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SOME ESTIMATES OF TOTAL TENSION AND THEIR APPLICATIONS

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Abstract

In this note, we give two best possible estimates of the total tension for a smooth map. Such estimates are established in terms of order of the map. Applications of such estimates to isometric immersions and to spectral geometry are given by applying an inequality obtained in [3].

1. Introduction.

Let M be a compact submanifold of a Euclidean m-space E^m. By applying the induced metric on M, the author introduced in [2] the notion of order of the submanifold. The notion of order is known to be closely related with the differential geometry of the submanifold (cf. [4]). In [5, 6] such notion was generalized to smooth maps of a compact Riemannian manifold into E^m. Some relations between the total tension and the order were obtained in [5, 6].

In this note, we will obtain two more relations between the total tension and the order of a map. Such relations are applied to obtain a best possible estimate of the total mean curvature of a spherical submanifold. By using a best possible inequality derived in [3], such relations were then applied to obtain some best possible eigenvalue estimates for minimal submanifolds in rank-one symmetric spaces.

2. Order of a Map.

Let M be a compact n-dimensional Riemannian manifold and Δ the Laplacian of M acting on the space $C^\infty(M)$ of smooth functions. Then Δ has an infinite discrete sequence of eigenvalues:

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty\,.$$

For each k ($k=0, 1, 2, \cdots$), the eigenspace $V_k=\{f\in C^\infty(M):\Delta f=\lambda_k f\}$ is finite-dimensional. With respect to the inner product $(f, g)=\int_M fg\,dV$ on $C^\infty(M)$, the

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decomposition $\sum_k V_k$ is orthogonal and dense in $C^\infty(M)$. Therefore, for each $f\in C^\infty(M)$, $f=f_0+\sum_{t\geqq 1}f_t$, where f_0 is a constant and f_t is the projection of f into V_t.

For a smooth map $x: M\to E^m$, we can apply the above decomposition to each coordinate function of M in E^m. Thus, we obtain the following spectral decomposition of the E^m-valued function x:

$$x=x_0+\sum_{t=1}^{\infty} x_t\,, \tag{2.1}$$

where x_0 is a constant vector which is the center of mass of x and x_t a vector with $\Delta x_t=\lambda_t x_t$.

If x is a non-constant map, then there exists a positive integer p such that $x_p\neq 0$ and $x=x_0+\sum_{t\geqq p}x_t$. If there are infinitely many nonzero x_t's in the decomposition (2.1), we put $q=\infty$. Otherwise, we let q be the largest integer such that $x_q\neq 0$ in the decomposition (2.1). In both cases we have

$$x=x_0+\sum_{t=p}^{q} x_t\,. \tag{2.2}$$

We call $[p, q]$ the *order of the map* x. The map is said to be of *finite type* if q is finite. Otherwise, the map is said to be of *infinite type*. More precisely, the map x is said to be of *k-type* ($k\in N\cup\{\infty\}$) if there exist exactly k nonzero x_t's ($t\geqq 1$) in the decomposition (2.2) (cf. [1, 2, 4]).

If $x: M\to E^m$ is an immersion and M equipped with the induced metric, then the submanifold M is said to be of k-type if the immersion does.

3. Total Tension.

If $\sigma: M\to N$ is a map between Riemannian manifolds, then the energy $e(\sigma)$ of σ is the real-valued function on M given by

$$e(\sigma)=\frac{1}{2}\operatorname{trace}(\sigma^* g')\,, \tag{3.1}$$

where g' is the metric on N. The energy $E(\sigma)$ of σ is defined by

$$E(\sigma)=\int_M e(\sigma)dV\,. \tag{3.2}$$

The Euler-Lagrange operator associated with E shall be written $\tau(\sigma)=\operatorname{div}(d\sigma)$ and called the *tension field* of σ. A map σ is harmonic if its tension field vanishes identically. The total tension of the map σ is defined by

$$\mathcal{T}(\sigma)=\int_M \|\tau\|^2 dV\,. \tag{3.3}$$

For a map $x: M\to E^m$, the *moment* of x is given by

(3.4) $$\mathcal{M}(x)=\int_M \langle x-x_0,\ x-x_0\rangle dV\,.$$

It is easy to verify that the moment of x is independent of the choice of the Euclidean coordinate system on E^m.

In this section, we give two best possible estimates of the total tension of a map $x: M\to E^m$.

The following result gives a *best possible lower bound* of total tension.

THEOREM 1. *Let $x: M\to E^m$ be a smooth non-constant map from a compact n-dimensional Riemannian manifold M into E^m. Then we have*

(3.5) $$\int_M \|\tau\|^2 dV \geqq 2(\lambda_1+\lambda_2)E(x)-\lambda_1\lambda_2\mathcal{M}(x)\,.$$

Equality sign holds if and only if x is either of 1*-type and of order* [1, 1] *or of order* [2, 2] *or x is of* 2*-type and of order* [1, 2].

Proof. Let $x: M\to E^m$ be a smooth non-constant map from M into E^m. Then we have

(3.6) $$x=x_0+\sum_{t=p}^{q} x_t\,,$$

where $[p, q]$ is the order of the map x. Since Δ is self-adjoint, we have $(x_t, x_s)=0$ for $t\neq s$. Thus, (3.6) gives

(3.7) $$\mathcal{M}(x)=(x-x_0,\ x-x_0)=\int_M \langle x-x_0,\ x-x_0\rangle dV=\sum_{t=p}^{q}(x_t,\ x_t)\,.$$

Moreover, from (3.1), (3.2) and (3.6), we find

(3.8) $$2E(x)=(dx,\ dx)=(x,\ \delta dx)=(x,\ \Delta x)\,,$$

which implies

(3.9) $$2E(x)=\sum_{t=p}^{q}\lambda_t(x_t,\ x_t)\,.$$

From the definition of tension field one may prove (cf. [7])

(3.10) $$\Delta x=-\tau(x)\,.$$

Thus, by applying (3.3), (3.6) and (3.10), we find

(3.11) $$\mathcal{T}(x)=(\Delta x,\ \Delta x)=\sum_{t=p}^{q}\lambda_t^2(x_t,\ x_t)\,.$$

Combining (3.7), (3.9) and (3.11), we obtain

$$\mathcal{T}(x)-2(\lambda_p+\lambda_{p+1})E(x)+\lambda_p\lambda_{p+1}\mathcal{M}(x) \tag{3.12}$$
$$=\sum_{t=p}^{q}(\lambda_t-\lambda_p)(\lambda_t-\lambda_{p+1})(x_t, x_t)\geqq 0\,.$$

This implies

$$\mathcal{T}(x)\geqq 2(\lambda_p+\lambda_{p+1})E(x)-\lambda_p\lambda_{p+1}\mathcal{M}(x)\,. \tag{3.13}$$

Since p is always greater than or equal to one, (3.13) gives inequality (3.5). If the equality sign of (3.5) holds, then (3.12) becomes an equality with $p=1$. Thus, from (3.12), we see that all of the x_t, $t>0$, vanish except $t=1, 2$. If either $x_1=0$ or $x_2=0$, x is of 1-type and of order [2, 2] or [1, 1]. Otherwise, x is of 2-type with order [1, 2]. This completes the proof of the theorem.

Remark 1. Given a compact Riemannian manifold M, there exist infinitely many smooth non-constant maps from M into E^m which satisfy equality sign of (3.5).

If x is of finite type, we also have the following *best possible upper bound* of total tension.

THEOREM 2. *If $x: M\to E^m$ is a smooth non-constant map of finite type, then we have*

$$\int_M \|\tau\|^2 dV\leqq 2(\lambda_p+\lambda_q)E(x)-\lambda_p\lambda_q\mathcal{M}(x)\,. \tag{3.15}$$

Equality sign holds if and only if x is of 1-type ($p=q$) or of 2-type.

Since this theorem can be proved in a way similar to that of Theorem 1, so we omit the proof.

4. Some Applications.

In this section we give some applications of Theorem 1. The following result gives a best possible estimate of total mean curvature for spherical submanifolds.

THEOREM 3. *Let $x: M\to S^{m-1}(r)\subset E^m$ be an isometric immersion of a compact n-dimensional Riemannian manifold M into a hypersphere $S^{m-1}(r)$ of radius r. Then the mean curvature vector H of M in E^m satisfies*

$$\int_M |H|^2 dV\geqq (1/n^2)\{n(\lambda_1+\lambda_2)-\lambda_1\lambda_2 r^2\}\operatorname{vol}(M)\,. \tag{4.1}$$

Equality sign holds if and only if M has constant mean curvature $|H|$ and M is of order [1, 1], [1, 2] or [2, 2] aud M is mass-symmetric.

Proof. Let $x: M \to S^{m-1}(r) \subset E^m$ be an isometric immersion. Then we have

$$\Delta x = -nH. \tag{4.2}$$

Thus, by combining with (3.10), we find

$$\langle \tau, \tau \rangle = n^2 \langle H, H \rangle. \tag{4.3}$$

On the other hand, since x is isometric, the energy $E(x)$ of x is given by

$$2E(x) = n \operatorname{vol}(M). \tag{4.4}$$

Therefore, by Theorem 1, (4.3) and (4.4), we obtain

$$n^2 \int_M |H|^2 dV \geqq n(\lambda_1 + \lambda_2) \operatorname{vol}(M) - \lambda_1 \lambda_2 \mathcal{M}(x). \tag{4.5}$$

Without loss of generality, we may assume that the hypersphere $S^{m-1}(r)$ is centered at the origin of E^m. Since M is immersed in $S^{m-1}(r)$, we have

$$\mathcal{M}(x) = \int_M \langle x, x \rangle dV - \int_M \langle x_0, x_0 \rangle dV \leqq r^2 \operatorname{vol}(M). \tag{4.6}$$

equality holding if and only if $x_0 = 0$. From (4.5) and (4.6), we get inequality (4.1).

If the equality sign of (4.1) holds, then both equality signs of (4.5) and (4.6) hold. Thus, x is of order [1, 1], [1, 2] or [2, 2] and $x_0 = 0$ *i.e.*, M is mass-symmetric in $S^{m-1}(r)$.

If x is of order [1, 1], we have $x = x_1$. Thus, (4.2) gives $-nH = \lambda_1 x$ which implies $n^2 \langle H, H \rangle = \lambda_1^2 r^2$. Thus, M has constant mean curvature. Similarly, if x is of order [2, 2], we have $n^2 \langle H, H \rangle = \lambda_2^2 r^2$ which also shows that M has constant mean curvature.

If x is of order [1, 2], then we have $x = x_1 + x_2$. Thus, $\Delta^2 x = (\lambda_1 + \lambda_2) \Delta x - \lambda_1 \lambda_2 x = -n(\lambda_1 + \lambda_2) H - \lambda_1 \lambda_2 x$. On the other hand, since M lies in $S^{m-1}(r)$, we also have $H = H' - (1/r)x$, where H' denotes the mean curvature vector of M in $S^{m-1}(r)$. Therefore, we find

$$\langle \Delta^2 x, x \rangle = nr(\lambda_1 + \lambda_2) - \lambda_1 \lambda_2 r^2 \tag{4.7}$$

which is a constant. On the other hand, by applying Lemma 4.2 of [4, p. 273], we also have

$$\langle \Delta^2 x, x \rangle = -\langle n \Delta H, x \rangle = n^2 \langle H, H \rangle / r^2. \tag{4.8}$$

Thus, from (4.7) and (4.8), we see that the mean curvature of M in E^m is also constant.

The converse follows easily from Theorem I, (4.3), (4.4), (4.5) and (4.6). This completes the proof of Theorem 3.

In the following, $\boldsymbol{F}$ denotes the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers, or the field $\boldsymbol{H}$ of quaternions. We put $d=1$, 2 or 4 according to $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$, respectively. We denote by $\boldsymbol{F}P^m$ the m-dimensional projective space over $\boldsymbol{F}$, and of constant sectional curvature 1 if $\boldsymbol{F}=\boldsymbol{R}$, of constant holomorphic sectional curvature 4 if $\boldsymbol{F}=\boldsymbol{C}$, and of constant quaternionic sectional curvature 4 if $\boldsymbol{F}=\boldsymbol{H}$, respectively.

By applying Theorem 3 and an inequality derived in [3], we have the following eigenvalue inequality for compact minimal submanifolds in projective spaces.

THEOREM 4. *Let M be a compact n-dimensional Riemannian manifold. If M admits an isometric minimal immersion into $\boldsymbol{F}P^m$, then we have*

$$\frac{m}{2(m+1)}\lambda_1\lambda_2 \geqq n(\lambda_1+\lambda_2-2n-2d)\,. \tag{4.9}$$

If $\boldsymbol{F}=\boldsymbol{H}$, then the equality holds if and only if $n=4m$ and $M=\boldsymbol{H}P^m$. If $\boldsymbol{F}=\boldsymbol{C}$, then the equality holds if and only if M is one of the following Einstein Hermitian symmetric spaces: $\boldsymbol{C}P^k(4)$, $\boldsymbol{C}P^k(2)$, Q^k, $\boldsymbol{C}P^k(4)\times\boldsymbol{C}P^k(4)$, $U(k+2)/U(k)\times U(2)$ $(k>2)$, $SO(10)/U(5)$, *and* $E_6/\mathrm{Spin}\,(10)\times T$, *with an appropriate metric, and m is given by* k, $k(k+3)/2$, $k+1$, $k(k+2)$, $k(k+3)/2$, 15, *and* 26, *respectively.*

Proof. Let $z={}^t(z_0, \cdots, z_m)\in\boldsymbol{F}^{m+1}$. We denote by $H(m+1;\boldsymbol{F})$ the space of all $(m+1)\times(m+1)$ Hermitian matrices over $\boldsymbol{F}$. On $H(m+1;\boldsymbol{F})$ we define an inner product $\langle\,,\rangle$ by $\langle A, B\rangle=(1/2)\,\mathrm{Re\,tr}\,(AB)$. On $\boldsymbol{F}^{m+1}$ we consider the metric $\langle z, z'\rangle=\mathrm{Re}\,({}^t\bar{z}z')$, ${}^t(\ )$ denotes the transpose. Let $S^{(m+1)d-1}$ denote the unit hypersphere of $\boldsymbol{F}^{m+1}$ defined by $\{z\in\boldsymbol{F}^{m+1}: \langle z, z\rangle=1\}$. Then the projective m-space $\boldsymbol{F}P^m$ can be regarded as the quotient space of the unit hypersphere obtained by identifying ${}^t(z_0, \cdots, z_m)$ with ${}^t(cz_0, \cdots, cz_m)$ with $c\in\boldsymbol{F}$ and $|c|=1$.

Define a mapping $\bar{\rho}: S^{(m+1)d-1}\to H(m+1;\boldsymbol{F})$ by

$$\bar{\rho}(z)=zz^*\,, \tag{4.10}$$

where $z^*={}^t\bar{z}$. If $\boldsymbol{F}=\boldsymbol{R}$, $\bar{\rho}$ defines an isometric immersion of S^m into $H(m+1;\boldsymbol{R})$ and it induces an isometric imbedding ρ of $\boldsymbol{R}P^m$ into $H(m+1;\boldsymbol{R})$. If $\boldsymbol{F}=\boldsymbol{C}$ or $\boldsymbol{H}$, $\bar{\rho}$ induces an isometric imbedding ρ of $\boldsymbol{F}P^m$ into $H(m+1;\boldsymbol{F})$.

If M admits a minimal isometric immersion into $\boldsymbol{F}P^m$, then by regarding $\boldsymbol{F}P^m$ as a submanifold of $\boldsymbol{H}(m+1;\boldsymbol{F})$ via ρ, we have the following best possible inequality of the mean curvature of M in $H(m+1;\boldsymbol{F})$ (Lemma 2 of [3] or Lemma 6.5 of [4, p. 152]):

$$|H|^2\leqq\frac{2(n+d)}{n}\,, \tag{4.11}$$

where H denotes the mean curvature vector of M in $H(m+1;\boldsymbol{F})$. Moreover, from Lemma 2 of [3], we know that equality sign of (4.11) holds if and only if M is a quaternion submanifold if $\boldsymbol{F}=\boldsymbol{H}$; M is a complex submanifold if $\boldsymbol{F}=\boldsymbol{C}$; and if $\boldsymbol{F}=\boldsymbol{R}$, then the equality sign of (4.11) holds automatically. On the other hand, it is known that $\boldsymbol{F}P^m$ is imbedded in a hypersphere of $H(m+1;\boldsymbol{F})$ with radius $r=\sqrt{m/2(m+1)}$ via ρ ([12] or cf. [3, 4]). Thus, by applying Theorem 3 and inequality (4.11), we may obtain

$$2(n+d)n \geqq n(\lambda_1+\lambda_2)-\frac{m}{2(m+1)}\lambda_1\lambda_2\,. \tag{4.12}$$

This implies (4.9).

If $\boldsymbol{F}=\boldsymbol{H}$ and the equality sign of (4.9) holds, then the equality sign of (4.11) holds. Hence, Lemma 2 of [3] shows that M is a quaternionic submanifold of $\boldsymbol{H}P^m$. Because the only quaternionic submanifolds of $\boldsymbol{H}P^m$ are quaternionic totally geodesic submanifolds. Thus, we conclude that M is a $\boldsymbol{H}P^{n/4}$. Now, since λ_1 and λ_2 of $\boldsymbol{H}P^k$ are given by $8(k+1)$ and $8(2k+3)$, respectively. Thus, we obtain $n=4m$. The converse of this is easy to verify.

If $\boldsymbol{F}=\boldsymbol{C}$ and the equality sign of (4.9) holds, then the equality sign of (4.11) holds. Thus, Lemma 2 of [3] implies that M is a Kaehler submanifold of $\boldsymbol{C}P^m$. Moreover, from Theorem 3, we see that either M is of 1-type in $H(m+1;\boldsymbol{C})$ or M is of 2-type and of order [1, 2]. If M is of 1-type, then by a result of Ros [10], we know that M is a totally geodesic $\boldsymbol{C}P^k$ $(2k=n)$. Since λ_1 and λ_2 of $\boldsymbol{C}P^k$ are given by $4(k+1)$ and $8(k+2)$, respectively, we find $n=2m$. If M is of 2-type and of order [1, 2], then we may apply a result of Ros-Udagawa [10, 13] about the classification of compact Kaehler submanifolds of $\boldsymbol{C}P^m$ of order [1, 2]. Such submanifolds are exactly non-totally geodesic Kaehler submanifolds which are Einsteinian and with parallel second fundamental form (cf. Proposition 3 of [13]). Furthermore, such Kaehler submanifolds were classified by Nakagawa and Takagi [9]; they are $\boldsymbol{C}P^k(2)$, Q^k, $\boldsymbol{C}P^k(4)\times\boldsymbol{C}P^k(4)$, $U(k+2)/U(k)\times U(2)$ $(k>2)$, $SO(10)/U(5)$ and $E_6/\mathrm{Spin}(10)\times T$ which lie fully in $\boldsymbol{C}P^m$ with m given respectively by $k(k+3)/2$, $k+1$, $k(k+2)$, $k(k+3)/2$, 15 and 26, respectively. Conversely, if M is one of Einstein Hermitian symmetric spaces and m is the corresponding integer, then by the known values of λ_1 and λ_2 of these spaces (see Table 1 below), we see that the equality sign of (4.9) holds. This completes the proof.

Remark 2. If $\boldsymbol{F}=\boldsymbol{R}$ and the equality sign of (4.9) holds, then M is of order [1, 2] in $H(m+1;\boldsymbol{R})$ by Theorem 3. If M is a projective space $\boldsymbol{F}P^k$ or the Cayley plane and if $\sigma: M\to S^N$ is the first standard imbedding of M, then it is clear that the composite immersion $\rho\circ\sigma: M\to S^N\to H(n+1;\boldsymbol{R})$ is of order [1, 2]. Moreover, if σ is full, then the equality sign of (4.9) holds. In view of Theorem 4 and [1], it seems to be interesting to classify all compact minimal submanifolds of $\boldsymbol{R}P^m$ which satisfy the equality sign of (4.9).

From Theorem 4 we also have the following.

COROLLARY 1. *If M is a compact n-dimensional minimal submanifold of $S^m(1)$, then we have*

$$m\lambda_1\lambda_2 \geqq 2n(m+1)\{\lambda_1+\lambda_2-2n-2\}. \tag{4.13}$$

Proof. If M admits a minimal isometric immersion into $S^m(1)$, then it admits a minimal isometric immersion into $\boldsymbol{RP}^m$. Thus, (4.13) follows immediately from Theorem 4.

Remark 3. Ros [11] obtain a best possible inequality between λ_1 and λ_2 similar to (4.13) with an additional assumption that M admits an order 1 minimal immersion in a sphere (see, also [8]).

Remark 4. In [10, 13], the Einstein Hermitian symmetric spaces given in Table 1 were characterized by their spectrum among all compact *Kaehler* submanifolds of $\boldsymbol{CP}^m$. By applying Theorem 4, we see that these manifolds can be characterized by their spectrum among all compact *minimal* submanifolds of $\boldsymbol{CP}^m$.

Table 1.

Submanifold	n	m	λ_1	λ_2
$\boldsymbol{CP}^k(4)$	$2k$	k	$4(k+1)$	$8(k+2)$
$\boldsymbol{CP}^k(2)$	$2k$	$\frac{1}{2}k(k+3)$	$2(k+1)$	$4(k+2)$
Q^k	$2k$	$k+1$	$4k$	$4(k+2)$
$\boldsymbol{CP}^k(4)\times\boldsymbol{CP}^k(4)$	$4k$	$4(k+2)$	$4(k+1)$	$8(k+1)$
$U(k+2)/U(k)\times U(2)$ $k>2$	$4k$	$\frac{1}{2}k(k+3)$	$4(k+2)$	$8(k+1)$
$SO(10)/U(5)$	20	15	32	48
$E_6/\mathrm{Spin}\,(10)\times T$	32	26	48	72

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