GENERAL FINSLER CONNECTIONS ON
A FINSLER VECTOR BUNDLE

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§ 1. Introduction.

In the present paper, we shall describe a fundamental concept of general Finsler connections combining vectorial Finsler connections introduced by the first author for Finsler vector bundles [2] with general connections introduced by the second author for differentiable manifolds [7] and [8] and treat some special examples.

§ 2. General Finsler connections on a Finsler vector bundle.

Let M be a real differentiable manifold of dimension n. Denote by TM the tangent bundle over M and by π the canonical projection of TM to M. Also, denote by $d\pi : T \backslash M \to TM$ the differential of π and define the vertical subbundle VTM of TTM as the kernel of $d\pi$. A complementary distribution N to VTM in TTM is called a *non-linear connection* on TM (see M. Matsumoto [4] p. 55). Of course, the fibres of vector bundles VTM and N are of the same dimension n. In this section we have the following range of indices: $i, j, k, \cdots = 1, 2, \cdots, n$; $a, b, c, \cdots = 1, 2, \cdots, p$.

In order to get a local feature of the notion of non-linear connection we consider a canonical chart $(\mathcal{U}, \varphi, R^{2n})$ on TM, where $\mathcal{U}$ is an open set of TM and $\varphi : \mathcal{U} \to R^{2n}$ is a diffeomorphism of $\mathcal{U}$ on $\varphi(\mathcal{U})$. For each $z \in \mathcal{U}$ denote by $(x^1, \cdots, x^n; y^1, \cdots, y^n)$ its coordinates in this chart. We take another local chart $(\varphi', \mathcal{U}', R^{2n})$ on TM such that $\mathcal{U} \cap \mathcal{U}' \neq \phi$. Then the non-linear connection N is given by n^2 differentable functions $N^i{}_j$ on each $\mathcal{U}$ satisfying

$$(2.1) \qquad N^{i'}{}_{j'} = \frac{\partial x^{i'}}{\partial x^i} \left(\frac{\partial x^j}{\partial x^{j'}} N^i{}_j + \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}} y^{k'} \right)$$

at all points of $\mathcal{U} \cap \mathcal{U}'$, where $N^{i'}{}_{j'}$ are the corresponding functions on $\mathcal{U}'$. We denote by $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\}$ the natural local field of frames on TM. Then $\left\{ \frac{\delta}{\delta x^i} \right\}$ given by

$$(2.2) \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j{}_i \frac{\partial}{\partial y^i},$$

is a local field of frames on the distribution N.

Now let $\pi_E : E \to TM$ be a vector bundle over TM with standard fibre R^p and $\Gamma(E)$ be the module of all differentiable cross-sections of E. Then we call E a *Finsler vector bundle* on M. A justification of the above name is the simple fact that all cross-sections of E are depending on both, point and direction on M. The vertical subbundle VTM and the non-linear connection N defined above are certainly Finsler vector bundles. Further, we consider a vector bundle morphism $P : E \to E$ and a *general connection* ∇ (Otsuki connection) on E with respect to P or a general connection with P as its component of the first order (see T. Otsuki [8] and [9]). Then by a recent work of N. Abe [1], a general connection on TM is a mapping $\nabla : \Gamma(TTM) \times \Gamma(E) \to \Gamma(E)$ satisfying

$$(2.3) \qquad \nabla_{fX+Y}S = f\nabla_X S + \nabla_Y S \quad \text{and}$$

$$(2.4) \qquad \nabla_X(fS + S') = (Xf)PS + f\nabla_X S + \nabla_X S',$$

for any S, $S' \in \Gamma(E)$, X, $Y \in \Gamma(TTM)$ and differentiable function f on TM.

We have now all the elements in order to introduce a new concept in Finsler geometry. We say that the triplet (N, P, ∇) is defining a *general Finsler connection* on the Finsler vector bundle E. If in particular, P is the identity morphism I_E on E, the general connection on E is just a linear connection on E and we obtain the concept of vectorial Finsler connection on E (see A. Bejancu [2]).

Since as it is well-known in Finsler geometry, local computations are always welcome, we give further the local alternative of the definition of a general Finsler connection on E. In order to do this we consider a vectorial chart $(\Phi, \mathcal{U}, R^p)$ on the vector bundle E, where $\mathcal{U}$ is an open set of TM and $\Phi : \pi_E^{-1} \to \mathcal{U} \times R^p$ is a diffeomorphism. Let $(\Phi', \mathcal{U}', R^p)$ be another vectorial chart such that $\mathcal{U} \cap \mathcal{U}' \neq \phi$. Denote by $\{S_a\}$ (resp. $S_{a'}$) the local basis of differentiable sections of E on $\mathcal{U}$ (resp. $\mathcal{U}'$). Then there exist p^2 real differentiable functions $G^a_{a'}$ defined on $\mathcal{U} \cap \mathcal{U}'$ and satisfying

$$(2.5) \qquad S_{a'} = G^a_{a'} S_a.$$

On the other hand, the addapted field of frames $\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i} \right\}$ and $\left\{ \frac{\delta}{\delta x^{i'}}, \frac{\partial}{\partial y^{i'}} \right\}$ on $\mathcal{U}$ and $\mathcal{U}'$ respectively satisfy

$$(2.6) \qquad \frac{\delta}{\delta x^{i'}} = \frac{\partial x^i}{\partial x^{i'}} \frac{\delta}{\delta x^i} ; \qquad \frac{\delta}{\delta y^{i'}} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial y^i}$$

Next, we define on each domain $\mathcal{U} \subset TM$ and $\pi_E^{-1}(\mathcal{U})$ the local chart of TM and the local vectorial chart of E, the differentiable functions $F_a{}^b{}_i$ and $C_a{}^b{}_i$ by

$$(2.7) \qquad \nabla_{\delta/\delta x^i} S_a = F_a{}^b{}_i S_b ; \qquad \nabla_{\partial/\partial y^i} S_a = C_a{}^b{}_i S_b.$$

Thus, given a general Finsler connection on E we have the quadruplet of differentiable functions $(N^i{}_j, F_a{}^b{}_i, C_a{}^b{}_i, P_a^b)$ on each open set $\mathcal{U} \subset TM$. Conversely, suppose on each domain $\mathcal{U} \subset TM$ there exists a quadruplet of differentiable functions $(N^i{}_j, F_a{}^b{}_i, C_a{}^b{}_i, P_a^b)$ satisfying (2.1) and

$$(2.8) \quad \begin{cases} P_{a'}^{b'} = \tilde{G}_b^{b'} P_a^b G_{a'}^a, \\[2mm] F_{a'}{}^{b'}{}_{i'} = \dfrac{\partial x^i}{\partial x^{i'}} \tilde{G}_b^{b'} \left(\dfrac{\delta G_{a'}^a}{\delta x^i} P_a^b + G_{a'}^a F_a{}^b{}_i \right), \\[2mm] C_{a'}{}^{b'}{}_{i'} = \dfrac{\partial x^i}{\partial x^{i'}} \tilde{G}_b^{b'} \left(\dfrac{\partial G_{a'}^a}{\partial y^i} P_a^b + G_{a'}^a C_a{}^b{}_i \right) \end{cases}$$

in each point of $\mathcal{U} \cap \mathcal{U}'$, where $(N^{i'}{}_{j'}, F_{a'}{}^{b'}{}_{i'}, C_{a'}{}^{b'}{}_{i'}, P_{a'}^{b'})$ is the quadruplet on $\mathcal{U}'$ and $(\tilde{G}_b^{b'})$ is the inverse matrix of $(G_{a'}^a)$. Then it is routine to show by means of (2.1) and (2.8) the existence of a general Finsler connection on E.

Thus we have

THEOREM 2.1. *Let E be a Finsler vector bundle on M. Then the existence of a general Finsler connection on E is equivalent to the existence of a quadruplet of differentiable functions $(N^i{}_j, F_a{}^b{}_i, C_a{}^b{}_i, P_a^b)$ on each domain $\mathcal{U}$ of TM satisfying (2.1) and (2.8).*

Let $GFC = (N, P, \nabla)$ be a general Finsler connection on the Finsler vector bundle E. Then the curvature tensor $\tilde{R}$ of ∇ is given by (see N. Abe [1])

$$(2.9) \quad \tilde{R}(X, Y)V = \nabla_X \nabla_Y PV - \nabla_Y \nabla_X PV - P(\nabla_{[X,Y]} PV)$$
$$- (\nabla_X I_E)\nabla_Y V + (\nabla_Y I_E)\nabla_X V,$$

for any $X, Y \in \Gamma(TTM)$ and $V \in \Gamma(E)$. Taking into account that in Finsler geometry people is accustomed to three local curvature tensors we define the local components $R_a{}^b{}_{ij}$, $P_a{}^b{}_{ij}$ and $S_a{}^b{}_{ij}$ of $\tilde{R}$ by

$$(2.10) \quad \begin{cases} \tilde{R}\left(\dfrac{\delta}{\delta x^j}, \dfrac{\delta}{\delta x^i}\right) S_a = R_a{}^b{}_{ij} S_b; \quad \tilde{R}\left(\dfrac{\partial}{\partial y^j}, \dfrac{\delta}{\delta x^i}\right) S_a = P_a{}^b{}_{ij} S_b; \\[3mm] \tilde{R}\left(\dfrac{\partial}{\partial y^j}, \dfrac{\partial}{\partial y^i}\right) S_a = S_a{}^b{}_{ij} S_b. \end{cases}$$

Then we denote the covariant components $\Lambda_a{}^b{}_i$, $\chi_a{}^b{}_i$ of the general Finsler connection ∇, the components $I_a{}^b{}_i$, $J_a{}^b{}_i$ of ∇I_E with respect to $\delta/\delta x^i$ and $\partial/\partial y^i$ respectively and the components of the torsion tensor of N as follows:

$$\Lambda_a{}^b{}_i = F_a{}^b{}_i - \frac{\delta P_a^b}{\delta x^i}; \qquad I_a{}^b{}_i = F_c{}^b{}_i P_a^c - P_c^b \Lambda_a{}^c{}_i,$$

$$(2.11) \qquad \chi_a{}^b{}_i = C_a{}^b{}_i - \frac{\partial P_a^b}{\partial y^i}; \qquad J_a{}^b{}_i = C_c{}^b{}_i P_a^c - P_c^b \chi_a{}^c{}_i,$$

$$R^k{}_{ij} = \frac{\delta N^k{}_i}{\delta x^j} - \frac{\delta N^k{}_j}{\delta x^i}.$$

Thus by using (2.9)-(2.11) we obtain (after a long computation) the local components of curvature tensor field $\tilde{R}$ given by

$$(2.12) \qquad R_a{}^b{}_{ij} = \left(\frac{\delta F_c{}^d{}_i}{\delta x^j} - \frac{\delta F_c{}^d{}_j}{\delta x^i}\right)P_a^c P_d^b + (F_c{}^d{}_i F_d{}^b{}_j - F_c{}^d{}_j F_d{}^b{}_i)P_a^c$$

$$+ I_c{}^b{}_i \Lambda_a{}^c{}_j - I_c{}^b{}_j \Lambda_a{}^c{}_i + R^k{}_{ij} C_c{}^d{}_k P_a^c P_d^b,$$

$$(2.13) \qquad P_a{}^b{}_{ij} = \left(\frac{\partial F_c{}^d{}_i}{\partial y^j} - \frac{\delta C_c{}^d{}_j}{\delta x^i}\right)P_a^c P_d^b + (F_c{}^d{}_i C_d{}^b{}_j - C_c{}^d{}_j F_d{}^b{}_i)P_a^c$$

$$+ I_c{}^b{}_i \chi_a{}^c{}_j - J_c{}^b{}_j \Lambda_a{}^c{}_i + \frac{\partial N^k{}_i}{\partial y^j} C_c{}^d{}_k P_a^c P_d^b,$$

$$(2.14) \qquad S_a{}^b{}_{ij} = \left(\frac{\partial C_c{}^d{}_i}{\partial y^j} - \frac{\partial C_c{}^d{}_j}{\partial y^i}\right)P_a^c P_d^b + (C_c{}^d{}_i C_d{}^b{}_j - C_c{}^d{}_j C_d{}^b{}_i)P_a^c$$

$$+ J_c{}^b{}_i \chi_a{}^c{}_j - J_c{}^b{}_j \chi_a{}^c{}_i.$$

It is easy to check that for the particular case $P = I_E$ from (2.12)-(2.14) we obtain the local components of the curvature tensor of a vectorial Finsler connection on E (see A. Bejancu [2]).

§3. General Finsler connections induced by vectorial Finsler connections.

Let $\pi_E: E \to TM$ and $\pi_F: F \to TM$ be two Finsler vector bundles on M. Suppose $VFC = (N, \nabla)$ is a vectorial Finsler connection on E. Then we consider the Whitney sum $G = E \oplus F$ and denote by P the projection morphism from G to E. Then we can define a general connection $\tilde{\nabla}$ on the whole G with respect to P by the following formula

$$(3.1) \qquad \tilde{\nabla}_X V = \nabla_X PV \qquad \text{for} \quad X \in \Gamma(TTM) \quad \text{and} \quad V \in \Gamma(G).$$

It is easy to verify that (3.1) defines a general connection on G. Thus, starting from the vectorial Finsler connection $VFC = (N, \nabla)$ on E we obtain on each Whitney sum G of E with any other Finsler vector bundle F, a general Finsler connection $GFC = (N, P, \tilde{\nabla})$ where $\tilde{\nabla}$ is given by (3.1). We call GFC the *induced general Finsler connection* by the vectorial Finsler connection VFC. Denote by $\tilde{R}$ and R the curvature tensors of $\tilde{\nabla}$ and ∇ respectively. Then the following theorem seems to us very important.

THEOREM 3.1. *The restriction of the curvature tensor of GFC to E is just the curvature tensor VFC, that is, we have*

$$(3.2) \qquad \tilde{R}(X, Y)V = R(X, Y)V \qquad \text{for any} \quad X, Y \in \Gamma(TTM), \quad V \in \Gamma(E).$$

Proof. First by a result of N. Abe [1] and (3.1) we have

(3.3) $(\tilde{\nabla}_X I_G)(\tilde{\nabla}_Y V) = \tilde{\nabla}_X (P \tilde{\nabla}_Y V) - P(\tilde{\nabla}_X \tilde{\nabla}_Y V)$

$$= \nabla_X (P \nabla_Y PV) - P(\nabla_X P \nabla_Y PV) \quad \text{for any} \quad X, Y \in \Gamma(TTM)$$

$$\text{and} \quad V \in \Gamma(E).$$

But $PV = V$ and $P \nabla_X V = \nabla_X V$ since $\nabla_X V \in \Gamma(E)$. Hence by (3.3) we get

(3.4) $(\tilde{\nabla}_X I_E)(\tilde{\nabla}_Y V) = 0.$

Thus, taking account of (3.4) in (2.9) we get our assertion.

Now, let $F^n = (M, F(x, y))$ be a Finsler space, where M is an n-dimensional differentiable manifold and $F(x, y)$ is the fundamental function of F^n (see M. Matsumoto [4]). Denote by $g_{ij}(x, y)$ the fundamental tensor field of F^n, that is we have

(3.5) $g_{ij}(x, y) = \dfrac{1}{2} \dfrac{\partial^2 F^2}{\partial y^i \partial y^j}.$

Suppose $g_{ij}(x, y)$ is positive definite, that is, it defines a Riemannian metric on the vectorial bundle VTM. Next we consider a Finsler connection FC on F^n which in fact is just a vectorial Finsler connection $FC = (N, \nabla)$ on VTM, (see A. Bejancu [2], [3]). The existence of a non linear connection N on TM implies $TTM = N \oplus VTM$. Hence by the above construction we get a general Finsler connection $VGFC = (N, v, \tilde{\nabla})$, $\tilde{\nabla}$ is given by (3.1) and v is the projection morphism of TTM to VTM. We call $VGFC$ the *vertical general Finsler connection* on TTM.

Now we define the isomorphism of vector bundles

(3.6) $L: N \to VTM$ by $L(X) = X^i \dfrac{\partial}{\partial y^i}$,   where $X = X^i \dfrac{\delta}{\delta x^i}.$

It is easy to check that L does not depend on the local coordinates (x^i, y^i) on TM. Then we define a linear connection ∇^* on the vector bundle N by

(3.7) $\nabla_X^* U = L^{-1}(\nabla_X LU)$ for any $X \in \Gamma(TTM)$, $U \in \Gamma(N).$

Next, denote by $\tilde{\nabla}^*$ the general connection induced by ∇^* on TTM. Hence we have

(3.8) $\tilde{\nabla}_X^* Y = \nabla_X^* hY$ for any $X, Y \in \Gamma(TTM($,

where h is the projection morphism of TTM to N. Thus we obtain another general Finsler connection $HGFC = (N, h, \tilde{\nabla}^*)$, whose curvature $\tilde{R}^*$ restricted to N is just the curvature tensor R^* of ∇^*. We call $HGFC$ the *horizontal general Finsler connection* on TTM.

As it is well-known, a Finsler connection $FC = (N^i_j, F_j^i{}_k, C_j^i{}_k)$ on F^n has five torsions $T_j^i{}_k$, $R^i{}_{jk}$, $P^i{}_{jk}$, $S^i{}_{jk}$ and $C_j^i{}_k$ given by

(3.9) $T_j^i{}_k = F_j^i{}_k - F_k^i{}_j;$

$$(3.10) \qquad R^i{}_{jk} = \frac{\delta N^i{}_j}{\delta x^k} - \frac{\delta N^i{}_k}{\delta x^j} \,;$$

$$(3.11) \qquad P^i{}_{jk} = \frac{\partial N^i{}_j}{\partial y^k} - F_k{}^i{}_j \,;$$

$$(3.12) \qquad S^i{}_{jk} = C_j{}^i{}_k - C_k{}^i{}_j \,;$$

$$\nabla_{\partial/\partial y^i} \frac{\partial}{\partial y^j} = C_j{}^k{}_i \frac{\partial}{\partial y^k} \,.$$

On the other hand, for any general connection on TTM we can define its torsion tensor field (see T. Otsuki [7] and H. Nemoto [6]). Then we have

THEOREM 3.2. *The torsion tensors of the horizontal and vertical general Finsler connections on TTM give all the torsions of the initial Finsler connection on F^n.*

Proof. First, we denote by T the torsion tensor field of the general connection $\tilde\nabla$. Thus we have

$$(3.13) \qquad T(X, Y) = \tilde\nabla_X Y - \tilde\nabla_Y X - v([X, Y]) \qquad \text{for any} \quad X, Y \in \Gamma(TTM).$$

Locally, we obtain

$$(3.14) \qquad T\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = R^k{}_{ij} \frac{\partial}{\partial y^k} \,; \qquad T\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}\right) = P^k{}_{ij} \frac{\partial}{\partial y^k} \,;$$

$$T\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = S^k{}_{ij} \frac{\partial}{\partial y^k} \,.$$

Next, we denote by T^* the torsion tensor field of $\tilde\nabla^*$. Then by using a similar formula to (3.13) we get

$$(3.15) \qquad T^*\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = T_i{}^k{}_j \frac{\delta}{\delta x^k} \,; \qquad T^*\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}\right) = C_i{}^k{}_j \frac{\delta}{\delta x^k} \,;$$

$$T^*\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = 0 \,.$$

Thus our assertion follows from (3.14) and (3.15).

Therefore, starting with a Finsler connection on F^n we obtain two general Finsler connections on TTM mentioned above. Now, we consider a general Finsler connection $GFC = (N, v, \tilde\nabla)$ on TTM with respect to the projection morphism v of TTM to VTM. Then we define a linear connection ∇ on VTM by

$$(3.16) \qquad \nabla_X V = v \tilde\nabla_X V \qquad \text{for any} \quad X \in \Gamma(TTM), \quad V \in \Gamma(VTM),$$

and obtain a Finsler connection $FC = (N, \nabla)$ on F^n. However, the vertical general Finsler connection induced by FC does not coincide in general with the initial

GFC. With respect to this problem, by (3.1) and (3.16) we easily obtain

THEOREM 3.3. *The vertical general Finsler connection induced by FC coincides with the initial general Finsler connection GFC if and only if we have:*
  (i) *the distribution VTM is parallel with respect to $\tilde{\nabla}$, i.e., we have*

$$\tilde{\nabla}_X V \in \Gamma(VTM) \quad for \ any \quad X \in \Gamma(TTM) \quad and \quad V \in \Gamma(VTM),$$

  (ii) *each section of N is parallel with respect to $\tilde{\nabla}$, i.e., we have*

$$\tilde{\nabla}_X H = 0 \quad for \ any \quad X \in \Gamma(TTM) \quad and \quad H \in \Gamma(N).$$

Remark 3.1. From the above study we can conclude that Finsler connections on a Finsler space are too particular geometrical objects. The set of Finsler connections on F^n is in a one to one correspondence with the set of general Finsler connections on TTM satisfying the very strong conditions of Theorem 3.3. This explains (in a way) why in many cases more conditions imposed to a Finsler connection imply the Riemannian case.

In particular we can consider the Cartan connection, Berwald connection, Rund connection, etc., and investigate properties of horizontal and vertical general Finsler connections induced by them on TTM. It is not the purpose of the present paper to perform such a study. However, we note a property for the Cartan connection.

By means of the Riemannian metric g on VTM we define a Riemannian metric on the vector bundle TTM similar to the Sasaki one and denote it also by g, that is

$$d s^2 = g_{ij}(x, y) d x^i d x^j + g_{ij}(x, y) \delta y^i \delta y^j,$$

where $\delta y^i = d y^i + N^i{}_j(x, y) d x^j$. Then the vertical and horizontal general Finsler connections on F^n corresponding to the Cartan connection satisfy

(3.17) $\tilde{\nabla}_X g = 0; \quad \tilde{\nabla}^*{}_X g = 0 \quad$ for any $\quad X \in \Gamma(TTM)$

and

(3.18) $T(vY, vZ) = T^*(hY, hZ) = 0 \quad$ for any $\quad Y, Z \in \Gamma(TTM).$

Thus it is possible to get more results on geometry of F^n via both general Finsler connections induced by Cartan connection. By this method we can perform invariant computations which certainly in Finsler geometry are welcome.

Remark 3.2. If we consider the general connection $\tilde{\nabla} + \tilde{\nabla}^*$ on TTM, it becomes affine, because $v + h = I_{TTM}$ and its torsion tensor coincides with the ones of the initial Finsler connection on F^n as easily seen from (3.13)-(3.15).

§4. General Finsler connections on a Finsler space.

In the previous sections we started with a vectorial Finsler connection (or in particular a Finsler connection) on a certain Finsler vector bundle and obtained general Finsler connections on larger vector bundles. It is the purpose of this section to study general Finsler connections on a Finsler space F^n, that is, general Finsler connections on VTM.

Let $GFC=(N^i{}_j, F_j{}^i{}_k, C_j{}^i{}_k, P^i_j)$ be a general Finsler connection on VTM. Then we call GFC a general Finsler connection on the Finsler space F^n. Since in this particular case we have

$$G^i_{i'} = \frac{\partial x^i}{\partial x^{i'}},$$

by (2.1) and (2.8) we get

$$(4.1) \quad \begin{cases} N^{i'}{}_{j'} = \dfrac{\partial x^{i'}}{\partial x^i}\Big(\dfrac{\partial x^j}{\partial x^{j'}} N^i{}_j + \dfrac{\partial^2 x^i}{\partial x^{j'}\partial x^{k'}} y^{k'}\Big), \\[2mm] P^{i'}_{j'} = \dfrac{\partial x^{i'}}{\partial x^i} P^i_j \dfrac{\partial x^j}{\partial x^{j'}}, \\[2mm] F_{j'}{}^{i'}{}_{k'} = \dfrac{\partial x^{i'}}{\partial x^i}\Big(\dfrac{\partial x^k}{\partial x^{k'}} \dfrac{\partial x^j}{\partial x^{j'}} F_j{}^i{}_k + \dfrac{\partial^2 x^j}{\partial x^{j'}\partial x^{k'}} P^i_j\Big), \\[2mm] C_{j'}{}^{i'}{}_{k'} = \dfrac{\partial x^k}{\partial x^{k'}} \dfrac{\partial x^j}{\partial x^{j'}} \dfrac{\partial x^{i'}}{\partial x^i} C_j{}^i{}_k. \end{cases}$$

Remark 4.1. If in particular, P is the identity morphism on VTM, from (4.1) we get just the transformation law for the coefficients of an usual Finsler connection on F^n (see R. Miron [5]).

Remark 4.2. From (4.1) it follows that $P^i_j(x, y)$ and $C_j{}^i{}_k(x, y)$ are Finsler tensor fields on F^n.

From (2.12)-(2.14) we obtain the components of the curvature tensors of a general Finsler connection on F^n given by

$$(4.2) \quad R_i{}^j{}_{kh} = \Big(\frac{\delta F_p{}^q{}_k}{\delta x^h} - \frac{\delta F_p{}^q{}_h}{\delta x^k}\Big)P^p_i P^j_q + (F_p{}^q{}_k F_q{}^j{}_h - F_p{}^q{}_h F_q{}^j{}_k)P^p_i$$

$$+ I_p{}^j{}_k \Lambda_i{}^p{}_h - I_p{}^j{}_h \Lambda_i{}^p{}_k - R^s{}_{hk} C_p{}^q{}_s P^p_i P^j_q,$$

$$(4.3) \quad P_i{}^j{}_{kh} = \Big(\frac{\partial F_p{}^q{}_k}{\partial y^h} - \frac{\delta C_p{}^q{}_h}{\delta x^k}\Big)P^p_i P^j_q + (F_p{}^q{}_k C_q{}^j{}_h - F_q{}^j{}_k C_p{}^q{}_h)P^p_i$$

$$+ I_p{}^j{}_k \chi_i{}^p{}_h - J_p{}^j{}_h \Lambda_i{}^p{}_k + \frac{\partial N^s{}_k}{\partial y^h} C_p{}^q{}_s P^p_i P^j_q,$$

$$(4.4) \quad S_i{}^j{}_{kh} = \Big(\frac{\partial C_p{}^q{}_k}{\partial y^h} - \frac{\partial C_p{}^q{}_h}{\partial y^k}\Big)P^p_i P^j_q + (C_p{}^q{}_k C_q{}^j{}_h - C_p{}^q{}_h C_q{}^j{}_k)P^p_i$$

$$+ J_p{}^j{}_k \chi_i{}^p{}_h - J_p{}^j{}_h \chi_i{}^p{}_k,$$

where $\varLambda_j{}^\imath{}_k$ and $\chi_j{}^\imath{}_k$ are obtained from (2.11), that is we have

$$(4.5) \qquad \varLambda_j{}^\imath{}_k = F_j{}^\imath{}_k - \frac{\delta P_j^i}{\delta x^k}\,; \qquad \chi_j{}^\imath{}_k = C_j{}^i{}_k - \frac{\partial P_j^i}{\partial y^k}\,.$$

Next we shall obtain the components of torsion tensor of the general Finsler connection GFC in a similar way as for the components of the torsion of a Finsler connection in Theorem 3.2. Thus, starting from a general Finsler connection $GFC = (N, P, \nabla)$ on VTM we construct a general connection $GFC^* = (N, L^{-1} \circ P \circ L, \nabla^*)$ on N by

$$(4.6) \qquad \nabla^*_X U = L^{-1}(\nabla_X LU) \qquad \text{for any} \quad X \in \varGamma(TTM) \quad \text{and} \quad U \in \varGamma(N),$$

where L is the isomorphism given by (3.6). Then we define the general connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ on TTM by

$$(4.7) \qquad\qquad\qquad \tilde{\nabla}_X Y = \nabla_X v Y \quad \text{and}$$

$$(4.8) \qquad\qquad\qquad \tilde{\nabla}^*{}_X Y = \nabla^*_X h Y$$

for any $X, Y \in \varGamma(TTM)$. It is easy to see that $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are general connections with respect to $P \circ v$ and $L^{-1} \circ P \circ L \circ h$ respectively. We denote by $\tilde{T}$ and $\tilde{T}^*$ the torsions of $\tilde{\nabla}$ and $\tilde{\nabla}^*$ respectively and by using similar formulas as (3.13) we obtain five tensor fields:

$$R^k{}_{\imath j} = \Big(\frac{\delta N_\imath^h}{\delta x^j} - \frac{\delta N_j^h}{\delta x^\imath}\Big) P_h^k\,; \qquad P^k{}_{\imath j} = \frac{\partial N_\imath^h}{\partial y^j} P_h^k - F_j{}^k{}_\imath\,;$$

$$T_\imath{}^k{}_j = F_\imath{}^k{}_j - F_j{}^k{}_\imath\,; \qquad S^k{}_{\imath j} = C_\imath{}^k{}_j - C_j{}^k{}_\imath \quad \text{and} \quad C_\imath{}^k{}_j,$$

which we call the torsions of the general Finsler connection GFC. As stated in Remark 3.2, these tensors constitutes also the torsion of the general connection $\tilde{\nabla} + \tilde{\nabla}^*$ on TTM.

Now we consider a Finsler tensor field $T^{ij}_{kh}(x, y)$ on F^n and look for its covariant differentiation. By a procedure performed by T. Otsuki in [7] we can define the (h)-*covariant derivative* of T^{ij}_{kh}

$$(4.8) \qquad T^{ij}_{kh,s} = P_p^i P_q^j \frac{\delta T^{pq}_{lm}}{\delta x^s} P_k^l P_h^m + T^{pq}_{lm} F_p{}^\imath{}_s P_q^j P_k^l P_h^m + T^{pq}_{lm} F_q{}^j{}_s P_p^i P_k^l P_h^m$$

$$- T^{pq}_{lm} \varLambda_k{}^l{}_s P_h^m P_p^i P_q^j - T^{pq}_{lm} \varLambda_h{}^m{}_s P_k^l P_p^i P_q^j$$

and its (v)-*covariant derivative* by

$$(4.9) \qquad T^{ij}_{kh|s} = P_p^i P_q^j \frac{\partial T^{pq}_{lm}}{\partial y^s} P_k^l P_h^m + T^{pq}_{lm} C_p{}^\imath{}_s P_q^j P_k^l P_h^m + T^{pq}_{lm} C_q{}^j{}_s P_p^i P_k^l P_h^m$$

$$- T^{pq}_{lm} \chi_k{}^l{}_s P_h^m P_p^i P_q^j - T^{pq}_{lm} \chi_h{}^m{}_s P_k^l P_p^i P_q^j\,.$$

Certainly, the (h)- and (v)-differentiation can be performed for an arbitrary Finsler tensor field with formulas which look like (4.8) and (4.9) for more indices.

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