T. OKAYASU
KODAI MATH. J.
10 (1987), 116—126

MINIMAL IMMERSIONS OF CURVATURE PINCHED 2-MANIFOLDS INTO SPHERES

BY TAKASHI OKAYASU

0. Introduction.

Let (M, g) be a closed, connected two-dimensional Riemannian manifold. We consider isometric minimal immersions $\phi: M \to S^N(1)$ into the N-dimensional unit sphere of the Euclidean space R^{N+1}. Let $S^2(K)$ be a sphere of constant curvature K and let $s \in N$, $K(s)=2[s(s+1)]^{-1}$. In [3] Borůvka constructed isometric minimal immersions $\psi_s: S^2(K(s)) \to S^{2s}(1)$. Later Calabi proved that any isometric minimal full immersion of $S^2(K)$ into $S^N(1)$ is congruent to some Ψ_s ([5]).

Because of Calabi's result Simon cojectured the following ([6]).

CONJECTURE. Let (M, g) be a closed, connected 2-manifold with curvature K. Let $s \in N$ and let $\phi: M \to S^N(1)$ be an isometric minimal immersion such that $K(s+1) \leqq K \leqq K(s)$. Then either $K \equiv K(s)$ or $K \equiv K(s+1)$ on (M, g) and $\phi = \psi_s$ or $\phi = \psi_{s+1}$, respectively.

The conjecture is true for $s=1$ and $s=2$ (cf. [7] for $s=1$ and $N=4$; [2] for $s=1$, N arbitrary; [6], [8] for $s=2$).

In this paper we give a partial positive answer to this conjecture for $s=3$; we prove:

THEOREM. *For a real number K_0 satisfying $1/10 < K_0 < 1/6$, put $K_1 = K_0 + (1/18)(1-6K_0)(10K_0-1)$ (then $1/10 < K_0 < K_1 < 1/6$). Let (M, g) be a closed, connected 2-dimensional Riemannian manifold with curvature K. Assume that $K_0 \leqq K \leqq K_1$. Then there exists no isometric minimal immersion $\phi: M \to S^N(1)$ for any N.*

In the course of the proof, we also get short proofs for the case $s=1, 2$.

The author would like to express hearty thanks to Professor S. Tanno for his advice and encouragement during the developement of this work.

Received October 14, 1986

1. Proof.

The main idea of our proof is to use the operators X, Y and H, which were introduced by Bryant for classifying minimal surfaces with constant Gaussian curvature in spheres ([4]), in our variable curvature case.

We describe the definitions of the operators X, Y and H following Bryant ([4]). Let (M^2, g) be an oriented, connected 2-dimensional Riemannian manifold. We let $\chi: \mathcal{F}\to M$ be the bundle of oriented or thonormal frames. Thus $f\in\mathcal{F}$ is a triple $f=(x, e_1, e_2)$, where $x\in M$ and $e_1, e_2\in T_xM$ form an oriented basis. The canonical 1-forms, ω^1, ω^2 on $\mathcal{F}$ are the unique 1-forms satisfying $d\chi=e_1\omega^1+e_2\omega^2$. Set $\omega=\omega^1+i\omega^2$, $\bar{\omega}=\omega^1-i\omega^2$. Let $\tau\to M$ be the complex line bundle of 1-forms which are multiples of ω and let $\tau^{-1}\to M$ be the complex line bundle of 1-forms which are multiples of $\bar{\omega}$. For $m\geqq 0$, let $\tau^m\to M$ (resp. $\tau^{-m}\to M$) be the m-th power of $\tau\to M$ (resp. $\tau^{-1}\to M$) as a complex line bundle. Using the identification $\omega^m=(\bar{\omega})^{-m}$ for all m, we have a canonical pairing $\tau^m\times\tau^k\to\tau^{m+k}$ for all m and k. Let $C^\infty(\tau^m)$ be the vector space consisting of all smooth sections of τ^m. If $\sigma\in C^\infty(\tau^m)$, then, on $\mathcal{F}$, we may write $\sigma=s(\omega)^m$ for a unique function s on $\mathcal{F}$. One easily computes that $ds=-mi\rho s+s'\omega+s''\bar{\omega}$ for some unique functions s' and s'' on $\mathcal{F}$, where ρ is the commection form. It is easy to see that the forms $s'(\omega)^{m+1}=\sigma'$ and $s''(\omega)^{m-1}=\sigma''$ are well-defined sections of τ^{m+1} and τ^{m-1} respectively. This allows us to define operators $\partial_m: C^\infty(\tau^m)\to C^\infty(\tau^{m+1})$ and $\bar{\partial}_m: C^\infty(\tau^m)\to C^\infty(\tau^{m-1})$ by $\partial_m\sigma=\sigma'$, $\bar{\partial}_m\sigma=\sigma''$. Let $I_m: C^\infty(\tau^m)\to C^\infty(\tau^m)$ be the identity map. Set $\mathcal{T}=\bigoplus_m C^\infty(\tau^m)$ as a Z-graded vector space and define the operators

$$X=\bigoplus_m\partial_m,\qquad Y=\bigoplus_m\bar{\partial}_m,\qquad H=\bigoplus_m m\cdot I_m\,.$$

Thus for a function f on M regarded as a cross section of τ^0, we get

$$Xf=\frac{1}{2}(e_1f-ie_2f)\omega\,,\qquad Yf=\frac{1}{2}(e_1f+ie_2f)\bar{\omega}\,. \tag{1}$$

Let $\langle\,,\rangle$ denote the standard inner product on R^{N+1}. We set $\mathcal{V}=R^{N+1}\otimes_R\mathcal{T}$ and extend the operators X, Y and H to $\mathcal{V}$ in the natural way. We also have a pairing $\langle\,,\rangle: \mathcal{V}\times\mathcal{V}\to\mathcal{T}$ extending the given $\langle\,,\rangle$ in the obvious fashion. We define conjugation in $\mathcal{V}$ by setting $\bar{\sigma}=\bar{s}(\omega)^{-m}$ for $\sigma=s(\omega)^m\in C^\infty(\tau^m)$. Thus we have $X\bar{\sigma}=\overline{Y\sigma}$, $Y\bar{\sigma}=\overline{X\sigma}$.

We have the following two propositions (see Proposition 1.1 and Proposition 1.2 in [4]).

PROPOSITION 1. *The operators X, Y and H satisfy*

$$[X, Y]=\left(-\frac{K}{2}\right)H\,, \tag{2}$$

$$\Delta=2(XY+YX)\,, \tag{3}$$

where K is the Gaussian *curvature of M and $\Delta: \mathcal{T}\to\mathcal{T}$ is the Laplace-Beltrami* *operator on each graded piece.*

PROPOSITION 2. *Let $\phi: M^2 \to S^N(1) \subset R^{N+1}$ be an isometric minimal immersion of an oriented 2-dimensional Riemannian manifold M. Then*

$$\langle \phi, \phi \rangle = 1; \tag{4}$$

$$\langle X\phi, X\phi \rangle = 0, \qquad \langle Y\phi, Y\phi \rangle = 0; \tag{5}$$

$$\langle X\phi, Y\phi \rangle = \frac{1}{2}; \tag{6}$$

$$\Delta\phi = -2\phi. \tag{7}$$

LEMMA 1. *Let (M^2, g) be an oriented 2-dimensional Riemannian manifold. If f is a smooth function on M, then*

$$\Delta f = 4XYf = 4YXf. \tag{8}$$

If f, h are smooth functions on M, then

$$Xf \cdot Yh + Yf \cdot Xh = \frac{1}{2} g(\operatorname{grad} f, \operatorname{grad} h). \tag{9}$$

Proof. Since $f \in \mathfrak{T}$ has degree 0, $Hf = 0$. Thus

$$\Delta f = 2(XYf + YXf), \qquad (XY - YX)f = = -\frac{K}{2} Hf = 0,$$

from which (8) follows. From (1) we obtain

$$Xf \cdot Yh = \frac{1}{4}(e_1 f - ie_2 f)(e_1 h + ie_2 h).$$

Therefore

$$Xf \cdot Yh + Yf \cdot Xh = \frac{1}{2}(e_1 f \cdot e_1 h + e_2 f \cdot e_2 h),$$

from which (9) follows.

LEMMA 2. *Let $\phi: M^2 \to S^N(1) \subset R^{N+1}$ be an isometric minimal immersion of an oriented 2-dimensional Riemannian manifold. Then*

$$XY\phi = YX\phi = -\frac{1}{2} f; \tag{10}$$

$$\langle \phi, X\phi \rangle = 0, \qquad \langle \phi, Y\phi \rangle = 0; \tag{11}$$

$$\langle X\phi, Y^2\phi \rangle = 0, \qquad \langle Y\phi, X^2\phi \rangle = 0; \tag{12}$$

$$XY^2\phi = \frac{K-1}{2} Y\phi, \qquad YX^2\phi = \frac{K-1}{2} X\phi; \tag{13}$$

(14) $$\langle X^2\phi,\ Y^2\phi\rangle=\frac{1-K}{4};$$

(15) $$\langle X^2\phi,\ Y^3\phi\rangle=-\frac{1}{4}YK,\qquad \langle Y^2\phi,\ X^3\phi\rangle=-\frac{1}{4}XK;$$

(16) $$XY^3\phi=\frac{1}{2}YK\cdot Y\phi+\frac{3K-1}{2}Y^2\phi,$$

$$YX^3\phi=\frac{1}{2}XK\cdot X\phi+\frac{3K-1}{2}X^2\phi;$$

(17) $$\langle X\phi,\ Y^3\phi\rangle=0,\qquad \langle Y\phi,\ X^3\phi\rangle=0;$$

(18) $$\langle X^3\phi,\ Y^3\phi\rangle=\frac{(1-K)(1-3K)}{8}-\frac{\Delta K}{16};$$

(19) $$\langle X^3\phi,\ Y^4\phi\rangle=\frac{9K-5}{8}YK-\frac{Y(\Delta K)}{16},$$

$$\langle Y^3\phi,\ X^4\phi\rangle=\frac{9K-5}{8}XK-\frac{X(\Delta K)}{16};$$

(20) $$XY^4\phi=\frac{1}{2}Y^2K\cdot Y\phi+2YK\cdot Y^2\phi+\frac{6K-1}{2}Y^3\phi,$$

$$YX^4\phi=\frac{1}{2}X^2K\cdot X\phi+2XK\cdot X^2\phi+\frac{6K-1}{2}X^3\phi;$$

(21) $$\langle X^4\phi,\ Y^4\phi\rangle=\frac{(1-K)(1-3K)(1-6K)}{16}-\frac{1}{16}\|\mathrm{grad}\,K\|^2$$

$$+\frac{15}{64}\Delta K^2-\frac{3}{16}\Delta K-\frac{1}{64}\Delta(\Delta K).$$

Proof. Applying Lemma 1 to each component of ϕ we obtain $\Delta\phi=4XY\phi=4YX\phi$. Then (10) follows immediately from (7). Operating X to $\langle\phi,\ \phi\rangle=1$, we get $\langle\phi,\ X\phi\rangle=0$. Since $\bar{\phi}=\phi$, we get $\langle\phi,\ Y\phi\rangle=0$. Operating Y to (6), we get $\langle YX\phi,\ Y\phi\rangle+\langle X\phi,\ Y^2\phi\rangle=0$. By (10) and (11) $\langle X\phi,\ Y^2\phi\rangle$ vanishes. The second equation of (12) is the conjugate of the first. By (2) and (10) we get

$$XY^2\phi=(XY)(Y\phi)=\Big(YX-\frac{K}{2}H\Big)(Y\phi)=Y(XY\phi)+\frac{K}{2}Y\phi$$

$$=-\frac{1}{2}Y\phi+\frac{K}{2}Y\phi=\frac{K-1}{2}Y\phi,$$

which proves (13). Operating X to $\langle X\phi,\ Y^2\phi\rangle=0$ and using (13) and (6), we get

$$\langle X^2\phi, Y^2\phi\rangle=-\langle X\phi, XY^2\phi\rangle=-\frac{K-1}{2}\langle X\phi, Y\phi\rangle=\frac{1-K}{4},$$

which proves (14). Operating Y to (14) and using (13) and (12), we get

$$-\frac{1}{4}YK=\langle YX^2\phi, Y^2\phi\rangle+\langle X^2\phi, Y^3\phi\rangle=\langle X^2\phi, Y^3\phi\rangle,$$

which proves (15). By (2) and (13) we get

$$XY^3\phi=XY(Y^2\phi)=\left(YX-\frac{K}{2}H\right)(Y^2\phi)=Y(XY^2\phi)+K\cdot Y^2\phi$$

$$=Y\left(\frac{K-1}{2}Y\phi\right)+K\cdot Y^2\phi=\frac{YK}{2}Y\phi+\frac{3K-1}{2}Y^2\phi,$$

which proves (16). Operating Y to $\langle X\phi, Y^2\phi\rangle=0$ and using (10), (11) and (5), we get

$$\langle X\phi, Y^3\phi\rangle=-\langle YX\phi, Y^2\phi\rangle=\frac{1}{2}\langle\phi, Y^2\phi\rangle$$

$$=\frac{1}{2}Y\langle\phi, Y\phi\rangle-\frac{1}{2}\langle Y\phi, Y\phi\rangle=0,$$

which proves (17). Operating X to $\langle X^2\phi, Y^3\phi\rangle=-(1/4)YK$ and using (8), (16), (12) and (14), we get

$$\langle X^3\phi, Y^3\phi\rangle=-\frac{1}{16}\Delta K-\langle X^2\phi, XY^3\phi\rangle$$

$$=-\frac{1}{16}\Delta K-\left\langle X^2\phi, \frac{1}{2}YK\cdot Y\phi+\frac{3K-1}{2}Y^2\phi\right\rangle$$

$$=-\frac{1}{16}\Delta K+\frac{(1-K)(1-3K)}{8},$$

which proves (18). Operating Y to (18) and using (16), (17) and (15), we get

$$\langle X^3\phi, Y^4\phi\rangle=-\frac{1}{16}Y(\Delta K)+\frac{3K-2}{4}YK-\langle YX^3\phi, Y^3\phi\rangle$$

$$=-\frac{1}{16}Y(\Delta K)+\frac{9K-5}{8}YK,$$

which proves (19). By (2) and (16) we get

$$XY^4\phi=\left(YX-\frac{K}{2}H\right)(Y^3\phi)=Y(XY^3\phi)+\frac{3K}{2}Y^3\phi$$

$$=\frac{1}{2}Y^2K\cdot Y\phi+2YK\cdot Y^2\phi+\frac{6K-1}{2}Y^3\phi.$$

This proves (20). Operating X to the first equation of (19) and using (8), (9) and (20), we get

$$\begin{aligned}\langle X^4\phi, Y^4\phi\rangle &= \frac{9}{32}\|\mathrm{grad}\, K\|^2 + \frac{9K-5}{32}\Delta K - \frac{1}{64}\Delta(\Delta K) - \langle X^3\phi, XY^4\phi\rangle \\ &= \frac{9}{32}\|\mathrm{grad}\, K\|^2 + \frac{9K-5}{32}\Delta K - \frac{1}{64}\Delta(\Delta K) \\ &\quad - \left\langle X^3\phi, \frac{1}{2}Y^2K\cdot Y\phi + 2YK\cdot Y^2\phi + \frac{6K-1}{2}Y^3\phi\right\rangle.\end{aligned}$$

By using (17), (15), (18) and $K\Delta K=(1/2)\Delta K^2-\|\mathrm{grad}\, K\|^2$, (21) follows easily from the last equation.

Now we can give proofs for the case $s=1, 2$. We briefly explain the case $s=2$. Let (M, g) be a closed 2-dimensional Riemannian manifold with $1/6\leqq K \leqq 1/3$. Let $\Phi: M\to S^N(1)\subset R^{N+1}$ be an isometric minimal immersion. We may assume M is orientable. Integration of (21) gives

$$\int_M\left\{\langle X^4\phi, Y^4\phi\rangle + \frac{(1-K)(1-3K)(6K-1)}{16} + \frac{1}{16}\|\mathrm{grad}\, K\|^2\right\} = 0\,.$$

Since $\langle X^4\phi, Y^4\phi\rangle = \langle X^4\phi, \overline{X^4\phi}\rangle \geqq 0$ and $1/6\leqq K\leqq 1/3$, the integrands on the left hand side are all non-negative. Therefore $K\equiv 1/3$ or $1/6$. Thus we get the conclusion from Calabi's theorem ([5]).

From now on we assume that $\phi: M\to S^N(1)\subset R^{N+1}$ is an isometric minimal immersion of a closed, connected 2-dimensional Riemannian manifold M. We may assume that M is orientable.

Set $\langle X^4\phi, Y^4\phi\rangle = F$. Operating Y to $\langle X^4\phi, Y^4\phi\rangle = F$ and using (20), we get

(22)
$$\begin{aligned}\langle X^4\phi, Y^5\phi\rangle &= YF - \langle YX^4\phi, Y^4\phi\rangle \\ &= YF - \frac{X^2K}{2}\cdot\langle X\phi, Y^4\phi\rangle - 2XK\langle X^2\phi, Y^4\phi\rangle \\ &\quad - \frac{6K-1}{2}\cdot\langle X^3\phi, Y^4\phi\rangle\,.\end{aligned}$$

On the other hand, by (17), (10), (11) and (5), we get

(23)
$$\begin{aligned}\langle X\phi, Y^4\phi\rangle &= -\langle YX\phi, Y^3\phi\rangle = \frac{1}{2}\langle \phi, Y^3\phi\rangle \\ &= \frac{1}{2}Y\langle\phi, Y^2\phi\rangle - \frac{1}{2}\langle Y\phi, Y^2\phi\rangle \\ &= \frac{1}{2}Y\{Y\langle\phi, Y\phi\rangle - \langle Y\phi, Y\phi\rangle\} - \frac{1}{4}Y\langle Y\phi, Y\phi\rangle \\ &= 0\,.\end{aligned}$$

By (15), (13), (17) and (9), we get

$$XK\langle X^2\phi,\ Y^4\phi\rangle = XK\{Y\langle X^2\phi,\ Y^3\phi\rangle - \langle YX^2\phi,\ Y^3\phi\rangle\} \tag{24}$$

$$= XK\left\{-\frac{1}{4}Y^2K - \frac{K-1}{2}\langle X\phi,\ Y^3\phi\rangle\right\}$$

$$= -\frac{1}{4}XK\cdot Y^2K$$

$$= -\frac{1}{4}\{Y(XK\cdot YK) - YXK\cdot YK\}$$

$$= -\frac{1}{16}Y(\|\mathrm{grad}\, K\|^2) + \frac{1}{16}\Delta K\cdot YK\,.$$

Substituting (23), (24) and (19) to (22), we obtain

$$\langle X^4\phi,\ Y^5\phi\rangle = YF + Y\left(\frac{1}{8}\|\mathrm{grad}\, K\|^2\right) - \frac{1}{8}\Delta K\cdot YK - \frac{(6K-1)(9K-5)}{16}YK + \frac{6K-1}{32}Y(\Delta K)\,. \tag{25}$$

Operating X to (25), we get

$$\langle X^5\phi,\ Y^5\phi\rangle + \langle X^4\phi,\ XY^5\phi\rangle \tag{26}$$

$$= XYF + \frac{1}{8}XY(\|\mathrm{grad}\, K\|^2) - \frac{1}{8}X(\Delta K\cdot YK) - \frac{1}{16}X\{(6K-1)(9K-5)YK\} + \frac{1}{32}X\{(6K-1)Y(\Delta K)\}\,.$$

Adding (26) to its conjugate, we get

$$2\langle X^5\phi,\ Y^5\phi\rangle + \langle X^4\phi,\ XY^5\phi\rangle + \langle Y^4\phi,\ YX^5\phi\rangle \tag{27}$$

$$= \frac{1}{2}\Delta F + \frac{1}{16}\Delta(\|\mathrm{grad}\, K\|^2) - \frac{1}{8}\{X(\Delta K\cdot YK) + Y(\Delta K\cdot XK)\}$$

$$- \frac{1}{16}\Big\{X\{(6K-1)(9K-5)YK\} + Y\{(6K-1)(9K-5)XK\}\Big\}$$

$$+ \frac{1}{32}\Big\{X\{(6K-1)Y(\Delta K)\} + Y\{(6K-1)X(\Delta K)\}\Big\}.$$

We need the following lemma.

LEMMA 3. *Let (M', g') be a closed, orientable 2-dimensional Riemannian manifold and f, h be functions on M'. Then*

$$\int_{M'}\{X(f\cdot Yh) + Y(f\cdot Xh)\} = 0\,.$$

Proof. By using (9) and Green's formula, we get

$$\int_{M'}\{X(f\cdot Yh)+Y(f\cdot Xh)\}$$

$$=\int_{M'}\{Xf\cdot Yh+Yf\cdot Xh+f\cdot XYh+f\cdot YXh\}$$

$$=\int_{M'}\left\{\frac{1}{2}g'(\mathrm{grad}\, f,\ \mathrm{grad}\, h)+\frac{1}{2}f\cdot\Delta h\right\}=0.$$

We integrate (27) and apply Lemma 3. Then we get

$$\int_{M'}\{2\langle X^5\phi,\ Y^5\phi\rangle+\langle X^4\phi,\ XY^5\phi\rangle+\langle Y^4\phi,\ YX^5\phi\rangle\}=0\,. \tag{28}$$

We compute $XY^5\phi$. By (20) we get

$$XY^5\phi=\left(YX-\frac{K}{2}H\right)(Y^4\phi)=Y(XY^4\phi)+2K\cdot Y^4\phi \tag{29}$$

$$=Y\left\{\frac{1}{2}Y^2K\cdot Y\phi+2YK\cdot Y^2\phi+\frac{6K-1}{2}Y^3\phi\right\}+2K\cdot Y^4\Phi$$

$$=\frac{1}{2}Y^3K\cdot Y\phi+\frac{5}{2}Y^2K\cdot Y^2\phi+5YK\cdot Y^3\phi+\frac{10K-1}{2}Y^4\phi\,.$$

By (29), (23), (19), (15) and (17), we get

$$\langle X^4\phi,\ XY^5\phi\rangle=\frac{1}{2}Y^3K\langle X^4\phi,\ Y\phi\rangle+\frac{5}{2}Y^2K\langle X^4\phi,\ Y^2\phi\rangle \tag{30}$$

$$+5YK\langle X^4\phi,\ Y^3\phi\rangle+\frac{10K-1}{2}\langle X^4\phi,\ Y^4\phi\rangle$$

$$=\frac{5}{2}Y^2K\{X\langle X^3\phi,\ Y^2\phi\rangle-\langle X^3\phi,\ XY^2\phi\rangle\}$$

$$+5YK\left\{\frac{9K-5}{8}XK-\frac{1}{16}X(\Delta K)\right\}+\frac{10K-1}{2}\langle X^4\phi,\ Y^4\phi\rangle$$

$$=-\frac{5}{8}Y^2K\cdot X^2K+\frac{5(9K-5)}{32}\|\mathrm{grad}\, K\|^2$$

$$-\frac{5}{16}YK\cdot X(\Delta K)+\frac{10K-1}{2}\langle X^4\phi,\ Y^4\phi\rangle\,.$$

From (28) and (30), we get

$$\int_M\left\{2\langle X^5\Phi,\ Y^5\Phi\rangle-\frac{5}{4}X^2K\cdot Y^2K+\frac{5(9K-5)}{16}\|\mathrm{grad}\, K\|^2\right. \tag{31}$$

$$\left.-\frac{5}{32}g(\mathrm{grad}\, K,\ \mathrm{grad}\,(\Delta K))+(10K-1)\langle X^4\phi,\ Y^4\phi\rangle\right\}$$

$$=0\,.$$

On the other hand, we have

$$\int_M X^2K\cdot Y^2K=\int_M \{X(XK\cdot Y^2K)-XK\cdot XY^2K\} \tag{32}$$

$$=\int_M \Big\{X\{Y(XK\cdot YK)-YXK\cdot YK\}-XK\cdot\Big(YX-\frac{K}{2}H\Big)(YK)\Big\}$$

$$=\int_M \Big\{\frac{1}{16}\Delta\|\mathrm{grad}\,K\|^2-\frac{1}{4}X(\Delta K\cdot YK)-\frac{1}{4}XK\cdot Y(\Delta K)-\frac{K}{2}XK\cdot YK\Big\}$$

$$=\int_M \Big\{-\frac{1}{4}\{X(\Delta K)\cdot YK+Y(\Delta K)\cdot XK\}-\frac{1}{16}(\Delta K)^2-\frac{K}{8}\|\mathrm{grad}\,K\|^2\Big\}$$

$$=\int_M \Big\{-\frac{1}{8}g\{(\mathrm{grad}\,\Delta K,\ \mathrm{grad}\,K)-\frac{1}{16}(\Delta K)^2-\frac{1}{8}K\|\mathrm{grad}\,K\|^2\Big\}$$

$$=\int_M \Big\{\frac{1}{16}(\Delta K)^2-\frac{K}{8}\|\mathrm{grad}\,K\|^2\Big\}.$$

From (31) and (32), we get

$$\int_M \Big\{2\langle X^5\phi,\ Y^5\phi\rangle+\frac{5}{64}(\Delta K)^2+\frac{95K-50}{32}\|\mathrm{grad}\,K\|^2 + (10K-1)\langle X^4,\ \phi Y^4\phi\rangle\Big\}=0. \tag{33}$$

By (21) we get from (33)

$$\begin{aligned} 0=&\int_M \Big\{2\langle X^5\Phi,\ Y^5\Phi\rangle+\frac{5}{64}(\Delta K)^2+\frac{95K-50}{32}\|\mathrm{grad}\,K\|^2 \\ &+\frac{10K-1}{2}\langle X^4\Phi,\ Y^4\Phi\rangle+\frac{10K-1}{2}\Big[\frac{(1-K)(1-3K)(1-6K)}{16} \\ &-\frac{1}{16}\|\mathrm{grad}\,K\|^2+\frac{15}{64}\Delta K^2-\frac{3}{16}\Delta K-\frac{1}{64}\Delta(\Delta K)\Big]\Big\} \\ =&\int_M \Big\{2\langle X^5\phi,\ Y^5\phi\rangle+\frac{1}{32}(1-K)(1-3K)(1-6K)(10K-1) \\ &+\frac{10K-1}{2}\langle X^4\Phi,\ Y^4\Phi\rangle+\frac{10K-19}{32}\|\mathrm{grad}\,K\|^2\Big\}. \end{aligned} \tag{34}$$

Let K_0, K_1 be constants. Then from (34) and (21) we get

$$\begin{aligned} 0=&\int_M \Big\{2\langle X^5\phi,\ Y^5\phi\rangle+\frac{1}{32}(1-K)(1-3K)(1-6K)(10K-1) \\ &+\frac{10(K-K_0)}{2}\langle X^4\phi,\ Y^4\phi\rangle+\frac{10K_0-1}{2}\langle X^4\phi,\ Y^4\phi\rangle \\ &+\frac{10(K-K_0)}{32}\|\mathrm{grad}\,K\|^2+\frac{10K_0-10}{32}\|\mathrm{grad}\,K\|^2\Big\} \end{aligned} \tag{35}$$

$$=\int_M\left\{2\langle X^5\phi,\ Y^5\phi\rangle+\frac{1}{32}(1-K)(1-3K)(1-6K)(10K-1+10K_0-1)\right.$$

$$\left.+5(K-K_0)\langle X^4\phi,\ Y^4\phi\rangle+\frac{5}{16}(K-K_0)\|\operatorname{grad} K\|^2-\frac{9}{16}\|\operatorname{grad} K\|^2\right\}.$$

On the other hand, by (18) we have

$$\int_M\left\{-\frac{9}{16}\|\operatorname{grad} K\|^2\right\}=\int_M\frac{9}{16}K\Delta K=\int_M\frac{9}{16}(K-K_1)\Delta K \tag{36}$$

$$=\int_M 9(K_1-K)\left\{\langle X^3\phi,\ Y^3\phi\rangle-\frac{1}{8}(1-K)(1-3K)\right\}$$

$$=\int_M\left\{9(K_1-K)\langle X^3\phi,\ Y^3\phi\rangle-\frac{9}{8}(1-K)(1-3K)(K_1-K)\right\}.$$

From (35) and (36) we get

$$0=\int_M\left\{2\langle X^5\phi,\ Y^5\phi\rangle+5(K-K_0)\langle X^4\phi,\ Y^4\phi\rangle\right. \tag{37}$$

$$+\frac{5}{16}(K-K_0)\|\operatorname{grad} K\|^2+9(K_1-K)\langle X^3\phi,\ Y^3\phi\rangle$$

$$\left.+\frac{1}{32}(1-K)(1-3K)[(1-6K)(10K+10K_0-2)+36(K-K_1)]\right\}.$$

Let $1/10<K_0<1/6$ and put $K_1=K_0+(1/18)(1-6K_0)(10K_0-1)$. Then $1/10<K_0<K_1<1/6$. We define $h(t)=(1-6t)(10t+10K_0-2)+36(t-K_1)$. Then $h(K_0)=0$ and $h(K_1)>0$. Thus $h(K)>0$ for $K_0<K\leqq K_1$.

We can rewrite (37) as follows.

$$0=\int_M\left\{2\langle X^5\phi,\ Y^5\phi\rangle+5(K-K_0)\langle X^4\phi,\ Y^4\phi\rangle\right. \tag{38}$$

$$+\frac{5}{16}(K-K_0)\|\operatorname{grad} K\|^2+9(K_1-K)\langle X^3\phi,\ Y^3\phi\rangle$$

$$\left.+\frac{1}{32}(1-K)(1-3K)h(K)\right\}.$$

The integrands of the right hand side are all non-negative under the assumption of the theorem. Therefore $(1-K)(1-3K)h(K)\equiv 0$. That is $K\equiv K_0$. Since $1/10<K_0<1/6$, according to Calabi's theorem ([4]), this is impossible. This completes the proof of the theorem. (q. e. d.).

Remarks. (1) Since $K_1-K_0=(1/18)(1-6K_0)(10K_0-1)=(1/18)\{-60(K_0-2/15)^2+1/15\}$, $\max(K_1-K_0)=1/(18\cdot 15)$. This value is 1/18 of 1/6−1/10.

(2) At present we need some additional assumption to prove the conjecture for $s\geqq 3$ ([1], [9]).

REFERENCES

[1] A. C. ASPERTI, A note on the minimal immersions of the two-sphere, preprint.
[2] K. BENKO, M. KOTHE, K. D. SEMMLER AND U. SIMON, Eigenvalues of the Laplacian and curvature, Colloq. Math., **42** (1979), 19-31.
[3] O. BORŮVKA, Sur les surfaces représentées par les fonctions sphèriques de premiere espèce, J. Math. Pure et Appl., **12** (1933), 337-383.
[4] R. L. BRYANT, Minimal surfaces of constant curvature in S^n, Trans. Amer. Math., **290** (1985), 259-271.
[5] E. CALABI, Minimal immersions of surfaces in Euclidean spheres, J. Diff. Geom., **1** (1967), 111-125.
[6] M. KOZLOWSKI AND U. SIMON, Minimal immersions of 2-manifolds into spheres, Math. Z., **186** (1984), 377-382.
[7] H. B. LAWSON, Local rigidity theorem for minimal hypersurfaces, Ann. Math., (2) **89** (1969), 187-197.
[8] T. OGATA, Minimal surfaces in a sphere with Gaussian curvature not less than 1/6, Tohoku Math. J., **37** (1985), 553-560.
[9] T. OGATA, U. Simon's conjecture on minimal surfaces in a sphere, preprint.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCINCE
HIROSAKI UNIVERSITY
BUNKYO-CHO 3 HIROSAKI
AOMORI-KEN 036 JAPAN