi_p\|_W\cdot|a|.$$

Next since W is contained in $U(q, M+1)$ and $g(\cdot, p)$ is positive harmonic on $U(q, M)$, Harnack's inequality implies that

$$\sup_{s\in W}g(s, p)-\inf_{s\in W}g(s, p)\leq\frac{4e}{e^2-1}\cdot g(q, p).$$

Here recall that $\|dg(\cdot, p)\|^2_{R-U(p,N)}=\int_{-\partial U(p,N)}g(\cdot, p)^*dg(\cdot, p)=2\pi N$ for every sufficiently large N (cf. the proof of Lemma 4). And since $g(\cdot, p)-t$ is Green's function on $\{r\in R: g(r, p)>t\}$ for every positive t, we can see that $\|dg(\cdot, p)\|^2_{\{r\in R: t<g(r,p)<t'\}}=2\pi(t'-t)$ for every t and t' with $0<t<t'$. Hence we conclude that

$$\|\varphi_p\|_W^2 \leq 4\pi \cdot \frac{4e}{e^2-1} \cdot g(q, p) < 25g(q, p),$$

which gives that

$$|g(q', p) - g(q, p)| < 5e^2 \cdot g(q, p)^{1/2} \cdot |a|.$$

The second inequality follows by recalling that $g(q, p) \leq M$, for $p \notin U(q, M)$.
q. e. d.

Remark. The author guess that, in case that $g(q, p) < 1$, we can show that $\|\varphi_p\|_W \leq A \cdot (g(q, p))^\lambda$ with some $\lambda > 1/2$ and a constant A (which may depend on p and R). Note that such λ should not be greater than 1.

§4. The proofs of Theorems 4 and 5.

For the proofs, the following lemma is crucial.

LEMMA 5. *There is an absolute constant C such that, for every q and M as in Theorem 1, and every $q' \in U(q, M+C)$, we can find an $M(q')$ such that $M \leq M(q') \leq M+C$, $U(q', M(q'))$ is simply connected and*

$$\text{(I)} \qquad U(q, M+C) < U(q', M(q')) < M(q, M).$$

Proof. Set $C = B+1$ with an absolute constant B in [4, Proposition 2], and apply [4, Proposition 2] to $h(z) = (1/2\pi) \cdot g(Z^{-1}(z), q')$ on $Z(W_0)$ with $W_0 = U(q, M+1/4) - \overline{U(q, M+C-1/4)}$. Then we have an $M(q')$ such that $\{s \in R : g(s, q') = M(q')\}$ is a simple closed curve in W_0 separating two boundary components of W_0.

Fix such an $M(q')$, then it is clear that $U(q', M(q'))$ is simply connected and satisfies (I). And since, in general, $g(\cdot, r)/M$ is the harmonic measure of $\partial U(r, M)$ in $R - \overline{U(r, M)}$, for every $r \in R$ and positive M, we can see that $M \leq M(q') \leq M+C$.
q. e. d.

Proof of Theorem 4. By Lemma 5, we can apply Theorem 3 with $q_1 \in U(q, M+C)$ and $M+C$. Then we have

$$\begin{aligned} |g(q_2, p) - g(q_1, p)| &\leq 5 \cdot (M+C)^{1/2} \cdot e^{M+C+4} \cdot \exp(-g(q_1, q_2)) \\ &\leq 5C \cdot e^{C+4}(M+1)^{1/2} \cdot e^M \cdot \exp(-g(q_1, q_2)) \end{aligned}$$

for every q_2 in $U(q_1, M+C+4)$.

On the other hand, if $q_1 \in U(q, M+2C+4)$, then $q \in U(q_1, M+2C+4)$ by symmetry, and hence again by Lemma 5 we see that $U(q, M+2C+4)$ is contained in $U(q_1, M+C+4)$. Hence the assertion holds with $C_0 = 2C+4$ and $C_1 = 5C \cdot e^{C+4}$.
q. e. d.

Next to show Theorem 5, we need the following

LEMMA 6. *Let p and q be distinct points on R such that $U(q, M-C)$ is simply connected with $M=g(q, p)$ and C given in Lemma 5. Then it holds that*

$$|g(q', p)-g(q, p)| \leq C' \cdot e^{M} \cdot \exp(-g(q', q))$$

for every $q' \in U(q, M+C+4)$, where C' is an absolute constant.

Proof. By Lemma 5 we have

$$U(q', M-C) \supset U(q, M) \supset U(q', M+C)$$

for every $q' \in U(q, M+C)$, which implies that

$$\sup_{s \in U(q, M+C)} g(s, p) \leq M+C, \quad \text{and}$$

$$\inf_{s \in U(q, M+C)} g(s, p) \geq M-C.$$

In particular, it holds that $\|\varphi_p\|^2_{U(q, M+C)} \leq 4\pi \cdot 2C$.

Hence by the same argument as in the proof of Theorem 3, we have

$$|g(q', p)-g(q, p)| < e^2 \cdot \|\varphi_p\|_W \cdot |a|$$

$$\leq e^{M+C+4} \cdot (8\pi C)^{1/2} \cdot \exp(-g(q', q))$$

for every $q' \in U(q, M+C+4)$, i.e., the assertion holds with $C'=e^{C+4} \cdot (8\pi C)^{1/2}$.
q. e. d.

Proof of Theorem 5. Suppose that $U(q, M-3C)$ is simply connected with $M=g(p, q)$. Then we can see by Lemma 5 that, for every $q_1 \in U(q, M+C)$, it holds that $M-C \leq M_1 = g(q_1, p) \leq M+C$ and $U(q_1, M-2C)$ is simply connected. Hence by Lemma 6 we have

$$|g(q_2, p)-g(q_1, p)| < C' \cdot e^{M_1} \cdot \exp(-g(q_1, q_2))$$

for every $q_2 \in U(q_1, M_1+C+4)$.

Thus as in the proof of Theorem 4, we can show that $C_2 = 3C+4$ and $C_3 = C' \cdot e^C$ are desired constants. q. e. d.

§5. Another application of Theorem 1.

In this section, we will use the same notation as in [5], and show that the remainder terms in the formulas (2) and (3) of [5, Theorem 2] can be estimated locally uniformly on R'_0 with respect to q and q'. Here we will discuss only (3), for the treatment of (2) is the same.

Let $F(q, q', t)$ be the remainder term, i.e.,

$$F(q, q', t)=g(q, q'; R_t)-g(q, q'; R_0)-\Big\{-\frac{1}{2}\cdot\log\Big(\frac{1}{t}\Big)\cdot G(q)\cdot G(q')$$

$$-t^2\cdot\mathrm{Re}\,[\eta\cdot(b_{0,q,1}(0)\cdot b_{0,q',2}(0)+b_{0,q,2}(0)\cdot b_{0,q',1}(0))]\Big\}.$$

Then we want to show the following

PROPOSITION. *When t tends to 0, $F(q, q', t)/t^2$ converges to 0 locally uniformly on R_0' with respect to q and q'.*

Proof. By the equation [5, §4 (14)], we can see (cf. [5, 293p]) that

$$F(q, q', t)=\frac{-1}{2\pi}\cdot\mathrm{Re}\Big[\sum_{n=1}^{\infty}c_{n,1}\cdot\oint_{\{|z_2|=t_0\}}(b_{t,q,2}(z_2)-b_{0,q,2}(z_2))\cdot(\eta t^2/z_2)^n dz_2$$

$$+\sum_{n=1}^{\infty}c_{n,2}\cdot\oint_{\{|z_1|=t_0\}}(b_{t,q,1}(z_1)-b_{0,q,1}(z_1))\cdot(\eta t^2/z_1)^n dz_1$$

$$+\sum_{n=2}^{\infty}(c_{n,1}\cdot e_{n,2}+c_{n,2}\cdot e_{n,1})(2\pi\sqrt{-1})(\eta t^2)^n\Big],$$

where $\phi(0, q)=\sum_{n=1}^{\infty}e_{n,j}z_j^{n-1}dz_j$ and $\phi(0, q')=\sum_{n=1}^{\infty}nc_{n,j}z_j^{n-1}dz_j$ on $\bar{U}_j(M_0)$ $(j=1, 2)$. Integrating over a suitable compact interval in $[t_0, \exp(-M_0))$, we see that

$$|F(q, q', t)|\leq A(t^2\cdot\|\phi(t, q)-\phi(0, q)\|_E\cdot\|\phi(0, q')\|_{U_0}$$

$$+t^4\|\phi(0, q)\|_{U_0}\cdot\|\phi(0, q')\|_{U_0})$$

with a suitable compact set E in $U_0=U_1(M_0)\cup U_2(M_0)$ and a constant A depending only on E.

Here by Theorem 1 (or, as is well-known), $\|\phi(0, q')\|_{U_0}$ is continuous, hence locally bounded on R_0' (as a function of q'). Hence the assertion follows from the following lemma. q. e. d.

LEMMA 7. *Set $F_t(q)=\|\phi(t, q)-\phi(0, q)\|_E$. Then $F_t(q)$ converges to 0 locally uniformly on R_0' as t tends to 0.*

Proof. First recall that $\lim_{t\to0}F_t(q)=0$ for every $q\in R_0'$ which follows by [5, Theorem 1]. And it suffices to show that, for every $q_0\in R_0'$ and every positive $\varepsilon>0$, there is a neighborhood V of q_0 in R_0' and a positive T such that $F_t(q)<\varepsilon$ for every $q\in V$ and $t\in[0, T]$.

To show this, note that

$$F_t(q)\leq 2\|dg_t(\cdot, q)-dg_t(\cdot, q_0)\|_E+F_t(q_0)$$

$$+2\|dg_0(\cdot, q)-dg_0(\cdot, q_0)\|_E.$$

Since $g_t(\cdot, q_0)$ converges to $g_0(\cdot, q_0)$ uniformly, at least, in some neighborhood V_0 of q_0 by [5, Corollary 1], we can apply Theorem 1 with the same M to both g_t and g_0 for every sufficiently small t. Hence we conclude that there is a neighborhood V_1 ($\subset V_0$) of q_0 and a T_0 (>0) such that for every $t \in [0, T_0]$ and $q \in V_1$ we have

$$\|dg_t(\cdot, q) - dg_t(\cdot, q_0)\|_E \leq C'' \cdot \exp(-g_t(q, q_0))$$

with a constant C'' depending only on M, from which the assertion follows easily. q. e. d.

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