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ON THE SPACES OF SELF HOMOTOPY EQUIVALENCES FOR FIBRE SPACES

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Introduction.

This paper contains a detailed account of the results announced in [21][†].

Let X be a connected CW complex with non-degenerate base point x_0 . In the following, by a CW complex we mean a connected CW complex with nondegenerate base point, unless otherwise stated. Denote by G(X) the space of self homotopy equivalences of X and $G_0(X)$ the space of self homotopy equivalences of (X, x_0) . In [19, 20], the author studied G(X) and $G_0(X)$ when X are certain product CW complexes. The main theme of this paper is to study $G_0(X)$ when X is a fibre space of a Hurewicz fibration: $F \xrightarrow{i} X \xrightarrow{p} B$. We call a Hurewicz fibration simply a fibration.

The first main result is the following:

THEOREM 1.5. Let E and B be CW complexes and let $p: E \rightarrow B$ be a fibration with fibre F. For a given n > 1, if F is (n-1)-connected and $\pi_i(B)=0$ for every $i \ge n$, then we have the following fibration:

$$\mathcal{G}(E \mod F) \longrightarrow G_0(E) \stackrel{\rho}{\longrightarrow} B'$$
,

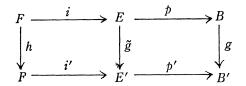
where B' is a space with the same weak homotopy type as $G_0(B) \times G_0(F)$ and $\mathcal{Q}(E \mod F)$ is the space of self fibre homotopy equivalences of E leaving the fibre F fixed.

For seeking the image of ρ in Theorem 1.5 we provide the following theorem by using Allaud's theory on the classification of fibre spaces [1].

THEOREM 3.1. Let F be a CW complex (not necessary connected). And let $(\xi, i): F \xrightarrow{i} E \xrightarrow{p} B$ and $(\xi', i'): F \xrightarrow{i'} E' \xrightarrow{p'} B'$ be two fibrations over CW complexes B and B' respectively. For given elements g of map_o(B, B') and h of G(F) there exists a fibration map $\tilde{g}: E \to E'$ such that the following diagram is semi-commutative:

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[†] A continuation of this paper was published as [22, 23], in which this paper is referred to as "On the spaces of self homotopy equivalences for fibre spaces I".



(the square on the left is homotopy commutative as maps to the fibre $p'^{-1}(b'_0)$ and the square on the right is commutative) if and only if we have

$$[\boldsymbol{\chi}_{\infty}(h)] \cdot [k] = [k'] \cdot [g],$$

where maps $k: (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$, $k': (B', b'_0) \rightarrow (B_{\infty}, b_{\infty})$ are corresponding maps to the fibrations (ξ, i) and (ξ', i') respectively and $\chi_{\infty}(h)$ is a self homotopy equivalence of (B_{∞}, b_{∞}) .

J. Siegel [12] studied the space $BG_0(E)$ where E is a CW complex of stable 2-stage Postnikov system. With respect to a CW complex E of general 2-stage Postnikov system, we have

THEOREM 3.4. For given 1 < m < n, let

$$F = K(\pi', n) \xrightarrow{i} E \xrightarrow{p} K(\pi, m) = B$$

be a fibration with a corresponding map $k: (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$. Then there exists a map $k': (B, b_0) \rightarrow (B'_{\infty}, b'_{\infty})$ such that $[Bj \circ k'] = [k]$. And we have

$$G_0(E) \underset{w}{\sim} R \times H^n(B, \pi') \times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi'), i),$$

where R is the subgroup of Aut $(\pi) \times Aut(\pi') = \varepsilon(B) \times \varepsilon(F)$ consisting of ([g], [h]) with

$$h_*([k'])=g^*([k']).$$

Here [k'] is regarded as an element of $H^{n+1}(B, \pi')$, g^* and h_* are the automorphisms of $H^{n+1}(B, \pi')$ induced by g and h respectively.

As is easily seen, a corollary of this is the following theorem proved by W. Shih [10] and Y. Nomura [9].

COROLLARY 3.5. Under the same hypothesis as Theorem 3.4, there exists the following exact sequence

$$1 \longrightarrow H^n(B, \pi') \longrightarrow \varepsilon(E) \longrightarrow R \longrightarrow 1$$
,

where $\varepsilon(E)$ is the group of homotopy classes of self homotopy equivalences of (E, e_0) and R is the same group as the group stated in Theorem 3.4.

§1. The function spaces and fibrations.

Throughout this paper, we shall work within the category of compactly generated Hausdorff spaces [15]. Let X and Y be spaces with non-degenerate base points. In the sequel by a base point we mean a non-degenerate base point. Then map (X, Y) will denote the space of maps of X to Y with the topology obtained by retopologizing the compact open topology and map₀(X, Y) will be the subspace of map (X, Y) of maps of X to Y preserving base points. Moreover, when k is a map of X to Y, we denote by map (X, Y; k) the path connected component of k in map (X, Y), and map₀(X, Y; k) is defined similarly.

In the following we simply call a Hurewicz fibration $p: E \rightarrow B$ a fibration $p: E \rightarrow B$ and write a fibration $p: E \rightarrow B$ with typical fibre F by a fibration

$$F \xrightarrow{i} E \xrightarrow{p} B$$
,

where *i* is a homotopy equivalence of *F* to the fibre $p^{-1}(b_0)$ over a base point b_0 of *B*.

Let $p: E \rightarrow B$ be a fibration. And let $i: A \rightarrow X$ be a closed cofibration. Then we have a fibration

$$p_{\#}: \operatorname{map}_{0}(A, E) \longrightarrow \operatorname{map}_{0}(A, B),$$

where $p_{\#}$ is induced by the projection $p: (E, e_0) \rightarrow (B, b_0)$. Also we have a fibration

$$i^*: \operatorname{map}_0(X, B) \longrightarrow \operatorname{map}_0(A, B)$$

where i^* is induced by the inclusion $i: (A, x_0) \rightarrow (X, x_0)$. We denote by $\operatorname{map}_0(X, B) \times \operatorname{map}_0(A, E)$ the fibred product of the fibration i^* and p_* . Then we define a map $\rho: \operatorname{map}_0(X, E) \rightarrow \operatorname{map}_0(X, B) \times \operatorname{map}_0(A, E)$ by

$$\rho(f) = (p_{\#}(f), i^{\#}(f)) = (p \circ f, f \circ i).$$

Then we have the following theorem [13, 16, 17].

THEOREM 1.1. Let $i: A \rightarrow X$ be a closed cofibration preserving base points. And let $p: E \rightarrow B$ be a fibration preserving base points. Then a map

$$\rho: \operatorname{map}_{0}(X, E) \longrightarrow \operatorname{map}_{0}(X, B) \times \operatorname{map}_{0}(A, E)$$

is a fibration.

Moreover we have the following lemmas which will be used later.

LEMMA 1.2. Let $p: E \rightarrow B$ be a fibration with fibre F and let $f: X \rightarrow B$ be a map. Assume that B is weakly contractible, then the fibre space f^*E of induced fibration by f has the same weak homotopy type as $X \times F$.

Proof. We may assume without loss of generality that both X and E are

path connected.

Let $i: F \to E$ and $j: F \to f^*E$ be the inclusion maps, and let $p_1: f^*E \to X$ and $p_2: f^*E \to E$ be the projections. Then we have the following commutative diagram

Since $\pi_n(B)$ is trivial for every n, $i_*: \pi_n(F) \rightarrow \pi_n(E)$ is an isomorphism for every n. Therefore $j_*: \pi_n(F) \rightarrow \pi_n(f^*E)$ is a monomorphism for every n and $p_{2*}: \pi_n(f^*E) \rightarrow \pi_n(E)$ is an epimorphism. Using these facts, we can easily see that the map $(p_1, p_2): f^*E \rightarrow X \times E$ induces isomorphisms of the corresponding homotopy groups. That is, f^*E is weakly homotopy equivalent to $X \times E$. Obviously $X \times F$ is weakly homotopy equivalent to $X \times E$. Thus f^*E has the same weak homotopy type as $X \times F$.

LEMMA 1.3. For given n > 1, let X be an (n-1)-connected CW complex and Y be a path connected space with $\pi_i(Y)=0$ for every $i \ge n$. Then $\operatorname{map}_0(X, Y)$ is weakly contractible.

Proof. From the hypothesis we see that $\operatorname{map}_0(X, Y)$ is path connected. Let $f: (S^j, *) \to (\operatorname{map}_0(X, Y), c)$ be a map where c denotes the constant map. Then we have its associated map $\tilde{f}: (S^j, *) \times (X, x_0) \to (Y, y_0)$ and the map $\tilde{f}: (X, x_0) \to (\mathcal{Q}^j(Y), c')$ associated with \tilde{f} where c' is the constant map. Obviously \tilde{f} is homotopic to the constant map. This implies that f is homotopically trivial.

Let E and B be CW complexes, and let $p: E \rightarrow B$ be a fibration with fibre F. Then, we see that F has the homotopy type of a CW complex (not necessary connected) [11, 14]. Moreover the inclusion $i: F = p^{-1}(b_0) \rightarrow E$ is a closed cofibration by the theorem of A. Strøm [17]. In Theorem 1.1, putting X=E, A=F, we have the following fibration

$$\rho: \operatorname{map}_0(E, E) \longrightarrow \operatorname{map}_0(E, B) \times \operatorname{map}_0(F, E).$$

About the fibred product $map_0(E, B) \times 'map_0(F, E)$, we have

LEMMA 1.4. Let E and B be CW complexes and let $p: E \rightarrow B$ be a fibration with fibre F. For a given n > 1, if F is (n-1)-connected and $\pi_i(B)=0$ for every $i \ge n$, then the fibred product $\operatorname{map}_0(E, B) \times \operatorname{map}_0(F, E)$ has the same weak homotopy type as $\operatorname{map}_0(B, B) \times \operatorname{map}_0(F, F)$.

Proof. First, we shall show that the fibred product $map_0(E, B) \times 'map_0(F, E)$

has the same weak homotopy type as $map_0(E, B) \times map_0(F, F)$. Second, we shall show that $map_0(E, B)$ has the same weak homotopy type as $map_0(B, B)$.

Let $p_{*}: \operatorname{map}_{0}(F, E) \rightarrow \operatorname{map}_{0}(F, B)$ be a fibration defined by $p_{*}(f) = p \circ f$. By Lemma 1.3 $\operatorname{map}_{0}(F, B)$ is weakly contractible. Consequently we have a fibration

$$\operatorname{map}_{\mathbf{0}}(F, F) \longrightarrow \operatorname{map}_{\mathbf{0}}(F, E) \xrightarrow{\begin{subarray}{c} \begin{subarray}{c} \begin{subaray}{c} \begin{subarray}{c} \$$

and so $map_0(F, F)$ is weakly homotopy equivalent to $map_0(F, E)$.

On the other hand, let $i^*: \operatorname{map}_0(E, B) \to \operatorname{map}_0(F, B)$ be a fibration defined by $i^*(f) = f \circ i$ where $i: F \to E$ is the inclusion, then the fibred product $\operatorname{map}_0(E, B) \times ' \operatorname{map}_0(F, E)$ is the fibre space $(i^*)^* \operatorname{map}_0(F, E)$ of induced fibration by i^* . Using Lemma 1.2, we have

$$\operatorname{map}_{0}(E, B) \times \operatorname{map}_{0}(F, E) \underset{w}{\sim} \operatorname{map}_{0}(E, B) \times \operatorname{map}_{0}(F, F).$$

Next we shall show

$$\operatorname{map}_{0}(E, B) \xrightarrow{w} \operatorname{map}_{0}(B, B).$$

Now, by our hypothesis we may assume that E is a subcomplex of B and the *n*-skeleton of B is contained in E. Then we have a fibration $p^*: \operatorname{map}_0(B, B) \to \operatorname{map}_0(E, B)$ which is a map induced by the inclusion $p: E \to B$.

Let $f: S^i \times (E, e_0) \to (B, b_0)$ be a map such that $f \mid * \times E = p$. Since $\pi_j(B) = 0$ for every $j \ge n$, by using obstruction theory f can be extended to a map $f': S^i \times (B, b_0) \to (B, b_0)$ such that $f' \mid * \times B = id_B$. This implies that

$$(p^{*})_{*}: \pi_{i}(\operatorname{map}_{0}(B, B), \imath d_{B}) \longrightarrow \pi_{i}(\operatorname{map}_{0}(E, B), p)$$

is surjective for every i>0. Similarly, we can see that $(p^*)_*$ is injective for every i>0. Hence

$$p^*: \operatorname{map}_0(B, B; id_B) \longrightarrow \operatorname{map}_0(E, B; p)$$

is a weak homotopy equivalence. Also, we see that $p^*: \operatorname{map}_0(B, B) \to \operatorname{map}_0(E, B)$ induces a bijective correspondence of the path connected components of $\operatorname{map}_0(B, B)$ onto the path connected components of $\operatorname{map}_0(E, B)$.

Furthermore, by the similar argument we see that p^* is a weak homotopy equivalence for each path connected component of map₀(*B*, *B*). We have

$$\operatorname{map}_{0}(E, B) \xrightarrow{\sim} \operatorname{map}_{0}(B, B).$$

Thus our proof is completed.

Let X be a CW complex. Then we denote by G(X) and $G_0(X)$ the space of self homotopy equivalences of X and the space of self homotopy equivalences of (X, x_0) , respectively.

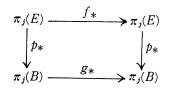
Under the same hypothesis as Lemma 1.4, we have

THEOREM 1.5. Let E and B be CW complexes and let $p: E \rightarrow B$ be a fibration with fibre F. For a given n > 1, if F is (n-1)-connected and $\pi_i(B)=0$ for every $i \ge n$, then we have the following fibration:

$$\mathscr{Q}(E \bmod F) \longrightarrow G_0(E) \stackrel{\rho}{\longrightarrow} B'$$

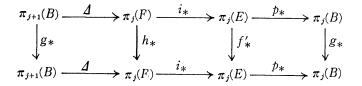
where B' is a subspace of $map_0(E, B) \times 'map_0(F, E)$ with the same weak homotopy type as $G_0(B) \times G_0(F)$ and $\mathcal{Q}(E \mod F)$ is the space of self fibre homotopy equivalences of E leaving the fibre F fixed.

Proof. Let f be an element of $G_0(E)$. Then by the above argument there exists a map $g: (B, b_0) \rightarrow (B, b_0)$ such that $p \circ f \simeq g \circ p$ rel e_0 . Consequently the following commutative diagram holds,



On the other hand, by our assumption we see that $p_*: \pi_j(E) \to \pi_j(B)$ is an isomorphism for every j < n. Since f is a self homotopy equivalence of (E, e_0) , $f_*: \pi_j(E) \to \pi_j(E)$ is an automorphism for every j. This implies that $g_*: \pi_j(B) \to \pi_j(B)$ is an automorphism for every j < n. Since B is a CW complex, by the theorem of J. H. C. Whitehead we obtain that g is a self homotopy equivalence of (B, b_0) . Thus g belongs to $G_0(B)$.

On the other hand, there exists a self map f' of (E, e_0) such that $f' \simeq f$ rel e_0 and $p \circ f' = g \circ p$, because $p: E \rightarrow B$ is a fibration and $p \circ f$ is homotopic to $g \circ p$ relative to e_0 . Put f'|F=h, then we have the following commutative diagram with exact rows,



Note that by our assumption for every $j \ge n$, $i_*: \pi_j(F) \to \pi_j(E)$ is an isomorphism and $\pi_j(B) = \pi_{j+1}(B) = 0$ and $f'_*: \pi_j(E) \to \pi_j(E)$ is an automorphism for every j, thus $h_*: \pi_j(F) \to \pi_j(F)$ is an automorphism for every $j \ge n$. Therefore h is a self homotopy equivalence of (F, e_0) .

From the argument of Lemma 1.4, there is a subspace B' of $\operatorname{map}_0(E, B) \times ' \operatorname{map}_0(F, E)$ which has the same weak homotopy type as $G_0(B) \times G_0(F)$ ($\subset \operatorname{map}_0(B, B) \times \operatorname{map}_0(F, F)$). Obviously we see from the definition of ρ that a

typical fibre $\rho^{-1}(p, i)$ is just $\mathcal{Q}(E \mod F)$. Finally we have a fibration:

$$\mathscr{G}(E \mod F) \longrightarrow G_0(E) \stackrel{\rho}{\longrightarrow} B' \xrightarrow{w} G_0(B) \times G_0(F).$$

§2. Fibration map theory.

Here we give a brief summary on a fibration map theory for fibrations which corresponds to the bundle map theory for principal bundles [3, 4, 5, 6].

Let $p: E \to B$ and $p': E' \to B'$ be fibrations with CW complexes B and B'. Let $\tilde{f}: E \to E'$ and $f: B \to B'$ be maps such that $p' \circ \tilde{f} = f \circ p$ and \tilde{f} carries each fibre of E into a fibre of E' by a homotopy equivalence. Then we call \tilde{f} a fibration map.

Let $\mathcal{Q}^*(E, E')$ be the space of fibration maps of E to E' and $\Phi: \mathcal{Q}^*(E, E') \rightarrow \max(B, B')$ be a map defined by $\Phi(\tilde{f})=f$ for each fibration map $\tilde{f}: E \rightarrow E'$ and its induced map $f: B \rightarrow B'$. Then we have a fibration

$$\Phi: \mathcal{G}^*(E, E') \longrightarrow \operatorname{map}(B, B').$$

Moreover, let B_0 be a subcomplex of B and put $p^{-1}(B_0) = E_0$. Let $\tilde{i}: E_0 \to E$ be the inclusion then $\tilde{i}^*: \mathcal{Q}^*(E, E') \to \mathcal{Q}^*(E_0, E')$ is a fibration. Let $\tilde{\alpha}: E_0 \to E'$ be a fixed fibration map which is extendable to a fibration map of E to E' and induces a map $\alpha: B_0 \to B$. Denote by $\mathcal{Q}^*_{\alpha}(E, E')$ a fibre $(\tilde{i}^*)^{-1}(\tilde{\alpha})$. Then we have a fibration $\Phi: \mathcal{Q}^*_{\alpha}(E, E') \to \max_{\alpha}(B, B')$, where $\max_{\alpha}(B, B')$ denotes the space of maps from B to B' whose restriction on B_0 is the map α . Furthermore, we denote by $\mathcal{Q}(E)$ the space of self fibre homotopy equivalences of E. Then every fibre $\Phi^{-1}(h)$ over h in the fibration $\Phi: \mathcal{Q}^*(E, E') \to \max(B, B')$ has the same homotopy type as $\mathcal{Q}(E)$. Also we have a fibration: $\mathcal{Q}(E \mod E_0) \to \mathcal{Q}(E) \xrightarrow{\tilde{i}^*} \mathcal{Q}(E_0)$, where $\mathcal{Q}(E \mod E_0)$ denotes the typical fibre $(\tilde{i}^*)^{-1}(id_{E_0})$.

Now, let $p: E \to B$ be a fibration with fibre F where B and F are CW complexes. Then there exists a universal fibration $p_{\infty}: E_{\infty} \to B_{\infty}$ with fibre F, where B_{∞} may be regarded as a classifying space BG(F) ([1], [7]). About this universal fibration we have the following Gottlieb's theorem which corresponds to Theorem (5.6) in [5].

THEOREM 2.1. Let $\tilde{k}: E \to E_{\infty}$ be a fibration map inducing a classifying map $k: B \to B_{\infty}$ for the fibration: $F \to E \xrightarrow{p} B$. Then the path connected component $\mathcal{G}^*_{k\circ\tilde{\iota}}(E, E_{\infty}; \tilde{k})$ in $\mathcal{G}^*_{k\circ\tilde{\iota}}(E, E_{\infty})$ containing \tilde{k} is weakly contractible.

Immediately we have the following

COROLLARY 2.2. $\mathcal{G}(E \mod E_0)$ has the same weak homotopy type as the loop space $\mathcal{Q} \max_{k \circ i}(B, B_{\infty}; k)$, where *i* denotes the inclusion of B_0 into *B*.

Remark 2.3. The fibration map theory stated above may be said that it is a replica of the bundle map theory initiated by I. M. James [6] and developed by D. H. Gottlieb [4, 5]. It seems that our theory has been known among experts but not appeared in the literatures. Note that the work of P. Booth, P. Heath, C. Morgan and R. Piccinini [2] is studying the same subject with a different approach.

Now, it should be noticed that for a CW complex B if

(1) B_{∞} is a homotopy associative *H*-space, or

(2) B is a suspension of a CW complex,

then $\Omega \operatorname{map}_0(B, B_\infty; k)$ has the same homotopy type as $\operatorname{map}_0(B, \Omega B_\infty)$. Thus, furthermore if F has the homotopy type of a CW complex, then we have

$$\mathscr{Q}(E \mod F) \xrightarrow{w} \max_{0} (B, \mathscr{Q}B_{\infty}) \xrightarrow{w} \max_{0} (B, G(F)).$$

Especially when F is an Eilenberg-MacLane complex $K(\pi, n)$ (n>1) we have the following

THEOREM 2.4. Let $p: E \rightarrow B$ be a fibration with fibre $F = K(\pi, n)$ (n > 1) such that B is a CW complex. Assume B is simply connected, then $\mathcal{G}(E \mod F)$ has the same weak homotopy type as

$$\operatorname{map}_{0}(B, K(\pi, n)) \underset{w}{\simeq} H^{n}(B, \pi) \times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi), i).$$

Proof. Let SG(F) be the path connected component of id_F in G(F). Since G(F) is a grouplike topological monoid and has the same weak homotopy type as $K(\pi, n) \times \operatorname{Aut}(\pi)$, where $\operatorname{Aut}(\pi)$ denotes the group of automorphisms of π , then we have the following fibration [7]:

$$BSG(F) \xrightarrow{Bj} BG(F) \xrightarrow{\lambda} K(\operatorname{Aut}(\pi), 1).$$

Here Bj may be regarded as the map between classifying spaces which is induced by the inclusion $j: SG(F) \rightarrow G(F)$.

Because B is a simply connected CW complex, we can easily see that $map_0(B, K(Aut(\pi), 1))$ is weakly contractible. Thus we have the following fibration:

$$\operatorname{map}_{0}(B, BSG(F); k') \xrightarrow{(Bj)_{\#}} \operatorname{map}_{0}(B, BG(F); k)$$
$$\xrightarrow{\lambda_{\#}} \operatorname{map}_{0}(B, K(\operatorname{Aut}(\pi), 1)),$$

where $[Bj \cdot k'] = [k]$. Consequently we have

$$\operatorname{map}_{0}(B, BSG(F); k') \xrightarrow{\sim} w \operatorname{map}_{0}(B, BG(F); k).$$

Note that $BSG(F) = K(\pi, n+1)$ is a homotopy associative H-space, then we have

by Corollary 2.2

$$\mathcal{Q}(E \mod F) \xrightarrow{w} \mathcal{Q} \operatorname{map}_{0}(B, BG(F); k)$$

$$\xrightarrow{w} \mathcal{Q} \operatorname{map}_{0}(B, BSG(F); k')$$

$$\xrightarrow{w} \mathcal{Q} \operatorname{map}_{0}(B, BSG(F); c)$$

$$\xrightarrow{w} \operatorname{map}_{0}(B, \mathcal{Q}BSG(F))$$

$$\xrightarrow{w} \operatorname{map}_{0}(B, K(\pi, n))$$

(for the last weak homotopy equivalence, see [7]),

where c is the constant map of B to BSG(F). By the theorem of J.C. Moore [8] it holds that

$$\operatorname{map}_{0}(B, K(\pi, n)) \simeq H^{n}(B, \pi) \times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi), i).$$

Thus our proof is completed.

§3. Applications.

In Theorem 1.5, we have essentially the following fibration:

$$\mathcal{Q}(E \bmod F) \longrightarrow G_0(E) \stackrel{\rho}{\longrightarrow} G_0(B) \times G_0(F) \,.$$

Hereafter we shall investigate the image of ρ .

For this purpose we shall recall Allaud's theory on the classification of fibre spaces [1, 7]. Let B be a space with base point b_0 . Then we write (ξ, i) the following fibration:

$$F \xrightarrow{i} E \xrightarrow{p} B$$
,

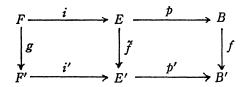
where $i: F \rightarrow p^{-1}(b_0)$ is a homotopy equivalence and the fibres are of the same homotopy type. Let (ξ', i') be another fibration:

$$F' \xrightarrow{i'} E' \xrightarrow{p'} B'$$

Define a map of (ξ, i) to (ξ', i') as a triple

$$g: F \longrightarrow F', \qquad \tilde{f}: E \longrightarrow E', \qquad f: (B, b_0) \longrightarrow (B', b'_0)$$

such that in the following diagram:



the square on the left is homotopy commutative as maps to the fibre $p'^{-1}(b'_0)$ and the square on the right is commutative. In the sequel the commutativity of this kind of diagram will be called briefly as semi-commutativity.

Now, let F be a CW complex (not necessary connected). Suppose given two fibrations (ξ, i) and (ξ', i') over CW complexes B and B' with the same fibre F

$$F \xrightarrow{i} E \xrightarrow{p} B,$$

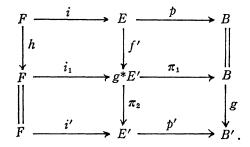
$$F \xrightarrow{i'} E' \xrightarrow{p'} B'.$$

there exists a map (h, f, g) of (ξ, i) to (ξ', i') such that h is a self homotopy equivalence of F.

We denote the induced fibration $(g^*\xi', i_1)$ as follows:

$$(g^*\xi', i_1): F \xrightarrow{i_1} g^*E' \xrightarrow{\pi_1} B.$$

Let $f': E \to g^*E'$ be a fibre homotopy equivalence given by f'(e) = (p(e), f(e)) for $e \in E$. Then we have the following semi-commutative diagram:



This implies that the fibration

$$(\boldsymbol{\xi}, i \circ h^{-1}): F \xrightarrow{i \circ h^{-1}} E \xrightarrow{p} B$$

is equivalent to the fibration $(g^*\xi', i_1)$.

On the other hand, by the main theorem of G. Allaud [1] there exists a map $\chi_E(h): (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$ corresponding to the fibration $(\xi, i \circ h^{-1})$. Obviously the homotopy class $[\chi_E(h)]$ is only dependent on the homotopy class [h]. Denote

by $(\boldsymbol{\xi}_{\infty}, i_{\infty})$ the universal fibration:

$$F \xrightarrow{i_{\infty}} E_{\infty} \xrightarrow{p_{\infty}} B_{\infty}$$

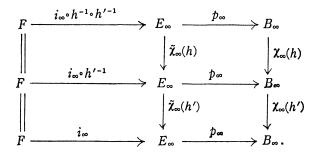
we have also a self map $\chi_{\infty}(h)$ of (B_{∞}, b_{∞}) corresponding to the fibration

$$(\xi_{\infty}, i_{\infty} \circ h^{-1}): F \xrightarrow{i_{\infty} \circ h^{-1}} E_{\infty} \xrightarrow{p_{\infty}} B_{\infty}.$$

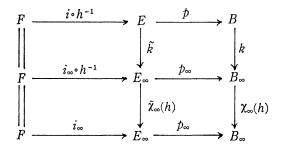
About map $\chi_{\infty}(h)$, we can get

$$[\boldsymbol{\chi}_{\infty}(h' \circ h)] = [\boldsymbol{\chi}_{\infty}(h')] \circ [\boldsymbol{\chi}_{\infty}(h)] \qquad (h, h' \in G(F))$$

by using the following semi-commutative diagram:



Consider the following semi-commutative diagram:

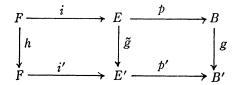


where $k: (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$ is a map corresponding to the fibration (ξ, i) . This implies that the map $\chi_{\infty}(h) \circ k: (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$ is a corresponding map to the fibration $(\xi, i \circ h^{-1})$. When $k': (B', b'_0) \rightarrow (B_{\infty}, b_{\infty})$ is a map corresponding to the fibration (ξ', i') , clearly we get a map $k' \circ g: (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$ corresponding to the fibration $(g^*\xi', i_1)$ which is equivalent to the fibration $(\xi, i \circ h^{-1})$. Consequently by the main theorem of G. Allaud again we have

$$[\boldsymbol{\chi}_{\infty}(h)] \cdot [k] = [k'] \cdot [g].$$

One can reverse the above argument. Thus we can obtain the following

THEOREM 3.1. Let F be a CW complex (not necessary connected). And let $(\xi, i): F \xrightarrow{i} E \xrightarrow{p} B$ and $(\xi', i'): F \xrightarrow{i'} E' \xrightarrow{p'} B'$ be two fibrations over CW complexes B and B' respectively. For given elements g of map₀(B, B') and h of G(F) there exists a fibration map $\tilde{g}: E \rightarrow E'$ such that the following diagram is semi-commutative:



(the square on the left is homotopy commutative as maps to the fibre $p'^{-1}(b'_0)$ and the square on the right is commutative) if and only if it holds that

$$[\boldsymbol{\chi}_{\infty}(h)] \cdot [k] = [k'] \cdot [g],$$

where maps $k: (B, b_0) \rightarrow (B_{\infty}, b_{\infty}), k': (B', b'_0) \rightarrow (B_{\infty}, b_{\infty})$ are corresponding ones to the fibrations (ξ, i) and (ξ', i') respectively and $\chi_{\infty}(h)$ is a self homotopy equivalence of (B_{∞}, b_{∞}) .

If a fibre F is simply connected, then this theorem yields the following result.

THEOREM 3.2. Under the same hypothesis as in Theorem 1.5, the image of $\rho: G_0(E) \rightarrow G_0(B) \times G_0(F)$ is just the union of the path connected components in $G_0(B) \times G_0(F)$ each of which contains (g, h) satisfying

$$[\boldsymbol{\chi}_{\infty}(h)] \cdot [k] = [k] \cdot [g],$$

where $k: (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$ is a map corresponding to the fibration: $F \xrightarrow{\imath} E \xrightarrow{p} B$.

About the map $\chi_{\infty}(h)$, when a fibre F is a complex $K(\pi, n)$ (n>1), we have the following

PROPOSITION 3.3. Let F be $K(\pi, n)$ (n>1) and let

$$F \xrightarrow{i_{\infty}} E_{\infty} \xrightarrow{p_{\infty}} B_{\infty} = BG(F)$$

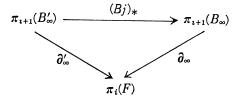
be the universal fibration with typical fibre F. Then for a given homotopy equivalence $h: F \rightarrow F$ we have

$$[\boldsymbol{\chi}_{\infty}(h) \circ Bj] = [Bj \circ h'],$$

where j is the inclusion of SG(F) to G(F), Bj is the map of $(BSG(F), b_{\infty})$ to $(BG(F), b_{\infty})$ and h' is a self map of $(BSG(F), b_{\infty})$ with $[h']=[h]\in Aut(\pi')$.

Proof. Let E'_{∞} denote the fibre space $(Bj)^*E_{\infty}$ of the induced fibration by $Bj: (B'_{\infty}, b'_{\infty}) \rightarrow (B_{\infty}, b_{\infty})$, where B'_{∞} is the classifying space BSG(F). First we shall show that E'_{∞} is contractible.

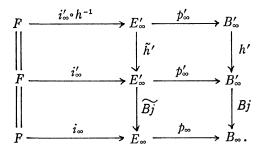
Now, since F is $K(\pi, n)$ (n>1), the boundary homomorphism $\partial_{\infty}: \pi_{i+1}(B_{\infty}) \to \pi_i(F)$ in the homotopy sequence of the universal fibration is an isomorphism if $i \ge 2$ and a monomorphism if i=1 ([1]). So, $\partial_{\infty}: \pi_{i+1}(B_{\infty}) \to \pi_i(F)$ is an isomorphism for every $i \ge 1$. Let us consider the homotopy sequence of the induced fibration $p'_{\infty}: E'_{\infty} \to B'_{\infty}$. Let ∂'_{∞} be the corresponding boundary homomorphism, then we have a commutative diagram:



Consequently we see that $\partial'_{\infty}: \pi_{i+1}(B'_{\infty}) \to \pi_i(F)$ is an isomorphism for every $i \ge 1$. This implies that $\pi_i(E'_{\infty})$ is trivial for every $i \ge 0$.

On the other hand, B'_{∞} is a CW complex and each fibre of the fibration $p'_{\infty}: E'_{\infty} \to B'_{\infty}$ has the same homotopy type as F which is a CW complex. By the theorem of Stasheff [11, 14] E'_{∞} has the same homotopy type as a CW complex. Therefore E'_{∞} is contractible.

Now, we have the following semi-commutative diagram:



This implies

$$[\boldsymbol{\chi}_{\boldsymbol{E}_{\boldsymbol{\omega}}'}(h)] = [Bj \cdot h']$$

and $[h] = [h'] \in \operatorname{Aut}(\pi')$ when we regard the homotopy classes [h] and [h'] as element of $\operatorname{Aut}(\pi')$. On the other hand, we have $[\chi_{E'_{\infty}}(h)] = [\chi_{\infty}(h) \circ Bj]$ because $Bj: (B'_{\infty}, b'_{\infty}) \to (B_{\infty}, b_{\infty})$ is the corresponding map of the fibration

$$(\boldsymbol{\xi}'_{\infty}, \, i'_{\infty}): F \xrightarrow{i'_{\infty}} E'_{\infty} \xrightarrow{p'_{\infty}} B'_{\infty}$$

Thus we have $[Bj \cdot h'] = [\chi_{\infty}(h) \cdot Bj].$

Now, using Proposition 3.3 combined with Theorem 1.5, 2.4 and 3.2, we obtain the following

THEOREM 3.4. For given 1 < m < n, let

$$F = K(\pi', n) \xrightarrow{i} E \xrightarrow{p} K(\pi, m) = B$$

be a fibration with a corresponding map $k: (B, b_0) \rightarrow (B_{\infty}, b_{\infty})$. Then there exists a map $k': (B, b_0) \rightarrow (B'_{\infty}, b'_{\infty})$ such that $[Bj \circ k'] = [k]$. And we have

$$G_0(E) \simeq \frac{1}{w} R \times H^n(B, \pi') \times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi'), i),$$

where R is the subgroup of Aut $(\pi) \times Aut (\pi') = \varepsilon(B) \times \varepsilon(F)$ consisting of ([g], [h]) with

$$h_*([k']) = g^*([k']).$$

Here [k'] is regarded as an element of $H^{n+1}(B, \pi')$, g^* and h_* are the automorphisms of $H^{n+1}(B, \pi')$ induced by g and h respectively.

Proof. Since B is a simply connected CW complex, there exists a map $k': (B, b_0) \rightarrow (B'_{\infty}, b'_{\infty})$ such that $[Bj \cdot k'] = [k]$.

Moreover, note that the following equalities hold [18, 19]:

$$G(B) \xrightarrow{w} \operatorname{Aut}(\pi) \times K(\pi, m), \qquad G_0(B) \xrightarrow{w} \operatorname{Aut}(\pi),$$
$$G(F) \xrightarrow{w} \operatorname{Aut}(\pi') \times K(\pi', n), \qquad G_0(F) \xrightarrow{w} \operatorname{Aut}(\pi').$$

By Theorem 1.5, 2.4 and 3.2 our proof is completed if we see that $[\chi_{\infty}(h) \circ k] = [k \circ g]$ is equivalent to

$$[h' \cdot k'] = [k' \cdot g].$$

By Proposition 3.3 we have

$$[\chi_{\infty}(h) \circ k] = [\chi_{\infty}(h) \circ Bj \circ k']$$
$$= [Bj \circ h' \circ k']$$
$$= [k \circ g]$$
$$= [Bj \circ k' \circ g].$$

Since the correspondence $(Bj)_*$ between based homotopy classes: $[B, B'_{\infty}]_0 \rightarrow [B, B_{\infty}]_0$ is bijective, we conclude that $[\chi_{\infty}(h) \circ k] = [k \circ g]$ is equivalent to

$$[h' \circ k'] = [k' \circ g].$$

Now, we must note the work of J. Siegel [12] where he studied the space $BG_0(E)$ under the same situation as the above theorem.

Moreover, note that the map $\rho: G_0(E) \to G_0(B) \times G_0(F)$ defined in Theorem 1.5 induces the homomorphism ρ_* of $\varepsilon(E)$ into $\varepsilon(B) \times \varepsilon(F)$, then we can easily see that the image of ρ_* is just R in Theorem 3.4 and the kernel of ρ_* may be regarded as $H^n(B, \pi')$. Thus as a corollary of Theorem 3.4 we have the following theorem proved by W. Shih [10] and Y. Nomura [9].

COROLLARY 3.5. Under the same hypothesis as Theorem 3.4, there exists the following exact sequence

 $1 \longrightarrow H^n(B, \pi') \longrightarrow \varepsilon(E) \longrightarrow R \longrightarrow 1$,

where R is the same group as the group stated in Theorem 3.4.

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