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ON THE SPACES OF SELF HOMOTOPY EQUIVALENCES FOR FIBRE SPACES

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Introduction.

This paper contains a detailed account of the results announced in [21]†.

Let X be a connected CW complex with non-degenerate base point x_0. In the following, by a CW complex we mean a connected CW complex with non-degenerate base point, unless otherwise stated. Denote by $G(X)$ the space of self homotopy equivalences of X and $G_0(X)$ the space of self homotopy equivalences of (X, x_0). In [19, 20], the author studied $G(X)$ and $G_0(X)$ when X are certain product CW complexes. The main theme of this paper is to study $G_0(X)$ when X is a fibre space of a Hurewicz fibration: $F\xrightarrow{i}X\xrightarrow{p}B$. We call a Hurewicz fibration simply a fibration.

The first main result is the following:

THEOREM 1.5. *Let E and B be CW complexes and let $p: E\to B$ be a fibration with fibre F. For a given $n>1$, if F is $(n-1)$-connected and $\pi_i(B)=0$ for every $i\geqq n$, then we have the following fibration:*

$$\mathcal{G}(E \operatorname{mod} F) \longrightarrow G_0(E) \xrightarrow{\rho} B',$$

where B' is a space with the same weak homotopy type as $G_0(B)\times G_0(F)$ and $\mathcal{G}(E \operatorname{mod} F)$ is the space of self fibre homotopy equivalences of E leaving the fibre F fixed.

For seeking the image of ρ in Theorem 1.5 we provide the following theorem by using Allaud's theory on the classification of fibre spaces [1].

THEOREM 3.1. *Let F be a CW complex (not necessary connected). And let $(\xi, i): F\xrightarrow{i}E\xrightarrow{p}B$ and $(\xi', i'): F\xrightarrow{i'} E' \xrightarrow{p'} B'$ be two fibrations over CW complexes B and B' respectively. For given elements g of* $\mathrm{map}_o(B, B')$ *and h of $G(F)$ there exists a fibration map $\tilde{g}: E \to E'$ such that the following diagram is semi-commutative:*

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† A continuation of this paper was published as [22, 23], in which this paper is referred to as "On the spaces of self homotopy equivalences for fibre spaces I".

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \downarrow h & & \downarrow \tilde{g} & & \downarrow g \\ F & \xrightarrow{i'} & E' & \xrightarrow{p'} & B' \end{array}$$

(the square on the left is homotopy commutative as maps to the fibre $p'^{-1}(b_0')$ *and the square on the right is commutative) if and only if we have*

$$[\chi_\infty(h)]\circ[k]=[k']\circ[g],$$

where maps $k:(B, b_0)\to(B_\infty, b_\infty)$, $k':(B', b_0')\to(B_\infty, b_\infty)$ *are corresponding maps to the fibrations* (ξ, i) *and* (ξ', i') *respectively and* $\chi_\infty(h)$ *is a self homotopy equivalence of* (B_∞, b_∞).

J. Siegel [12] studied the space $BG_0(E)$ where E is a CW complex of stable 2-stage Postnikov system. With respect to a CW complex E of general 2-stage Postnikov system, we have

THEOREM 3.4. *For given* $1<m<n$, *let*

$$F=K(\pi', n)\xrightarrow{i} E \xrightarrow{p} K(\pi, m)=B$$

be a fibration with a corresponding map $k:(B, b_0)\to(B_\infty, b_\infty)$. *Then there exists a map* $k':(B, b_0)\to(B_\infty', b_\infty')$ *such that* $[Bj\circ k']=[k]$. *And we have*

$$G_0(E)\underset{w}{\simeq} R\times H^n(B, \pi')\times\prod_{i=1}^{n-1}K(H^{n-i}(B, \pi'), i),$$

where R *is the subgroup of* $\mathrm{Aut}(\pi)\times\mathrm{Aut}(\pi')=\varepsilon(B)\times\varepsilon(F)$ *consisting of* $([g], [h])$ *with*

$$h_*([k'])=g^*([k']).$$

Here $[k']$ *is regarded as an element of* $H^{n+1}(B, \pi')$, g^* *and* h_* *are the automorphisms of* $H^{n+1}(B, \pi')$ *induced by* g *and* h *respectively.*

As is easily seen, a corollary of this is the following theorem proved by W. Shih [10] and Y. Nomura [9].

COROLLARY 3.5. *Under the same hypothesis as Theorem* 3.4, *there exists the following exact sequence*

$$1\longrightarrow H^n(B, \pi')\longrightarrow\varepsilon(E)\longrightarrow R\longrightarrow 1,$$

where $\varepsilon(E)$ *is the group of homotopy classes of self homotopy equivalences of* (E, e_0) *and* R *is the same group as the group stated in Theorem* 3.4.

§1. The function spaces and fibrations.

Throughout this paper, we shall work within the category of compactly generated Hausdorff spaces [15]. Let X and Y be spaces with non-degenerate base points. In the sequel by a base point we mean a non-degenerate base point. Then map(X, Y) will denote the space of maps of X to Y with the topology obtained by retopologizing the compact open topology and $\mathrm{map}_0(X, Y)$ will be the subspace of map(X, Y) of maps of X to Y preserving base points. Moreover, when k is a map of X to Y, we denote by map$(X, Y; k)$ the path connected component of k in map(X, Y), and $\mathrm{map}_0(X, Y; k)$ is defined similarly.

In the following we simply call a Hurewicz fibration $p: E \to B$ a fibration $p: E \to B$ and write a fibration $p: E \to B$ with typical fibre F by a fibration

$$F \xrightarrow{i} E \xrightarrow{p} B,$$

where i is a homotopy equivalence of F to the fibre $p^{-1}(b_0)$ over a base point b_0 of B.

Let $p: E \to B$ be a fibration. And let $i: A \to X$ be a closed cofibration. Then we have a fibration

$$p_{\#}: \mathrm{map}_0(A, E) \longrightarrow \mathrm{map}_0(A, B),$$

where $p_{\#}$ is induced by the projection $p: (E, e_0) \to (B, b_0)$. Also we have a fibration

$$i^{\#}: \mathrm{map}_0(X, B) \longrightarrow \mathrm{map}_0(A, B),$$

where $i^{\#}$ is induced by the inclusion $i: (A, x_0) \to (X, x_0)$. We denote by $\mathrm{map}_0(X, B) \times' \mathrm{map}_0(A, E)$ the fibred product of the fibration $i^{\#}$ and $p_{\#}$. Then we define a map $\rho: \mathrm{map}_0(X, E) \to \mathrm{map}_0(X, B) \times' \mathrm{map}_0(A, E)$ by

$$\rho(f) = (p_{\#}(f), i^{\#}(f)) = (p \circ f, f \circ i).$$

Then we have the following theorem [13, 16, 17].

THEOREM 1.1. *Let $i: A \to X$ be a closed cofibration preserving base points. And let $p: E \to B$ be a fibration preserving base points. Then a map*

$$\rho: \mathrm{map}_0(X, E) \longrightarrow \mathrm{map}_0(X, B) \times' \mathrm{map}_0(A, E)$$

is a fibration.

Moreover we have the following lemmas which will be used later.

LEMMA 1.2. *Let $p: E \to B$ be a fibration with fibre F and let $f: X \to B$ be a map. Assume that B is weakly contractible, then the fibre space f^*E of induced fibration by f has the same weak homotopy type as $X \times F$.*

Proof. We may assume without loss of generality that both X and E are path connected.

Let $i: F \to E$ and $j: F \to f^*E$ be the inclusion maps, and let $p_1: f^*E \to X$ and $p_2: f^*E \to E$ be the projections. Then we have the following commutative diagram

$$\begin{array}{ccccc} \pi_n(F) & \xrightarrow{j_*} & \pi_n(f^*E) & \xrightarrow{p_{1*}} & \pi_n(X) \\ \| & & \downarrow{\scriptstyle p_{2*}} & & \downarrow{\scriptstyle f_*} \\ \pi_n(F) & \xrightarrow{i_*} & \pi_n(E) & \xrightarrow{p_*} & \pi_n(B) . \end{array}$$

Since $\pi_n(B)$ is trivial for every n, $i_*: \pi_n(F) \to \pi_n(E)$ is an isomorphism for every n. Therefore $j_*: \pi_n(F) \to \pi_n(f^*E)$ is a monomorphism for every n and $p_{2*}: \pi_n(f^*E) \to \pi_n(E)$ is an epimorphism. Using these facts, we can easily see that the map $(p_1, p_2): f^*E \to X \times E$ induces isomorphisms of the corresponding homotopy groups. That is, f^*E is weakly homotopy equivalent to $X \times E$. Obviously $X \times F$ is weakly homotopy equivalent to $X \times E$. Thus f^*E has the same weak homotopy type as $X \times F$.

LEMMA 1.3. *For given $n > 1$, let X be an $(n-1)$-connected CW complex and Y be a path connected space with $\pi_i(Y) = 0$ for every $i \geqq n$. Then* $\text{map}_0(X, Y)$ *is weakly contractible.*

Proof. From the hypothesis we see that $\text{map}_0(X, Y)$ is path connected. Let $f: (S^j, *) \to (\text{map}_0(X, Y), c)$ be a map where c denotes the constant map. Then we have its associated map $\tilde{f}: (S^j, *) \times (X, x_0) \to (Y, y_0)$ and the map $\bar{f}: (X, x_0) \to (\Omega^j(Y), c')$ associated with $\tilde{f}$ where c' is the constant map. Obviously $\bar{f}$ is homotopic to the constant map. This implies that f is homotopically trivial.

Let E and B be CW complexes, and let $p: E \to B$ be a fibration with fibre F. Then, we see that F has the homotopy type of a CW complex (not necessary connected) [11, 14]. Moreover the inclusion $i: F = p^{-1}(b_0) \to E$ is a closed cofibration by the theorem of A. Strøm [17]. In Theorem 1.1, putting $X = E$, $A = F$, we have the following fibration

$$\rho: \text{map}_0(E, E) \longrightarrow \text{map}_0(E, B) \times' \text{map}_0(F, E) .$$

About the fibred product $\text{map}_0(E, B) \times' \text{map}_0(F, E)$, we have

LEMMA 1.4. *Let E and B be CW complexes and let $p: E \to B$ be a fibration with fibre F. For a given $n > 1$, if F is $(n-1)$-connected and $\pi_i(B) = 0$ for every $i \geqq n$, then the fibred product* $\text{map}_0(E, B) \times' \text{map}_0(F, E)$ *has the same weak homotopy type as* $\text{map}_0(B, B) \times \text{map}_0(F, F)$.

Proof. First, we shall show that the fibred product $\text{map}_0(E, B) \times' \text{map}_0(F, E)$

has the same weak homotopy type as $\mathrm{map}_0(E, B)\times\mathrm{map}_0(F, F)$. Second, we shall show that $\mathrm{map}_0(E, B)$ has the same weak homotopy type as $\mathrm{map}_0(B, B)$.

Let $p_{\#}: \mathrm{map}_0(F, E)\rightarrow\mathrm{map}_0(F, B)$ be a fibration defined by $p_{\#}(f)=p\circ f$. By Lemma 1.3 $\mathrm{map}_0(F, B)$ is weakly contractible. Consequently we have a fibration

$$\mathrm{map}_0(F, F)\longrightarrow\mathrm{map}_0(F, E)\xrightarrow{p_{\#}}\mathrm{map}_0(F, B),$$

and so $\mathrm{map}_0(F, F)$ is weakly homotopy equivalent to $\mathrm{map}_0(F, E)$.

On the other hand, let $i^{\#}: \mathrm{map}_0(E, B)\rightarrow\mathrm{map}_0(F, B)$ be a fibration defined by $i^{\#}(f)=f\circ i$ where $i: F\rightarrow E$ is the inclusion, then the fibred product $\mathrm{map}_0(E, B)\times'\mathrm{map}_0(F, E)$ is the fibre space $(i^{\#})^{*}\mathrm{map}_0(F, E)$ of induced fibration by $i^{\#}$. Using Lemma 1.2, we have

$$\mathrm{map}_0(E, B)\times'\mathrm{map}_0(F, E)\underset{w}{\simeq}\mathrm{map}_0(E, B)\times\mathrm{map}_0(F, F).$$

Next we shall show

$$\mathrm{map}_0(E, B)\underset{w}{\simeq}\mathrm{map}_0(B, B).$$

Now, by our hypothesis we may assume that E is a subcomplex of B and the n-skeleton of B is contained in E. Then we have a fibration $p^{\#}: \mathrm{map}_0(B, B)\rightarrow\mathrm{map}_0(E, B)$ which is a map induced by the inclusion $p: E\rightarrow B$.

Let $f: S^i\times(E, e_0)\rightarrow(B, b_0)$ be a map such that $f|*\times E=p$. Since $\pi_j(B)=0$ for every $j\geqq n$, by using obstruction theory f can be extended to a map $f': S^i\times(B, b_0)\rightarrow(B, b_0)$ such that $f'|*\times B=id_B$. This implies that

$$(p^{\#})_*: \pi_i(\mathrm{map}_0(B, B), id_B)\longrightarrow\pi_i(\mathrm{map}_0(E, B), p)$$

is surjective for every $i>0$. Similarly, we can see that $(p^{\#})_*$ is injective for every $i>0$. Hence

$$p^{\#}: \mathrm{map}_0(B, B; id_B)\longrightarrow\mathrm{map}_0(E, B; p)$$

is a weak homotopy equivalence. Also, we see that $p^{\#}: \mathrm{map}_0(B, B)\rightarrow\mathrm{map}_0(E, B)$ induces a bijective correspondence of the path connected components of $\mathrm{map}_0(B, B)$ onto the path connected components of $\mathrm{map}_0(E, B)$.

Furthermore, by the similar argument we see that $p^{\#}$ is a weak homotopy equivalence for each path connected component of $\mathrm{map}_0(B, B)$. We have

$$\mathrm{map}_0(E, B)\underset{w}{\simeq}\mathrm{map}_0(B, B).$$

Thus our proof is completed.

Let X be a CW complex. Then we denote by $G(X)$ and $G_0(X)$ the space of self homotopy equivalences of X and the space of self homotopy equivalences of (X, x_0), respectively.

Under the same hypothesis as Lemma 1.4, we have

THEOREM 1.5. *Let E and B be CW complexes and let $p: E \to B$ be a fibration with fibre F. For a given $n>1$, if F is $(n-1)$-connected and $\pi_i(B)=0$ for every $i \geqq n$, then we have the following fibration*:

$$\mathcal{Q}(E \bmod F) \longrightarrow G_0(E) \xrightarrow{\rho} B',$$

where B' is a subspace of $\mathrm{map}_0(E, B) \times' \mathrm{map}_0(F, E)$ with the same weak homotopy type as $G_0(B) \times G_0(F)$ and $\mathcal{Q}(E \bmod F)$ is the space of self fibre homotopy equivalences of E leaving the fibre F fixed.

Proof. Let f be an element of $G_0(E)$. Then by the above argument there exists a map $g: (B, b_0) \to (B, b_0)$ such that $p \circ f \simeq g \circ p$ rel e_0. Consequently the following commutative diagram holds,

$$\begin{array}{ccc}
\pi_j(E) & \xrightarrow{f_*} & \pi_j(E) \\
\downarrow p_* & & \downarrow p_* \\
\pi_j(B) & \xrightarrow{g_*} & \pi_j(B)
\end{array}$$

On the other hand, by our assumption we see that $p_*: \pi_j(E) \to \pi_j(B)$ is an isomorphism for every $j<n$. Since f is a self homotopy equivalence of (E, e_0), $f_*: \pi_j(E) \to \pi_j(E)$ is an automorphism for every j. This implies that $g_*: \pi_j(B) \to \pi_j(B)$ is an automorphism for every $j<n$. Since B is a CW complex, by the theorem of J. H. C. Whitehead we obtain that g is a self homotopy equivalence of (B, b_0). Thus g belongs to $G_0(B)$.

On the other hand, there exists a self map f' of (E, e_0) such that $f' \simeq f$ rel e_0 and $p \circ f' = g \circ p$, because $p: E \to B$ is a fibration and $p \circ f$ is homotopic to $g \circ p$ relative to e_0. Put $f'|F = h$, then we have the following commutative diagram with exact rows,

$$\begin{array}{ccccccc}
\pi_{j+1}(B) & \xrightarrow{\Delta} & \pi_j(F) & \xrightarrow{i_*} & \pi_j(E) & \xrightarrow{p_*} & \pi_j(B) \\
\downarrow g_* & & \downarrow h_* & & \downarrow f'_* & & \downarrow g_* \\
\pi_{j+1}(B) & \xrightarrow{\Delta} & \pi_j(F) & \xrightarrow{i_*} & \pi_j(E) & \xrightarrow{p_*} & \pi_j(B)
\end{array}$$

Note that by our assumption for every $j \geqq n$, $i_*: \pi_j(F) \to \pi_j(E)$ is an isomorphism and $\pi_j(B) = \pi_{j+1}(B) = 0$ and $f'_*: \pi_j(E) \to \pi_j(E)$ is an automorphism for every j, thus $h_*: \pi_j(F) \to \pi_j(F)$ is an automorphism for every $j \geqq n$. Therefore h is a self homotopy equivalence of (F, e_0).

From the argument of Lemma 1.4, there is a subspace B' of $\mathrm{map}_0(E, B) \times'$ $\mathrm{map}_0(F, E)$ which has the same weak homotopy type as $G_0(B) \times G_0(F)$ $(\subset \mathrm{map}_0(B, B) \times \mathrm{map}_0(F, F))$. Obviously we see from the definition of ρ that a typical fibre $\rho^{-1}(p, i)$ is just $\mathcal{G}(E \bmod F)$. Finally we have a fibration:

$$\mathcal{G}(E \bmod F) \longrightarrow G_0(E) \stackrel{\rho}{\longrightarrow} B' \underset{w}{\simeq} G_0(B) \times G_0(F).$$

§ 2. Fibration map theory.

Here we give a brief summary on a fibration map theory for fibrations which corresponds to the bundle map theory for principal bundles [3, 4, 5, 6].

Let $p: E \to B$ and $p': E' \to B'$ be fibrations with CW complexes B and B'. Let $\tilde{f}: E \to E'$ and $f: B \to B'$ be maps such that $p' \circ \tilde{f} = f \circ p$ and $\tilde{f}$ carries each fibre of E into a fibre of E' by a homotopy equivalence. Then we call $\tilde{f}$ a fibration map.

Let $\mathcal{G}^*(E, E')$ be the space of fibration maps of E to E' and $\Phi: \mathcal{G}^*(E, E') \to \mathrm{map}(B, B')$ be a map defined by $\Phi(\tilde{f}) = f$ for each fibration map $\tilde{f}: E \to E'$ and its induced map $f: B \to B'$. Then we have a fibration

$$\Phi: \mathcal{G}^*(E, E') \longrightarrow \mathrm{map}(B, B').$$

Moreover, let B_0 be a subcomplex of B and put $p^{-1}(B_0) = E_0$. Let $\tilde{i}: E_0 \to E$ be the inclusion then $\tilde{i}^\#: \mathcal{G}^*(E, E') \to \mathcal{G}^*(E_0, E')$ is a fibration. Let $\tilde{\alpha}: E_0 \to E'$ be a fixed fibration map which is extendable to a fibration map of E to E' and induces a map $\alpha: B_0 \to B$. Denote by $\mathcal{G}^*_{\tilde{\alpha}}(E, E')$ a fibre $(\tilde{i}^\#)^{-1}(\tilde{\alpha})$. Then we have a fibration $\Phi: \mathcal{G}^*_{\tilde{\alpha}}(E, E') \to \mathrm{map}_\alpha(B, B')$, where $\mathrm{map}_\alpha(B, B')$ denotes the space of maps from B to B' whose restriction on B_0 is the map α. Furthermore, we denote by $\mathcal{G}(E)$ the space of self fibre homotopy equivalences of E. Then every fibre $\Phi^{-1}(h)$ over h in the fibration $\Phi: \mathcal{G}^*(E, E') \to \mathrm{map}(B, B')$ has the same homotopy type as $\mathcal{G}(E)$. Also we have a fibration: $\mathcal{G}(E \bmod E_0) \to \mathcal{G}(E) \stackrel{\tilde{i}^\#}{\longrightarrow} \mathcal{G}(E_0)$, where $\mathcal{G}(E \bmod E_0)$ denotes the typical fibre $(\tilde{i}^\#)^{-1}(id_{E_0})$.

Now, let $p: E \to B$ be a fibration with fibre F where B and F are CW complexes. Then there exists a universal fibration $p_\infty: E_\infty \to B_\infty$ with fibre F, where B_∞ may be regarded as a classifying space $BG(F)$ ([1], [7]). About this universal fibration we have the following Gottlieb's theorem which corresponds to Theorem (5.6) in [5].

THEOREM 2.1. *Let $\tilde{k}: E \to E_\infty$ be a fibration map inducing a classifying map $k: B \to B_\infty$ for the fibration: $F \to E \stackrel{p}{\to} B$. Then the path connected component $\mathcal{G}^*_{\tilde{k} \circ \tilde{i}}(E, E_\infty; \tilde{k})$ in $\mathcal{G}^*_{\tilde{k} \circ \tilde{i}}(E, E_\infty)$ containing $\tilde{k}$ is weakly contractible.*

Immediately we have the following

COROLLARY 2.2. *$\mathcal{G}(E \bmod E_0)$ has the same weak homotopy type as the loop space $\Omega\, \mathrm{map}_{k \circ i}(B, B_\infty; k)$, where i denotes the inclusion of B_0 into B.*

Remark 2.3. The fibration map theory stated above may be said that it is a replica of the bundle map theory initiated by I. M. James [6] and developed by D. H. Gottlieb [4, 5]. It seems that our theory has been known among experts but not appeared in the literatures. Note that the work of P. Booth, P. Heath, C. Morgan and R. Piccinini [2] is studying the same subject with a different approach.

Now, it should be noticed that for a CW complex B if

(1) B_∞ is a homotopy associative H-space, or

(2) B is a suspension of a CW complex,

then $\Omega\,\mathrm{map}_0(B, B_\infty; k)$ has the same homotopy type as $\mathrm{map}_0(B, \Omega B_\infty)$. Thus, furthermore if F has the homotopy type of a CW complex, then we have

$$\mathcal{G}(E \bmod F) \underset{w}{\simeq} \mathrm{map}_0(B, \Omega B_\infty) \underset{w}{\simeq} \mathrm{map}_0(B, G(F)).$$

Especially when F is an Eilenberg-MacLane complex $K(\pi, n)$ $(n>1)$ we have the following

THEOREM 2.4. *Let $p: E\to B$ be a fibration with fibre $F=K(\pi, n)$ $(n>1)$ such that B is a CW complex. Assume B is simply connected, then $\mathcal{G}(E \bmod F)$ has the same weak homotopy type as*

$$\mathrm{map}_0(B, K(\pi, n)) \underset{w}{\simeq} H^n(B, \pi)\times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi), i).$$

Proof. Let $SG(F)$ be the path connected component of id_F in $G(F)$. Since $G(F)$ is a grouplike topological monoid and has the same weak homotopy type as $K(\pi, n)\times \mathrm{Aut}(\pi)$, where $\mathrm{Aut}(\pi)$ denotes the group of automorphisms of π, then we have the following fibration [7]:

$$BSG(F) \xrightarrow{Bj} BG(F) \xrightarrow{\lambda} K(\mathrm{Aut}(\pi), 1).$$

Here Bj may be regarded as the map between classifying spaces which is induced by the inclusion $j: SG(F)\to G(F)$.

Because B is a simply connected CW complex, we can easily see that $\mathrm{map}_0(B, K(\mathrm{Aut}(\pi), 1))$ is weakly contractible. Thus we have the following fibration:

$$\mathrm{map}_0(B, BSG(F); k') \xrightarrow{(Bj)_\#} \mathrm{map}_0(B, BG(F); k) \xrightarrow{\lambda_\#} \mathrm{map}_0(B, K(\mathrm{Aut}(\pi), 1)),$$

where $[Bj\circ k']=[k]$. Consequently we have

$$\mathrm{map}_0(B, BSG(F); k') \underset{w}{\simeq} \mathrm{map}_0(B, BG(F); k).$$

Note that $BSG(F)=K(\pi, n+1)$ is a homotopy associative H-space, then we have by Corollary 2.2

$$\begin{aligned}\mathcal{G}(E \bmod F) &\underset{w}{\simeq} \Omega\,\mathrm{map}_0(B, BG(F)\,;\, k)\\ &\underset{w}{\simeq} \Omega\,\mathrm{map}_0(B, BSG(F)\,;\, k')\\ &\underset{w}{\simeq} \Omega\,\mathrm{map}_0(B, BSG(F)\,;\, c)\\ &\underset{w}{\simeq} \mathrm{map}_0(B, \Omega BSG(F))\\ &\underset{w}{\simeq} \mathrm{map}_0(B, K(\pi, n))\end{aligned}$$

(for the last weak homotopy equivalence, see [7]),

where c is the constant map of B to $BSG(F)$. By the theorem of J. C. Moore [8] it holds that

$$\mathrm{map}_0(B, K(\pi, n)) \underset{w}{\simeq} H^n(B, \pi) \times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi), i)\,.$$

Thus our proof is completed.

§ 3. Applications.

In Theorem 1.5, we have essentially the following fibration:

$$\mathcal{G}(E \bmod F) \longrightarrow G_0(E) \xrightarrow{\ \rho\ } G_0(B) \times G_0(F)\,.$$

Hereafter we shall investigate the image of ρ.

For this purpose we shall recall Allaud's theory on the classification of fibre spaces [1, 7]. Let B be a space with base point b_0. Then we write (ξ, i) the following fibration:

$$F \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B\,,$$

where $i: F \to p^{-1}(b_0)$ is a homotopy equivalence and the fibres are of the same homotopy type. Let (ξ', i') be another fibration:

$$F' \xrightarrow{\ i'\ } E' \xrightarrow{\ p'\ } B'\,.$$

Define a map of (ξ, i) to (ξ', i') as a triple

$$g: F \longrightarrow F', \qquad \tilde{f}: E \longrightarrow E', \qquad f: (B, b_0) \longrightarrow (B', b_0')$$

such that in the following diagram:

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ F' & \xrightarrow{i'} & E' & \xrightarrow{p'} & B' \end{array}$$

the square on the left is homotopy commutative as maps to the fibre $p'^{-1}(b_0')$ and the square on the right is commutative. In the sequel the commutativity of this kind of diagram will be called briefly as semi-commutativity.

Now, let F be a CW complex (not necessary connected). Suppose given two fibrations (ξ, i) and (ξ', i') over CW complexes B and B' with the same fibre F

$$F \xrightarrow{i} E \xrightarrow{p} B\,,$$

$$F \xrightarrow{i'} E' \xrightarrow{p'} B'\,,$$

there exists a map (h, f, g) of (ξ, i) to (ξ', i') such that h is a self homotopy equivalence of F.

We denote the induced fibration $(g^*\xi', i_1)$ as follows:

$$(g^*\xi', i_1): F \xrightarrow{i_1} g^*E' \xrightarrow{\pi_1} B\,.$$

Let $f': E \to g^*E'$ be a fibre homotopy equivalence given by $f'(e)=(p(e), f(e))$ for $e \in E$. Then we have the following semi-commutative diagram:

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f'} & & \| \\ F & \xrightarrow{i_1} & g^*E' & \xrightarrow{\pi_1} & B \\ \| & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle g} \\ F & \xrightarrow{i'} & E' & \xrightarrow{p'} & B'\,. \end{array}$$

This implies that the fibration

$$(\xi, i\circ h^{-1}): F \xrightarrow{i\circ h^{-1}} E \xrightarrow{p} B$$

is equivalent to the fibration $(g^*\xi', i_1)$.

On the other hand, by the main theorem of G. Allaud [1] there exists a map $\chi_E(h): (B, b_0) \to (B_\infty, b_\infty)$ corresponding to the fibration $(\xi, i\circ h^{-1})$. Obviously the homotopy class $[\chi_E(h)]$ is only dependent on the homotopy class $[h]$. Denote by (ξ_∞, i_∞) the universal fibration:

$$F \xrightarrow{i_\infty} E_\infty \xrightarrow{p_\infty} B_\infty ,$$

we have also a self map $\chi_\infty(h)$ of (B_∞, b_∞) corresponding to the fibration

$$(\xi_\infty, i_\infty \circ h^{-1}) : F \xrightarrow{i_\infty \circ h^{-1}} E_\infty \xrightarrow{p_\infty} B_\infty .$$

About map $\chi_\infty(h)$, we can get

$$[\chi_\infty(h' \circ h)] = [\chi_\infty(h')] \circ [\chi_\infty(h)] \qquad (h,\ h' \in G(F))$$

by using the following semi-commutative diagram:

$$\begin{array}{ccccc}
F & \xrightarrow{i_\infty \circ h^{-1} \circ h'^{-1}} & E_\infty & \xrightarrow{p_\infty} & B_\infty \\
\| & & \downarrow \tilde{\chi}_\infty(h) & & \downarrow \chi_\infty(h) \\
F & \xrightarrow{i_\infty \circ h'^{-1}} & E_\infty & \xrightarrow{p_\infty} & B_\infty \\
\| & & \downarrow \tilde{\chi}_\infty(h') & & \downarrow \chi_\infty(h') \\
F & \xrightarrow{i_\infty} & E_\infty & \xrightarrow{p_\infty} & B_\infty .
\end{array}$$

Consider the following semi-commutative diagram:

$$\begin{array}{ccccc}
F & \xrightarrow{i \circ h^{-1}} & E & \xrightarrow{p} & B \\
\| & & \downarrow \tilde{k} & & \downarrow k \\
F & \xrightarrow{i_\infty \circ h^{-1}} & E_\infty & \xrightarrow{p_\infty} & B_\infty \\
\| & & \downarrow \tilde{\chi}_\infty(h) & & \downarrow \chi_\infty(h) \\
F & \xrightarrow{i_\infty} & E_\infty & \xrightarrow{p_\infty} & B_\infty
\end{array}$$

where $k : (B, b_0) \to (B_\infty, b_\infty)$ is a map corresponding to the fibration (ξ, i). This implies that the map $\chi_\infty(h) \circ k : (B, b_0) \to (B_\infty, b_\infty)$ is a corresponding map to the fibration $(\xi, i \circ h^{-1})$. When $k' : (B', b_0') \to (B_\infty, b_\infty)$ is a map corresponding to the fibration (ξ', i'), clearly we get a map $k' \circ g : (B, b_0) \to (B_\infty, b_\infty)$ corresponding to the fibration $(g^*\xi', i_1)$ which is equivalent to the fibration $(\xi, i \circ h^{-1})$. Consequently by the main theorem of G. Allaud again we have

$$[\chi_\infty(h)] \circ [k] = [k'] \circ [g] .$$

One can reverse the above argument. Thus we can obtain the following

THEOREM 3.1. *Let F be a CW complex (not necessary connected). And let $(\xi, i): F \xrightarrow{i} E \xrightarrow{p} B$ and $(\xi', i'): F \xrightarrow{i'} E' \xrightarrow{p'} B'$ be two fibrations over CW complexes B and B' respectively. For given elements g of* $\mathrm{map}_0(B, B')$ *and h of $G(F)$ there exists a fibration* map *$\tilde{g}: E \to E'$ such that the following diagram is semi-commutative:*

$$\begin{array}{ccccc} F & \xrightarrow{\quad i \quad} & E & \xrightarrow{\quad p \quad} & B \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle g} \\ F & \xrightarrow{\quad i' \quad} & E' & \xrightarrow{\quad p' \quad} & B' \end{array}$$

(the square on the left is homotopy commutative as maps to the fibre $p'^{-1}(b_0')$ and the square on the right is commutative) if and only if it holds that

$$[\chi_\infty(h)]\circ[k]=[k']\circ[g],$$

where maps $k:(B, b_0)\to(B_\infty, b_\infty)$, $k':(B', b_0')\to(B_\infty, b_\infty)$ are corresponding ones to the fibrations (ξ, i) and (ξ', i') respectively and $\chi_\infty(h)$ is a self homotopy equivalence of (B_∞, b_∞).

If a fibre F is simply connected, then this theorem yields the following result.

THEOREM 3.2. *Under the same hypothesis as in Theorem* 1.5, *the image of $\rho: G_0(E)\to G_0(B)\times G_0(F)$ is just the union of the path connected components in $G_0(B)\times G_0(F)$ each of which contains (g, h) satisfying*

$$[\chi_\infty(h)]\circ[k]=[k]\circ[g],$$

where $k:(B, b_0)\to(B_\infty, b_\infty)$ is a map corresponding to the fibration: $F \xrightarrow{i} E \xrightarrow{p} B$.

About the map $\chi_\infty(h)$, when a fibre F is a complex $K(\pi, n)$ $(n>1)$, we have the following

PROPOSITION 3.3. *Let F be $K(\pi, n)$ $(n>1)$ and let*

$$F \xrightarrow{i_\infty} E_\infty \xrightarrow{p_\infty} B_\infty = BG(F)$$

be the universal fibration with typical fibre F. Then for a given homotopy equivalence $h: F\to F$ we have

$$[\chi_\infty(h)\circ Bj]=[Bj\circ h'],$$

where j is the inclusion of $SG(F)$ to $G(F)$, Bj is the map of $(BSG(F), b'_\infty)$ to $(BG(F), b_\infty)$ and h' is a self map of $(BSG(F), b'_\infty)$ with $[h']=[h]\in \mathrm{Aut}(\pi')$.

Proof. Let E'_∞ denote the fibre space $(Bj)^*E_\infty$ of the induced fibration by $Bj:(B'_\infty, b'_\infty)\to(B_\infty, b_\infty)$, where B'_∞ is the classifying space $BSG(F)$. First we shall show that E'_∞ is contractible.

Now, since F is $K(\pi, n)$ $(n>1)$, the boundary homomorphism $\partial_\infty: \pi_{i+1}(B_\infty) \to\pi_i(F)$ in the homotopy sequence of the universal fibration is an isomorphism if $i\geqq 2$ and a monomorphism if $i=1$ ([1]). So, $\partial_\infty: \pi_{i+1}(B_\infty)\to\pi_i(F)$ is an isomorphism for every $i\geqq 1$. Let us consider the homotopy sequence of the induced fibration $p'_\infty: E'_\infty\to B'_\infty$. Let ∂'_∞ be the corresponding boundary homomorphism, then we have a commutative diagram:

$$\begin{array}{ccc} \pi_{i+1}(B'_\infty) & \xrightarrow{(Bj)_*} & \pi_{i+1}(B_\infty) \\ & {\scriptstyle \partial'_\infty}\searrow \quad \swarrow{\scriptstyle \partial_\infty} & \\ & \pi_i(F) & \end{array}$$

Consequently we see that $\partial'_\infty: \pi_{i+1}(B'_\infty)\to\pi_i(F)$ is an isomorphism for every $i\geqq 1$. This implies that $\pi_i(E'_\infty)$ is trivial for every $i\geqq 0$.

On the other hand, B'_∞ is a CW complex and each fibre of the fibration $p'_\infty: E'_\infty\to B'_\infty$ has the same homotopy type as F which is a CW complex. By the theorem of Stasheff [11, 14] E'_∞ has the same homotopy type as a CW complex. Therefore E'_∞ is contractible.

Now, we have the following semi-commutative diagram:

$$\begin{array}{ccccc} F & \xrightarrow{i'_\infty\circ h^{-1}} & E'_\infty & \xrightarrow{p'_\infty} & B'_\infty \\ \| & & \downarrow{\scriptstyle \tilde{h}'} & & \downarrow{\scriptstyle h'} \\ F & \xrightarrow{i'_\infty} & E'_\infty & \xrightarrow{p'_\infty} & B'_\infty \\ \| & & \downarrow{\scriptstyle \widetilde{Bj}} & & \downarrow{\scriptstyle Bj} \\ F & \xrightarrow{i_\infty} & E_\infty & \xrightarrow{p_\infty} & B_\infty . \end{array}$$

This implies

$$[\chi_{E'_\infty}(h)]=[Bj\circ h']$$

and $[h]=[h']\in\mathrm{Aut}(\pi')$ when we regard the homotopy classes $[h]$ and $[h']$ as element of $\mathrm{Aut}(\pi')$. On the other hand, we have $[\chi_{E'_\infty}(h)]=[\chi_\infty(h)\circ Bj]$ because $Bj:(B'_\infty, b'_\infty)\to(B_\infty, b_\infty)$ is the corresponding map of the fibration

$$(\xi'_\infty, i'_\infty): F \xrightarrow{i'_\infty} E'_\infty \xrightarrow{p'_\infty} B'_\infty .$$

Thus we have $[Bj\circ h'] = [\chi_\infty(h)\circ Bj]$.

Now, using Proposition 3.3 combined with Theorem 1.5, 2.4 and 3.2, we obtain the following

THEOREM 3.4. *For given* $1<m<n$, *let*

$$F=K(\pi', n) \xrightarrow{i} E \xrightarrow{p} K(\pi, m)=B$$

be a fibration with a corresponding map $k:(B, b_0)\to(B_\infty, b_\infty)$. *Then there exists a map* $k':(B, b_0)\to(B'_\infty, b'_\infty)$ *such that* $[Bj\circ k']=[k]$. *And we have*

$$G_0(E) \underset{w}{\simeq} R\times H^n(B, \pi')\times\prod_{i=1}^{n-1} K(H^{n-i}(B, \pi'), i),$$

where R is the subgroup of $\mathrm{Aut}(\pi)\times\mathrm{Aut}(\pi')=\varepsilon(B)\times\varepsilon(F)$ *consisting of* $([g], [h])$ *with*

$$h_*([k'])=g^*([k']).$$

Here $[k']$ *is regarded as an element of* $H^{n+1}(B, \pi')$, g^* *and* h_* *are the automorphisms of* $H^{n+1}(B, \pi')$ *induced by g and h respectively.*

Proof. Since B is a simply connected CW complex, there exists a map $k':(B, b_0)\to(B'_\infty, b'_\infty)$ such that $[Bj\circ k']=[k]$.

Moreover, note that the following equalities hold [18, 19]:

$$G(B) \underset{w}{\simeq} \mathrm{Aut}(\pi)\times K(\pi, m), \qquad G_0(B) \underset{w}{\simeq} \mathrm{Aut}(\pi),$$

$$G(F) \underset{w}{\simeq} \mathrm{Aut}(\pi')\times K(\pi', n), \qquad G_0(F) \underset{w}{\simeq} \mathrm{Aut}(\pi').$$

By Theorem 1.5, 2.4 and 3.2 our proof is completed if we see that $[\chi_\infty(h)\circ k]=[k\circ g]$ is equivalent to

$$[h'\circ k']=[k'\circ g].$$

By Proposition 3.3 we have

$$\begin{aligned}[\chi_\infty(h)\circ k] &= [\chi_\infty(h)\circ Bj\circ k'] \\ &= [Bj\circ h'\circ k'] \\ &= [k\circ g] \\ &= [Bj\circ k'\circ g].\end{aligned}$$

Since the correspondence $(Bj)_*$ between based homotopy classes: $[B, B'_\infty]_0\to[B, B_\infty]_0$ is bijective, we conclude that $[\chi_\infty(h)\circ k]=[k\circ g]$ is equivalent to

$$[h'\circ k']=[k'\circ g].$$

Now, we must note the work of J. Siegel [12] where he studied the space $BG_0(E)$ under the same situation as the above theorem.

Moreover, note that the map $\rho: G_0(E)\to G_0(B)\times G_0(F)$ defined in Theorem 1.5 induces the homomorphism ρ_* of $\varepsilon(E)$ into $\varepsilon(B)\times\varepsilon(F)$, then we can easily see that the image of ρ_* is just R in Theorem 3.4 and the kernel of ρ_* may be regarded as $H^n(B, \pi')$. Thus as a corollary of Theorem 3.4 we have the following theorem proved by W. Shih [10] and Y. Nomura [9].

COROLLARY 3.5. *Under the same hypothesis as Theorem* 3.4, *there exists the following exact sequence*

$$1 \longrightarrow H^n(B, \pi') \longrightarrow \varepsilon(E) \longrightarrow R \longrightarrow 1,$$

where R is the same group as the group stated in Theorem 3.4.

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