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KODAI MATH. J.
10 (1987), 102—107

NEW CHARACTERIZATIONS OF φ-SYMMETRIC SPACES

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1. Introduction.

Sasakian φ-symmetric spaces have been introduced and studied by T. Takahashi in [6] and the notion of a φ-geodesic symmetry, used to define them, has been explored further in [2]. In this note we give three characterizations of Sasakian φ-symmetric spaces. The first is technical in nature and is analogous to the fact that a Kähler manifold is locally symmetric if and only if $(\nabla_X R)_{XJXXJX}=0$ for all X. The second is geometric in nature and is analogous to the following characteristic property of symmetric spaces. Let m be the midpoint of a sufficiently short geodesic segment joining p and q. Then the geodesic spheres of radius equal to the distance $d(m, p)$ centered at p and q have the same shape operator at m. Finally, we have previously observed [2] that a complete, simply connected locally φ-symmetric space is a naturally reductive homogeneous space. We shall show conversely that a naturally reductive homogeneous space with an invariant Sasakian structure is φ-symmetric.

2. Sasakian manifolds and φ-symmetric spaces.

A C^∞ manifold M^{2n+1} is said to be an *almost contact manifold* if the structural group of its tangent bundle is reducible to $\mathcal{U}(n)\times 1$. It is well-known that such a manifold admits a tensor field φ of type (1, 1), a vector field ξ and a 1-form η satisfying

$$\eta(\xi)=1, \qquad \varphi^2=-I+\eta\otimes\xi.$$

These conditions imply that $\varphi\xi=0$ and $\eta\circ\varphi=0$. Moreover, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y)=g(X, Y)-\eta(X)\eta(Y)$$

for any tangent vector fields X and Y; note that this implies that $\eta(X)=g(X, \xi)$. M together with these structure tensors is said to be an *almost contact metric manifold.* If now these structure tensors satisfy

$$(1) \qquad (\nabla_X\varphi)Y=g(X, Y)\xi-\eta(Y)X,$$

Received September 8, 1986

where ∇ denotes the Riemannian connection of g, M is said to be a *Sasakian manifold*. It is easy to see from (1) that

$$\nabla_X \xi = -\varphi X \tag{2}$$

from which it follows that ξ is a Killing vector field. The curvature tensor

$$R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

of a Sasakian manifold satisfies

$$R_{X\xi}Y = \eta(Y)X - g(X, Y)\xi. \tag{3}$$

For a general reference to the above ideas see [1], [8].

A geodesic γ on a Sasakian manifold is said to be a *φ-geodesic* if $\eta(\gamma')=0$. From (2) it is easy to see that a geodesic which is initially orthogonal to ξ remains orthogonal to ξ. A local diffeomorphism s_m of M, $m \in M$, is said to be a *φ-geodesic symmetry* if its domain U is such that, for every φ-geodesic $\gamma(s)$ such that $\gamma(0)$ lies in the intersection of U with the integral curve of ξ through m,

$$(s_m \circ \gamma)(s) = \gamma(-s)$$

for all s with $\gamma(\pm s) \in U$, s being the arc length [6]. At the point m the differential s_{m*} of s_m is given by

$$s_{m*}(m) = -I + 2\eta \otimes \xi.$$

In [6] Takahashi introduced the notion of a *locally φ-symmetric* space by reguiring that

$$\varphi^2 (\nabla_V R)_{XY} Z = 0$$

for all vector fields V, X, Y, Z orthogonal to ξ. On the other hand he defined a *globally φ-symmetric space* by requiring that any φ-geodesic symmetry be extendable to a global automorphism of M and that the Killing vector field ξ generate a global one-parameter subgroup of isometries.

Let $\tilde{\mathcal{U}}$ be a neighborhood on M on which ξ is regular. Then, as is well-known, the fibration $\tilde{\mathcal{U}} \to \mathcal{U} = \tilde{\mathcal{U}}/\xi$ gives a Kähler structure (J, G) on the base manifold $\mathcal{U}$. Among the main results of [6] are the following:

PROPOSITION 1. *A Sasakian manifold is a locally φ-symmetric space if and only if each Kähler manifold, which is the base space of a local fibering, is a Hermitian locally symmetric space.*

PROPOSITION 2. *A necessary and sufficient condition for a Sasakian manifold to be a locally φ-symmetric space is that it admits a φ-geodesic symmetry, at every point, which is a local automorphism.*

PROPOSITION 3. *A complete, connected, simply connected Sasakian locally φ-*

symmetric space is a globally φ-symmetric space.

If M is a Sasakian manifold which is also a homogeneous space, we say that M has an *invariant Sasakian structure* if the structure tensors (φ, ξ, η, g) are invariant by the group of isometries acting transitively on M. For M homogeneous we have a local homogeneous structure T on M, *i.e.* a tensor field of type (1, 2) such that with respect to the connection $\bar{\nabla}$ defined by

$$\bar{\nabla}_X Y = \nabla_X Y + T_X Y,$$

g, R and T are parallel. If now M has an invariant Sasakian structure, then the structure tensors φ, ξ, η are also parallel with respect to the canonical connection $\bar{\nabla}$. If moreover the homogeneous manifold is naturally reductive then $T_X X=0$ for all tangent vectors X. For a general reference to these ideas see [7].

We close this section with several notational matters and an important lemma from Kähler geometry. We denote by $d(p, q)$ the distance with respect to the metric g between points p and q. We denote by $T_p(m)$ the shape operator of the geodesic sphere, centered at p, at the point m. (p is assumed to be sufficiently close to m.) For the curvature tensor and its covariant derivative we write R_{XYZW} for $g(R_{XY}Z, W)$ and $(\nabla_U R)_{XYZW}$ for $g((\nabla_U R)_{XY}Z, W)$.

In the case of Kähler manifolds we denote as before the structure tensors by (J, G) and we denote by D the Riemannian connection of G and by $\underline{R}$ its curvature tensor. We have ([4], [5])

LEMMA 4. *A Kähler manifold is a Hermitian locally symmetric space if and only if*

$$(D_X \underline{R})_{XJXXJX}=0$$

for all vector fields X.

3. Characterizations of φ-symmetric spaces.

We now give three results on the characterization of φ-symmetric Sasakian manifolds.

THEOREM 5. *A Sasakian manifold M is locally φ-symmetric if and only if*

$$(\nabla_U R)_{U\varphi UU\varphi U}=0 \tag{4}$$

for all vector fields U orthogonal to ξ.

Proof. The necessity is clear, so we prove only the sufficiency. Consider the local fibration $\tilde{\mathcal{U}} \to \mathcal{U} = \tilde{\mathcal{U}}/\xi$ as before. For a vector field X on $\mathcal{U}$ we denote its horizontal lift with respect to the connection form η by X^*. Then Takahashi shows in [6] that

$$((D_V\underline{R})_{XY}Z)^*=-\varphi^2((\nabla_{V^*}R)_{X^*Y^*}Z^*)$$

for any vector fields V, X, Y, Z on $\mathcal{V}$. Taking the inner product with a horizontal lift W^* we have

$$G((D_V\underline{R})_{XY}Z, W)=g((\nabla_{V^*}R)_{X^*Y^*}Z^*, W^*).$$

If now (4) holds, setting $U=X^*$ we have

$$(D_X\underline{R})_{XJXXJX}=0.$$

Thus from Lemma 4 we see that $\mathcal{V}$ is a Hermitian locally symmetric space and hence by Proposition 1, M is locally φ-symmetric.

THEOREM 6. *A Sasakian manifold M is locally φ-symmetric if and only if for every point $m\in M$ and every φ-geodesic γ through m we have the following property: For every $p\in\gamma$ such that p and $s_m(p)$ lie in a normal neighborhood of m, the shape operators at m of the geodesic spheres of radius $d(m, p)$ centered at p and $s_m(p)$ "commute" with $s_{m*}(m)$, i.e.*

$$s_{m*}(m)\circ T_p(m)=T_{s_m(p)}(m)\circ s_{m*}(m). \tag{5}$$

Proof. If M is locally φ-symmetric, then by Proposition 2 each s_m is an automorphism of the Sasakian structure; in particular each s_m is an isometry and hence the condition (5) is satisfied.

Conversely we shall show that it is in fact enough to assume that (5) holds when applied to the vector φU at m, U being the unit tangent field to the geodesic γ. A power series expansion of the shape operator $T_p(m)$ with coefficients at m may be derived using the formula

$$T_m(p)=-(A'A^{-1})(r), \qquad r=d(m, p),$$

where A satisfies the Jacobi equation

$$A''-R\circ A=0$$

with initial values $A(0)=0$, $A'(0)=I$. (See for example Section 3 in [3].) One obtains

$$T_p(m)=\frac{1}{r}I+\frac{1}{3}R(m)r+\frac{1}{12}R'(m)r^2+0(r^3),$$

where the meaning of the coefficients is for $T_p(m)X$:

$$R(m)X=R_{UX}U, \qquad R'(m)X=(\nabla_U R)_{UX}U.$$

Now denote $s_{m*}(m)=-I+2\eta\otimes\xi$ by S. Then using the power series expansions of $T_p(m)$ and $T_{s_m(p)}(m)$ in (5) and applying both sides to φU, we have

$$(\nabla_{SU}R)_{SUS\varphi U}SU=S(\nabla_U R)_{U\varphi U}U$$

from which

$$(\nabla_U R)_{U\varphi U}U = -(\nabla_U R)_{U\varphi U}U + 2\eta(\nabla_U R_{U\varphi U}U)\xi\,.$$

Taking the inner product of this last expression with φU we have

$$(\nabla_U R)_{U\varphi UU\varphi U} = 0$$

for any U orthogonal to ξ and the result follows from Theorem 5.

THEOREM 7. *Let M be a complete, connected, simply connected Sasakian manifold. Then M is a globally φ-symmetric space if and only if M is a naturally reductive homogeneous space with invariant Sasakian structure.*

Proof. We remarked earlier that Proposition 3 shows that a connected, complete, simply connected locally φ-symmetric space is a globally φ-symmetric space. In [2] it was noted that a complete, simply connected locally φ-symmetric space is a naturally reductive homogeneous space. In particular, an explicit tensor field T of type (1, 2) such that $T_X X = 0$ was given for which $\bar{\nabla} = \nabla + T$ gives $\bar{\nabla} R = \bar{\nabla} g = \bar{\nabla} T = \bar{\nabla}\varphi = 0$, etc.

Conversely, if M is a naturally reductive homogeneous space with invariant Sasakian structure, then there exists a tensor field T of type (1, 2) such that with respect to the connection $\bar{\nabla} = \nabla + T$, g, R, T, φ, etc. are parallel and $T_X X = 0$ for all tangent vectors X. Now, let γ be a φ-geodesic and U its unit tangent field. Expanding $\bar{\nabla} R = 0$ we have

$$\begin{aligned}(\nabla_U R)_{U\varphi UU\varphi U} &= 2R_{T_U U\varphi UU\varphi U} + 2R_{UT_U\varphi UU\varphi U}\\ &= 2R_{UT_U\varphi UU\varphi U}\,.\end{aligned}$$

Further, from $\bar{\nabla}\varphi = 0$ we have, using (1),

$$\begin{aligned}0 &= (\nabla_U\varphi)U + T_U\varphi U - \varphi T_U U\\ &= \xi + T_U\varphi U\,.\end{aligned}$$

Therefore, from (3),

$$R_{UT_U\varphi UU\varphi U} = -g(R_{U\xi}U,\, \varphi U) = g(\xi,\, \varphi U) = 0$$

giving

$$(\nabla_U R)_{U\varphi UU\varphi U} = 0$$

and the result follows.

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