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# ON A FAMILY OF INTEGRAL OPERATORS RELATED TO FRACTIONAL CALCULUS

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## I. Introduction.

Let $\mathcal{F}$ denote the class of analytic functions $f$ regular in the unit disk $E=\{|z|<1\}$ and normalized at the origin by $f(0)=0$ and $f'(0)=1$. On the other hand, let $\sigma$ be a probability measure supported by the unit interval $I=[0, 1]$. Then the linear integral operator $\mathcal{L}$ is defined on $\mathcal{F}$ by the expression

$$\mathcal{L}f(z)=\int_I \frac{f(zt)}{t}\,d\sigma(t).$$

It is readily seen that $f\in\mathcal{F}$ implies $\mathcal{L}f\in\mathcal{F}$.

Let the Taylor expansion of $f\in\mathcal{F}$ be given by

$$f(z)=\sum_{\nu=1}^{\infty} c_\nu z^\nu \qquad \text{with } c_1=1.$$

Then substitution followed by termwise integration yields

$$\mathcal{L}f(z)=\sum_{\nu=1}^{\infty} \alpha_\nu c_\nu z^\nu$$

where $\{\alpha_\nu\}_{\nu=1}^{\infty}$ is the moment sequence with respect to $\sigma$ defined by

$$\alpha_\nu=\int_I t^{\nu-1}d\sigma(t) \qquad (\nu=1, 2, \cdots),$$

which is decreasing and nonnegative; in particular, $\alpha_1=1$.

The iteration $\{\mathcal{L}^n\}_{n=0}^{\infty}$ arises automatically by $\mathcal{L}^0=\mathrm{id}$, $\mathcal{L}^n=\mathcal{L}\mathcal{L}^{n-1}$ $(n=1, 2, \cdots)$ or by

$$\mathcal{L}^n f(z)=\sum_{\nu=1}^{\infty} \alpha_\nu{}^n c_\nu z^\nu.$$

We discussed in [2, 3] the problem of interpolating the sequence $\{\mathcal{L}^n\}$ into a family $\{\mathcal{L}^\lambda\}$ depending on a continuous parameter $\lambda$ in such a way that the additivity $\mathcal{L}^\lambda\mathcal{L}^\mu=\mathcal{L}^{\lambda+\mu}$ remains valid. We then derived several properties of the family thus introduced, and observed the simplest distinguished case gener-

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ated by a special probability measure $\sigma(t)=t$ especially in detail.

In the present paper we shall mainly observe the case generated by the probability measure involving a real parameter $a$ defined by

$$\sigma(t\,;a)=t^{a} \qquad \text{with } a>0\,.$$

The measure $\sigma(t\,;1)=t$ is indeed included as a particular one but it plays an exceptional role occasionally in certain sense.

## 2. Family generated by $t^a$.

We now suppose that a probability measure $\sigma$ possesses the density $\rho$:

$$\sigma(t)=\int_0^t \rho(\tau)d\tau\,; \qquad \rho(\tau)\geqq 0\,, \qquad \int_I \rho(\tau)d\tau=1\,.$$

The operator generated by this measure will be denoted by $\mathcal{L}[\rho]$:

$$\mathcal{L}[\rho]f(z)=\int_I \frac{f(zt)}{t}\rho(t)dt\,.$$

We begin with the following lemma:

LEMMA 1. *The product of two operators becomes* $\mathcal{L}[p]\mathcal{L}[q]=\mathcal{L}[\rho]$ *where*

$$\rho(t)=\int_t^1 p(s)q\Big(\frac{t}{s}\Big)\frac{ds}{s}\,.$$

*Proof.* Direct calculation shows that

$$\begin{aligned}\mathcal{L}[p]\mathcal{L}[q]f(z)&=\int_I p(s)\frac{ds}{s}\int_I f(zs\tau)q(\tau)\frac{d\tau}{\tau}\\ &=\int_I p(s)\frac{ds}{s}\int_0^s f(zt)q\Big(\frac{t}{s}\Big)\frac{dt}{t}=\int_I f(zt)\frac{dt}{t}\int_t^1 p(s)q\Big(\frac{t}{s}\Big)\frac{ds}{s}\\ &=\int_I \frac{f(zt)}{t}\rho(t)dt=\mathcal{L}[\rho]f(z)\end{aligned}$$

with $\rho$ stated in the lemma.

REMARK. If we put $t=e^{-v}$ and accordingly $p(t)=P(v)$, $q(t)=Q(v)$ and $\rho(t)=R(v)$, then the expression in the lemma becomes

$$R(v)=\int_0^v P(u)Q(v-u)du\,.$$

This shows that $R$ is the convolution of $P$ and $Q$: $R=P*Q$.

LEMMA 2. *Let every member* $\mathcal{L}^{\lambda}$ *of the family* $\{\mathcal{L}^{\lambda}\}_{\lambda>0}$ *be generated by a* *measure with the density* $\rho_\lambda$: $\mathcal{L}^\lambda=\mathcal{L}[\rho_\lambda]$. *Then the additivity* $\mathcal{L}^\lambda\mathcal{L}^\mu=\mathcal{L}^{\lambda+\mu}$ *is characterized by*

$$\int_t^1\rho_\lambda(s)\rho_\mu\left(\frac{t}{s}\right)\frac{ds}{s}=\rho_{\lambda+\mu}(t)\,.$$

*Proof.* In view of Lemma 1 we have $\mathcal{L}[\rho_\lambda]\mathcal{L}[\rho_\mu]=\mathcal{L}[\rho]$ with

$$\rho(t)=\int_t^1\rho_\lambda(s)\rho_\mu\left(\frac{t}{s}\right)\frac{ds}{s}\,.$$

Hence the additivity is characterized by the condition that $\mathcal{L}[\rho]f(z)=\mathcal{L}[\rho_{\lambda+\mu}]f(z)$ holds for any $f\in\mathfrak{F}$. This condition applied, for instance, to $f(z)=z(1-z)^{-1}\in\mathfrak{F}$ yields, by comparing the coefficients of $z^\nu$,

$$\int_I t^{\nu-1}\rho(t)dt=\int_I t^{\nu-1}\rho_{\lambda+\mu}(t)dt\,.$$

In view of the unicity of the solution of moment problem, we obtain $\rho=\rho_{\lambda+\mu}$. Conversely, if $\rho=\rho_{\lambda+\mu}$, it is evident that the additivity holds.

Now, we observe the probability measure defined by $\sigma(t\,;a)=t^a$ with $a>0$.

THEOREM 1. *The additive family of operators generated by* $\sigma(t\,;a)=t^a$ *with* $a>0$ *is given by the probability measure* $\sigma_\lambda(t\,;a)$ *with the density* $\rho_\lambda(t\,;a)$ *defined by*

$$\sigma_\lambda(t\,;a)=\int_0^t\rho_\lambda(\tau\,;a)d\tau\,,\qquad \rho_\lambda(t\,;a)=\frac{a^\lambda}{\Gamma(\lambda)}t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda-1}.$$

*Proof.* The condition stated in Lemma 2 can be verified by direct calculation. In fact, we have

$$\begin{aligned}
&\int_t^1\rho_\lambda(s\,;a)\rho_\mu\left(\frac{t}{s}\,;a\right)\frac{ds}{s}\\
&=\frac{a^{\lambda+\mu}}{\Gamma(\lambda)\Gamma(\mu)}\int_t^1 s^{a-1}\left(\log\frac{1}{s}\right)^{\lambda-1}\left(\frac{t}{s}\right)^{a-1}\left(\log\frac{s}{t}\right)^{\mu-1}\frac{ds}{s}\\
&=\frac{a^{\lambda+\mu}}{\Gamma(\lambda)\Gamma(\mu)}t^{a-1}\int_t^1\left(\log\frac{1}{s}\right)^{\lambda-1}\left(\log\frac{s}{t}\right)^{\mu-1}\frac{ds}{s}\\
&=\frac{a^{\lambda+\mu}}{\Gamma(\lambda)\Gamma(\mu)}t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda+\mu-1}\int_0^1 u^{\lambda-1}(1-u)^{\mu-1}du \qquad \left[\log\frac{1}{s}=u\log\frac{1}{t}\right]\\
&=\frac{a^{\lambda+\mu}}{\Gamma(\lambda)\Gamma(\mu)}t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda+\mu-1}\cdot\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}=\rho_{\lambda+\mu}(t\,;a)\,.
\end{aligned}$$

The assertion may be proved alternatively as follows. In fact, since the moment with respect to $\sigma(t\,;a)$ is equal to

$$\alpha_\nu(a)=\int_I t^{\nu-1}d\sigma(t\,;\,a)=\frac{a}{\nu+a-1}\,,$$

it is sufficient to show that the moment with respect to the measure $\sigma_\lambda(t\,;\,a)$ stated in the theorem is equal to $\alpha_\nu(a)^\lambda$, what is an immediate consequence of a familiar formula

$$\int_I t^{\kappa-1}\Big(\log\frac{1}{t}\Big)^{\lambda-1}dt=\frac{\Gamma(\lambda)}{\kappa^\lambda}\,.$$

According to Theorem 1 we shall denote in the following lines $\mathcal{L}[\rho_\lambda(t\,;\,a)]$ briefly by $\mathcal{L}(a)^\lambda$:

$$\mathcal{L}(a)^\lambda f(z)=\frac{a^\lambda}{\Gamma(\lambda)}\int_I f(zt)t^{a-2}\Big(\log\frac{1}{t}\Big)^{\lambda-1}dt\,.$$

The behaviors of the general family $\{\mathcal{L}^\lambda\}$ as $\lambda\to+0$ and $\lambda\to\infty$ have been shown in [2]. But, in case of $\sigma(t\,;\,a)$, since the extreme exceptional cases do not appear, we can state the following theorem:

THEOREM 2. *The limit relations*

$$\lim_{\lambda\to+0}\mathcal{L}(a)^\lambda f(z)=f(z)\qquad\textit{and}\qquad\lim_{\lambda\to\infty}\mathcal{L}(a)^\lambda f(z)=z$$

*hold for every $f\in\mathfrak{F}$ in $E$ uniformly in the wider sense.*

On the other hand, the behaviors as $a\to+0$ and $a\to\infty$ become as follows:

THEOREM 3. *The limit relations*

$$\lim_{a\to+0}\mathcal{L}(a)^\lambda f(z)=z\qquad\textit{and}\qquad\lim_{a\to\infty}\mathcal{L}(a)^\lambda f(z)=f(z)$$

*hold for every $f\in\mathfrak{F}$ in $E$ uniformly in the wider sense.*

*Proof.* Let $z$ be restricted on any fixed compact in $E$. Then both $|f(zt)/t-z|$ and $|f(zt)/t-f(z)|$ possess for every $t\in I$ a bound $M$, say. First, we have

$$\mathcal{L}(a)^\lambda f(z)-z=\frac{a^\lambda}{\Gamma(\lambda)}\int_I\Big(\frac{f(zt)}{t}-z\Big)t^{a-1}\Big(\log\frac{1}{t}\Big)^{\lambda-1}dt\,.$$

For any $\varepsilon>0$ there exists $\tau\in(0,1)$ such that $|f(zt)/t-z|<\varepsilon/2$ as $0\leqq t<\tau$, and hence for $a<1$

$$|\mathcal{L}(a)^\lambda f(z)-z|<\frac{\varepsilon}{2}\frac{a^\lambda}{\Gamma(\lambda)}\int_0^\tau t^{a-1}\Big(\log\frac{1}{t}\Big)^{\lambda-1}dt+M\tau^{a-1}\frac{a^\lambda}{\Gamma(\lambda)}\int_\tau^1\Big(\log\frac{1}{t}\Big)^{\lambda-1}dt\,.$$

The first summand of this estimate is always less than $\varepsilon/2$, while the second summand becomes less than $\varepsilon/2$ provided $a$ is sufficiently near to zero. This leads to the first relation in the theorem. Next, we have

$$\mathcal{L}(a)^{\lambda}f(z)-f(z)=\frac{a^{\lambda}}{\Gamma(\lambda)}\int_{I}\left(\frac{f(zt)}{t}-f(z)\right)t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda-1}dt.$$

For any $\varepsilon>0$ there exists $\tau\in(0, 1)$ such that $|f(zt)/t-f(z)|<\varepsilon/2$ as $1-\tau<t\leqq 1$, and hence for $a>1$

$$|\mathcal{L}(a)^{\lambda}f(z)-f(z)|<M(1-\tau)^{a-1}\frac{a^{\lambda}}{\Gamma(\lambda)}\int_{0}^{1-\tau}\left(\log\frac{1}{t}\right)^{\lambda-1}dt$$

$$+\frac{\varepsilon}{2}\frac{a^{\lambda}}{\Gamma(\lambda)}\int_{1-\tau}^{1}t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda-1}dt.$$

Since the second summand of this estimate is always less than $\varepsilon/2$ and the first summand becomes less than $\varepsilon/2$ for $a$ large enough, the second relation follows.

Though the proof given here has been based on the integral representation for $\mathcal{L}(a)^{\lambda}f(z)$, a rather brief proof may be given by referring to its series expansion.

### 3. Relation to integration operator.

We have pointed out in [2] that the operator $\mathcal{L}$ with general $\sigma$ and the differentiation with respect to $\log z$ are commutative. Further, in particular case generated by $\sigma(t)=\sigma(t, 1)$, the operation $\mathcal{L}(1)^{\lambda}$ can be represented in the form

$$\mathcal{L}(1)^{\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_{\infty}^{\log z}f(\zeta)(\log z-\log\zeta)^{\lambda-1}d\log\zeta.$$

Here the integration is taken along the half straight line on the $\log\zeta$-plane which is parallel to the real axis and contained in the left half-plane $\{\mathrm{Re}\log\zeta<0\}$. Thus, this operator coincides with the fractional integration of order $\lambda$ with respect to $\log z$. In particular, $\mathcal{L}(1)$ is just the inverse operator of $d/d\log z$.

The last-mentioned fact is peculiar to the case $a=1$. The corresponding property in the case $a\neq 1$ is stated as in the following theorem.

THEOREM 4. *The operator $\mathcal{L}(a)$ with $a\neq 1$ coincides with the integration with respect to $w=a(a-1)^{-1}z^{a-1}$ followed by multiplication of $z^{-(a-1)}$, the non-integral power being understood to mean the principal branch. More precisely, we have*

$$\mathcal{L}(a)f(z)=\begin{cases}\dfrac{1}{z^{a-1}}\displaystyle\int_{\infty}^{w}F(\omega)d\omega & (0<a<1),\\[2ex] \dfrac{1}{z^{a-1}}\displaystyle\int_{0}^{w}F(\omega)d\omega & (a>1),\end{cases}$$

*where $w=a(a-1)^{-1}z^{a-1}$, $F(w)=f((a^{-1}(a-1)w)^{1/(a-1)})$ and the integration paths are* *the half straight line* $\{\arg\omega=\pi-(1-a)\arg z,\ \infty>|\omega|>|w|\}$ *for* $0<a<1$ *and the segment* $\{\arg\omega=(a-1)\arg z,\ 0<|\omega|<|w|\}$ *for* $a>1$, *respectively.*

*Proof.* The operator $\mathcal{L}(a)$ is, by definition, given by

$$\mathcal{L}(a)f(z)=a\int_I \frac{f(zt)}{t}t^{a-1}dt=\frac{a}{z^{a-1}}\int_0^z f(\zeta)\zeta^{a-2}d\zeta,$$

the last integration being taken along the segment from 0 to $z$. We have only to change the variable of the last integral by $d\omega=a\zeta^{a-2}d\zeta$, or more concretely, by $\omega=a(a-1)^{-1}\zeta^{a-1}$. When $\zeta$ runs along the segment from 0 to $z$, $\omega$ runs along the respective integration path stated in the theorem.

It will be seen that the relation

$$\frac{d}{d\log z}\mathcal{L}(a)f(z)=af(z)-(a-1)\mathcal{L}(a)f(z)$$

holds for any $a>0$. This may be regarded as a straightforward generalization of the already mentioned relation $(d/d\log z)\mathcal{L}(1)f(z)=f(z)$ corresponding to $a=1$. Here we state it in slightly general form:

THEOREM 5. *For any* $a>0$ *and* $\lambda\geqq 1$, *we have*

$$\frac{d}{d\log z}\mathcal{L}(a)^\lambda=a\mathcal{L}(a)^{\lambda-1}-(a-1)\mathcal{L}(a)^\lambda,$$

$\mathcal{L}(a)^0$ *being understood to be the identity operator.*

*Proof.* By differentiating the defining equation of $\mathcal{L}(a)^\lambda f(z)$, we obtain

$$\frac{d}{d\log z}\mathcal{L}(a)^\lambda f(z)=\frac{a^\lambda}{\Gamma(\lambda)}\int_I zf'(zt)t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda-1}dt$$

which becomes after integration by parts

$$\frac{d}{d\log z}\mathcal{L}(a)^\lambda f(z)=\frac{a^\lambda}{\Gamma(\lambda)}\left[f(zt)t^{a-1}\left(\log\frac{1}{t}\right)^{\lambda-1}\right]_0^1$$

$$-\int_I f(zt)\left(-(\lambda-1)t^{a-2}\left(\log\frac{1}{t}\right)^{\lambda-2}+(a-1)t^{a-2}\left(\log\frac{1}{t}\right)^{\lambda-1}\right)dt.$$

Here we remember $f\in\mathfrak{F}$ and $a>0$. We get for $\lambda=1$

$$\frac{d}{d\log z}\mathcal{L}(a)f(z)=a\left(f(z)-(a-1)\int_I f(zt)t^{a-2}dt\right)$$

$$=af(z)-(a-1)\mathcal{L}(a)f(z),$$

while we obtain for $\lambda>1$

$$\frac{d}{d\log z}\mathcal{L}(a)^{\lambda}f(z)=\frac{a^{\lambda}}{\Gamma(\lambda)}\Big((\lambda-1)\int_I f(zt)t^{a-2}\Big(\log\frac{1}{t}\Big)^{\lambda-2}dt$$

$$-(a-1)\int_I f(zt)t^{a-2}\Big(\log\frac{1}{t}\Big)^{\lambda-1}dt\Big)$$

$$=a\mathcal{L}(a)^{\lambda-1}f(z)-(a-1)\mathcal{L}(a)^{\lambda}f(z).$$

In the following lines we shall consider the relation of $\mathcal{L}(a)$ to the ordinary integration operator $\mathcal{J}$ defined by

$$\mathcal{J}f(z)=\int_0^z f(\zeta)d\zeta.$$

For that purpose we attempt to derive the expression for $\mathcal{L}(a)$ in terms of $\mathcal{J}$ and its iterations. For the sake of brevity we make use of Pochhammer's symbol

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\prod_{j=0}^{n-1}(x+j)\qquad(n=0,1,\cdots),$$

the empty product denoting unity; in particular, $(x)_0=1$ even for $x=0$.

THEOREM 6. *For any $a>0$ we have*

$$\mathcal{L}(a)=a\sum_{\kappa=1}^{\infty}\frac{(2-a)_{\kappa-1}}{z^k}\mathcal{J}^{\kappa}.$$

*In particular, when $a=k>1$ is an integer, the right hand expression reduces to finite sum consisting of the beginning $k-1$ terms.*

*Proof*. Since $|z-\zeta|<|z|$ holds on the integration path in the expression for $\mathcal{L}(a)$ except at $\zeta=0$, we have

$$\zeta^{a-2}=z^{a-2}\Big(1-\frac{z-\zeta}{z}\Big)^{a-2}$$

$$=z^{a-2}\sum_{\kappa=0}^{\infty}(-1)^{\kappa}\binom{a-2}{\kappa}\Big(\frac{z-\zeta}{z}\Big)^{\kappa}$$

$$=z^{a-2}\sum_{\kappa=1}^{\infty}\frac{(2-a)_{\kappa-1}}{(\kappa-1)!}\frac{1}{z^{\kappa-1}}(z-\zeta)^{\kappa-1}.$$

Substitution followed by termwise integration yields

$$\mathcal{L}(a)f(z)=a\sum_{\kappa=1}^{\infty}\frac{(2-a)_{\kappa-1}}{z^{\kappa}}\frac{1}{(\kappa-1)!}\int_0^z f(\zeta)(z-\zeta)^{\kappa-1}d\zeta$$

$$=a\sum_{\kappa=1}^{\infty}\frac{(2-a)_{\kappa-1}}{z^{\kappa}}\mathcal{J}^{\kappa}f(z).$$

When $a=k>1$ is an integer, then $(2-k)_{\kappa-1}$ vanishes for every $\kappa\geqq k$. ——The case $a=1$ is exceptional in the sense that every term in the summand for $\mathcal{L}(1)$ does not vanish.

It may be noted that, for integral value of $a$, the relation in the theorem can also be inductively varified, by making use of integration by parts. On the other hand, it is remarked that the operator $\mathcal{L}(2)$ was observed by Libera [4] and Livingston [5] and that $\mathcal{L}(a)$ for integer $a>1$ was studied by Bernardi [1], both in connection with some classes of univalent functions.

**4. Generalization.**

By relaxing the restriction that the referring probability measure $\sigma$ is a monomial, we now consider a probability measure defined by a power series

$$\sigma(t)=\sum_{k=1}^{\infty}\omega_k t^k$$

with convergence radius greater than unity: $\limsup_{k\to\infty}\sqrt[k]{|\omega_k|}<1$. In view of the condition that $\sigma$ is a probability measure, we have to suppose

$$\rho(t)=\sigma'(t)=\sum_{k=1}^{\infty}k\omega_k t^{k-1}\geqq 0 \quad (t\in I), \qquad \sigma(1)=\sum_{k=1}^{\infty}\omega_k=1.$$

THEOREM 7. *Let $\sigma$ satisfy the just mentioned conditions. Then the operator $\mathcal{L}[\rho]$ defined by*

$$\mathcal{L}[\rho]f(z)=\int_I \frac{f(zt)}{t}\rho(t)dt \quad (f\in\mathfrak{F})$$

*is represented in terms of the ordinary integration operator $\mathcal{J}$ in the form*

$$\mathcal{L}[\rho]=\sum_{\kappa=1}^{\infty}(-1)^{\kappa-1}\frac{\varphi^{(\kappa-1)}(1)}{z^{\kappa}}\mathcal{J}^{\kappa},$$

*where $\varphi$ is defined by*

$$\varphi(t)=\frac{\rho(t)}{t}=\sum_{k=1}^{\infty}k\omega_k t^{k-2}.$$

*Proof*. By substituting the expressions for $\mathcal{L}(k)$ ($k=1, 2, \cdots$) derived in Theorem 6, we obtain

$$\begin{aligned}\mathcal{L}[\rho]&=\sum_{k=1}^{\infty}\omega_k\mathcal{L}(k)\\ &=\omega_1\sum_{\kappa=1}^{\infty}\frac{(\kappa-1)!}{z^{\kappa}}\mathcal{J}^{\kappa}+\sum_{k=2}^{\infty}\omega_k k\sum_{\kappa=1}^{k-1}(-1)^{\kappa-1}\frac{(k-2)!}{(k-\kappa-1)!\,z^{\kappa}}\mathcal{J}^{\kappa}\\ &=\sum_{\kappa=1}^{\infty}\frac{\Phi_{\kappa}}{z^{\kappa}}\mathcal{J}^{\kappa}, \qquad \text{say.}\end{aligned}$$

The coefficients of the last expression are given by

$$\begin{aligned}\Phi_\kappa&=(\kappa-1)!\,\omega_1+(-1)^{\kappa-1}\sum_{k=\kappa+1}^{\infty}k\frac{(k-2)!}{(k-\kappa-1)!}\omega_k\\&=(-1)^{\kappa-1}\Big[\frac{d^{\kappa-1}}{dt^{\kappa-1}}\Big(\frac{\omega_1}{t}+\sum_{k=2}^{\infty}k\omega_k t^{k-2}\Big)\Big]^{t=1}\\&=(-1)^{\kappa-1}\varphi^{(\kappa-1)}(1),\end{aligned}$$

and hence the desired result.

The result just derived can be more slightly generalized with respect to the referring measure $\sigma$.

THEOREM 8. *Let a probability measure $\sigma$ be given by*

$$\sigma(t)=\int_0^\infty t^a d\tau(a)$$

*where a measure $\tau$ defined on* $(0, \infty)$ *satisfies the conditions*

$$\rho(t)=\sigma'(t)=\int_0^\infty at^{a-1}d\tau(a)\geqq 0\quad (t\in I),\qquad \sigma(1)=\int_0^\infty d\tau(a)=1.$$

*Then we have*

$$\mathcal{L}[\rho]=\sum_{\kappa=1}^{\infty}(-1)^{\kappa-1}\frac{\varphi^{(\kappa-1)}(1)}{z^\kappa}\mathcal{G}^\kappa,$$

*where $\varphi$ is defined by*

$$\varphi(t)=\frac{\rho(t)}{t}=\int_0^\infty at^{a-2}d\tau(a).$$

*Proof*. The proof proceeds quite similar as for the previous theorem. In fact, we have

$$\begin{aligned}\mathcal{L}[\rho]f(z)&=\int_I\frac{f(zt)}{t}d\int_0^\infty t^a d\tau(a)\\&=\int_0^\infty d\tau(a)\int_I\frac{f(zt)}{t}at^{a-1}dt=\int_0^\infty\mathcal{L}(a)f(z)d\tau(a).\end{aligned}$$

Hence, by substituting the expression for $\mathcal{L}(a)$ derived in Theorem 6, we obtain

$$\mathcal{L}[\rho]=\int_0^\infty a\sum_{\kappa=1}^{\infty}\frac{(2-a)_{\kappa-1}}{z^\kappa}\mathcal{G}^\kappa d\tau(a)=\sum_{\kappa=1}^{\infty}\frac{\Phi_k}{z^\kappa}\mathcal{G}^k,\qquad \text{say}.$$

The coefficients of the last expression are given by

$$\Phi_k=\int_0^\infty a(2-a)_{\kappa-1}d\tau(a)$$

$$=(-1)^{\kappa-1}\left[\frac{d^{\kappa-1}}{dt^{\kappa-1}}\int_0^\infty at^{a-2}d\tau(a)\right]^{t=1}=(-1)^{\kappa-1}\varphi^{(\kappa-1)}(1),$$

and hence the result.

REMARK. Throughout this section the restriction $\rho(t)=\sigma'(t)\geqq 0$ $(t\in I)$ is really inessential, since the discussions concern to derive relations involving equality alone. From this standpoint we supplement here an example concerning Theorem 7.

We consider $\sigma'$ (of indefinite sign) given by

$$\sigma(t)=A_{2m}\int_0^t \tau P_{2m}(\tau)d\tau \qquad (m\geqq 1),$$

where $P_{2m}$ denotes the Legendre polynomial of degree $2m$ and $A_{2m}$ is the normalization factor determined by $\sigma(1)=1$. By means of Rodrigues formula we get after repeated integration by parts

$$\frac{1}{A_{2m}}=\int_I \tau P_{2m}(\tau)d\tau=\frac{(-1)^{m-1}}{2^{2m}}\frac{(2m-2)!}{(m-1)!(m+1)!}.$$

By making use of a familiar formula

$$P_n(t)=\sum_{\nu=0}^{n}(-1)^\nu\frac{(n+\nu)!}{\nu!^2(n-\nu)!}\left(\frac{1-t}{2}\right)^\nu,$$

we get after repeated differentiation

$$P_n^{(\kappa-1)}(t)=\sum_{\nu=\kappa-1}^{n}(-1)^\nu\frac{(n+\nu)!}{\nu!^2(n-\nu)!}\left(-\frac{1}{2}\right)^{\kappa-1}\frac{\nu!}{(\nu-\kappa+1)!}\left(\frac{1-t}{2}\right)^{\nu-\kappa+1}.$$

Thus, for $\varphi(t)=A_{2m}P_{2m}(t)$ we obtain the value of $\varphi^{(\kappa-1)}$ and finally

$$\mathcal{L}[\sigma']=(-1)^{m-1}2^{2m}\frac{(m-1)!(m+1)!}{(2m-2)!}\sum_{\kappa=1}^{2m+2}\frac{(-1)^{\kappa-1}\cdot(2m+\kappa-1)!}{(\kappa-1)!(2m-\kappa+1)!}\frac{1}{2^{\kappa-1}}\frac{1}{z^\kappa}\mathcal{J}^\kappa.$$

For $m=0$, we have $P_0(\tau)=1$, $A_0=2$; $\sigma(t)=t^2$ and

$$\mathcal{L}[\sigma']=\mathcal{L}(2)=\frac{2}{z}\mathcal{J},$$

while for $m=1/2$, we have $P_1(\tau)=\tau$, $A_1=3$; $\sigma(t)=t^3$ and

$$\mathcal{L}[\sigma']=\mathcal{L}(3)=3\frac{1}{z}\mathcal{J}-\frac{1}{z^2}\mathcal{J}^2.$$

However, the case with odd suffix greater than 1 has been rejected, since we would have $1/A_n=0$ for any odd integer $n\geqq 3$.

In this occasion we state another remark. The discussions developed in § 2 and § 3 for the case generated by

$$\rho(t\,;\,a)=at^{a-1}, \qquad \rho_{\lambda}(t\,;\,a)=\frac{a^{\lambda}}{\Gamma(\lambda)}t^{a-1}\Big(\log\frac{1}{t}\Big)^{\lambda-1}$$

will be generalized formally to the case

$$\rho(t\,;\,a,\,b)=\frac{a^{b}}{\Gamma(b)}t^{a-1}\Big(\log\frac{1}{t}\Big)^{b-1}.$$

However, the latter can be reduced essentially to the former. In fact, we have only to take into account the relation

$$\rho_{\lambda}(t\,;\,a,\,b)=\rho_{b\lambda}(t\,;\,a,\,1)=\rho_{b\lambda}(t\,;\,a).$$

## 5. Hadamard product.

The Hadamard product $*$ of two power series

$$\varphi(z)=\sum_{\nu=1}^{\infty}a_{\nu}z^{\nu}, \qquad \psi(z)=\sum_{\nu=1}^{\infty}b_{\nu}z^{\nu}$$

is defined by

$$\varphi*\psi(z)=\sum_{\nu=1}^{\infty}a_{\nu}b_{\nu}z^{\nu}.$$

It is readily seen that $\varphi,\,\psi\in\mathfrak{F}$ implies $\varphi*\psi\in\mathfrak{F}$ and the particular function

$$\chi(z)=\frac{z}{1-z}=\sum_{\nu=1}^{\infty}z^{\nu}$$

plays the role of unit function with respect to the operation $*$ in the class $\mathfrak{F}$; namely, $f*\chi=\chi*f=f$ $(f\in\mathfrak{F})$.

On the other hand, any operator $\mathcal{L}$ under consideration satisfies

$$\mathcal{L}(\varphi*\psi)(z)=\int_I\frac{(\varphi*\psi)(zt)}{t}d\sigma(t)=\int_I\Big(\varphi(z)*\frac{\psi(zt)}{t}\Big)d\sigma(t)$$

$$=\varphi(z)*\int_I\frac{\psi(zt)}{t}d\sigma(t)=(\varphi*\mathcal{L}\psi)(z),$$

whence follows, in particular,

$$\mathcal{L}f=\mathcal{L}(f*\chi)=f*\mathcal{L}\chi.$$

Thus, the action of $\mathcal{L}$ on any function $f\in\mathfrak{F}$ is reduced to the Hadamard product of $f$ with $\mathcal{L}\chi$.

If we consider, for instance, the operator $\mathcal{L}(a)$ defined in § 2, we have an expansion of $\mathcal{L}(a)\chi$ in the form

$$\mathcal{L}(a)\chi(z)=a\sum_{\nu=1}^{\infty}\frac{z^{\nu}}{\Gamma(\nu+a)}.$$

On the other hand, we have derived an expression for $\mathcal{L}(a)$ in terms of $\{\mathcal{J}^{\kappa}\}$, which, in particular, yields

$$\sum_{\nu=1}^{\infty}\frac{z^{\nu}}{\Gamma(\nu+a)}=\sum_{\kappa=1}^{\infty}\frac{(2-a)_{\kappa-1}}{z^{\kappa}}\mathcal{J}^{\kappa}\chi(z)$$

valid for $a>0$. Now, as readily seen directly, $\mathcal{J}^{\kappa}\chi$ is expressed by the expansion

$$\mathcal{J}^{\kappa}\chi(z)=z^{\kappa}\sum_{\nu=1}^{\infty}\frac{z^{\nu}}{(\nu+1)_{\kappa}}.$$

By substituting this into the above relation and comparing the coefficients of $z^{\nu}$, we obtain an identity

$$\frac{1}{\Gamma(\nu+a)}=\sum_{\kappa=1}^{\infty}\frac{(2-a)_{\kappa-1}}{(\nu+1)_{\kappa}}\qquad(\nu=1,\ 2,\ \cdots).$$

Now, $\mathcal{J}^{\kappa}\chi(z)$ is for any integer $\kappa\geqq 0$ an elementary function. We have, for instance,

$$\mathcal{J}\chi(z)=\log\frac{1}{1-z}-z,\qquad \mathcal{J}^{2}\chi(z)=-(1-z)\log\frac{1}{1-z}+z-\frac{z^{2}}{2}.$$

For any integer $\kappa\geqq 2$ we can derive similar explicit expression in the form

$$\mathcal{J}^{\kappa}\chi(z)=\frac{(-1)^{\kappa-1}}{(\kappa-1)!}(1-z)^{\kappa-1}\log\frac{1}{1-z}+\frac{(-1)^{\kappa-1}}{\kappa!}(1-z)^{\kappa}$$
$$+\frac{(-1)^{\kappa-1}}{(\kappa-1)!}\sum_{j=2}^{\kappa-1}\frac{1}{j}\cdot(1-z)^{\kappa-1}+\sum_{j=0}^{\kappa-2}+\frac{(-1)^{j}}{j!}\frac{1}{(\kappa-j)!(\kappa-j-1)}(1-z)^{j},$$

the empty sum being to be understood zero. It is verified, for instance, by induction though the actual calculation is somewhat troublesome.

## 6. Distortion inequalities.

In the previous paper [2], we discussed some distortion properties on the family $\{\mathcal{L}^{\lambda}\}$ generated by a general measure $\sigma$, and specialized them in the case of $\{\mathcal{L}(1)^{\lambda}\}$. We supplement here these results by observing the family $\{\mathcal{L}(a)^{\lambda}\}$ with $a>0$.

First, for a fixed pair $f,\ g\in\mathfrak{F}$ we consider the quantities $M$ and $N$ defined by

$$M(r;\ a,\ \lambda,\ \mu)=\max_{|z|=r}|\mathcal{L}(a)^{\lambda}f(z)-\mathcal{L}(a)^{\mu}g(z)|,$$

$$N(r;\ a,\ \lambda)=\max_{|z|=r}|\mathcal{L}(a)^{\lambda}f(z)-z|.$$

THEOREM 9. *For any* $f,\ g\in\mathfrak{F}$ *the quantity* $M(r;\ a,\ \lambda+\delta,\ \mu+\delta)$ *decreases with* *respect to* $\delta\geqq 0$. *More precisely, for* $\delta'>\delta\geqq 0$ *we have*

$$\left(\frac{a+1}{a}\right)^{\delta'}M(r;a,\lambda+\delta',\mu+\delta')\leqq\left(\frac{a+1}{a}\right)^{\delta}M(r;a,\lambda+\delta,\mu+\delta).$$

*Proof.* As shown in [2], we have

$$M(r;a,\lambda+\delta,\mu+\delta)\leqq M(r;a,\lambda,\mu)\int_I t\,d\sigma_\delta(t;a).$$

The last factor of the right hand member is in the present case equal to

$$\int_I t\,d\sigma_\delta(t;a)=\frac{a^\delta}{\Gamma(\delta)}\int_I t^a\left(\log\frac{1}{t}\right)^{\delta-1}dt=\left(\frac{a}{a+1}\right)^\delta.$$

Let $0\leqq\delta<\delta'$. Then, by replacing $\lambda$, $\mu$ and $\delta$ in the above inequality by $\lambda+\delta$, $\mu+\delta$ and $\delta'-\delta$, respectively, we obtain the desired result.

COROLLARY. *For any* $f\in\mathfrak{F}$ *the quantity* $((a+1)/a)^\delta N(r;a,\lambda+\delta)$ *decreases with respect to* $\delta\geqq 0$.

*Proof.* Since $L(a)^\mu z$ becomes $z$ for any $\mu$, the quantity $M(r;a,\lambda,\mu)$ reduces to $N(r;a,\lambda)$ provided $g(z)=z$. Hence, the assertion follows from the theorem by only substituting $g(z)=z$.

By the way, it follows from the Corollary that

$$\left(\frac{a+1}{a}\right)^\delta N(r;a,\lambda+\delta)\leqq N(r;a,\lambda).$$

If we replace here both $\lambda$ and $\delta$ by $\lambda/2$, we get

$$N(r;a,\lambda)\leqq\left(\frac{a}{a+1}\right)^{\lambda/2}N\left(r;a,\frac{\lambda}{2}\right).$$

In view of this inequality, we see that the first limit relation stated in Theorem 3 is again verified.

Next, for a fixed $f\in\mathfrak{F}$ we observe the quantities $h$ and $H$ defined by

$$\begin{matrix}h_\lambda(r;a)\\ H_\lambda(r;a)\end{matrix}=\begin{matrix}\min_{|z|=r}\\ \max_{|z|=r}\end{matrix}\operatorname{Re}\frac{\mathcal{L}(a)^\lambda f(z)}{z}.$$

THEOREM 10. *For any* $f\in\mathfrak{F}$ *and* $\delta>0$ *we have*

$$h_{\lambda+\delta}(r;a)\geqq h_\lambda(r;a)+\Phi(\delta,a)(1-h_\lambda(r;a)),$$

$$H_{\lambda+\delta}(r;a)\leqq H_\lambda(r;a)-\Phi(\delta,a)(H_\lambda(r;a)-1)$$

*where* $\Phi$ *is given by*

$$\Phi(\delta, a)=1-2a^{\delta}\sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}}{(\nu+a-1)^{\delta}}.$$

*The equality sign in either estimation does not appear for any* $r\in(0, 1)$ *unless* $f(z)=z$. *If, in particular,* $a=k$ *is a positive integer,* $\Phi(\delta, k)$ *is expressible in the form*

$$\Phi(\delta, k)=1+2(-1)^{k-1}k^{\delta}\Big((1-2^{1-\delta})\zeta(\delta)+\sum_{\kappa=1}^{k}\frac{(-1)^{\kappa-1}}{\kappa^{\delta}}\Big),$$

$\zeta$ *denoting Riemann zeta function.*

*Proof.* The inequalities having been generally shown in [2], it suffices to verify the expression for $\Phi$. We first have

$$\begin{aligned}\Phi(\delta, a)&=\int_I\frac{1-t}{1+t}d\sigma_{\delta}(t; a)\\&=\frac{a^{\delta}}{\Gamma(\delta)}\int_I\frac{1-t}{1+t}t^{a-1}\Big(\log\frac{1}{t}\Big)^{\delta-1}dt\\&=\frac{a^{\delta}}{\Gamma(\delta)}\int_I\Big(1-2\sum_{\nu=2}^{\infty}(-1)^{\nu}t^{\nu-1}\Big)t^{a-1}\Big(\log\frac{1}{t}\Big)^{\delta-1}dt\\&=1-2a^{\delta}\sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}}{(\nu+a-1)^{\delta}}.\end{aligned}$$

Next, in view of the formula

$$\sum_{\kappa=1}^{\infty}\frac{(-1)^{\kappa-1}}{\kappa^{\delta}}=(1-2^{1-\delta})\zeta(\delta),$$

we get for a positive integer $k$ the relation

$$\begin{aligned}\sum_{\nu=2}^{\infty}\frac{(-1)^{\nu}}{(\nu+k-1)^{\delta}}&=\sum_{\kappa=k+1}^{\infty}\frac{(-1)^{\kappa-k+1}}{\kappa^{\delta}}=(-1)^{k}\Big(\sum_{\kappa=1}^{\infty}-\sum_{k=1}^{k}\Big)\frac{(-1)^{\kappa-1}}{\kappa^{\delta}}\\&=(-1)^{k}(1-2^{1-\delta})\zeta(\delta)+(-1)^{k}\sum_{\kappa=1}^{k}\frac{(-1)^{\kappa}}{\kappa^{\delta}}.\end{aligned}$$

By substituting this in the above expression for $\Phi(\delta, a)$ with $a=k$, we obtain its desired expression.

COROLLARY. *We have*

$$h_{\lambda+\delta}(r; a)\geqq h_{\lambda}(r; a)+(1-e^{-\Phi'(0,a)\delta})(1-h_{\lambda}(r; a)),$$

$$H_{\lambda+\delta}(r; a)\leqq H_{\lambda}(r; a)-(1-e^{-\Phi'(0,a)\delta})(H_{\lambda}(r; a)-1),$$

*where* $\Phi'$ *is given by*

$$\Phi'(0, a)=\left[\frac{\partial}{\partial\delta}\Phi(\delta, a)\right]^{\delta=+0}$$

$$=2\lim_{\delta\to+0}\sum_{\nu=2}^{\infty}(-1)^{\nu}\frac{\log(\nu+a-1)-\log a}{(\nu+a-1)^{\delta}}.$$

*If, in particular, $a=k$ is a positive integer, then*

$$\Phi'(0, k)=2(-1)^{k-1}\log\frac{k!\sqrt{\pi/2}}{(2^{[k/2]}\cdot[k/2]!)^2}-\log k.$$

*Proof.* We first note that $\Phi(+0, a)=0$. In fact, by means of integration by parts, we get

$$\Phi(\delta, a)=\frac{a^{\delta}}{\Gamma(\delta)}\int_I\frac{1-t}{1+t}t^{a-1}\left(\log\frac{1}{t}\right)^{\delta-1}dt$$

$$=\frac{a^{\delta}}{\Gamma(\delta+1)}\left\{\left[-\frac{1-t}{1+t}t^a\left(\log\frac{1}{t}\right)^{\delta}\right]_0^1+\int_I\frac{d}{dt}\left(\frac{1-t}{1+t}t^a\right)\cdot\left(\log\frac{1}{t}\right)^{\delta}dt\right\}$$

$$=\frac{a^{\delta}}{\Gamma(\delta+1)}\int_I\frac{d}{dt}\left(\frac{1-t}{1+t}t^a\right)\cdot\left(\log\frac{1}{t}\right)^{\delta}dt,$$

whence readily follows

$$\Phi(+0, a)=\int_I\frac{d}{dt}\left(\frac{1-t}{1+t}t^a\right)dt=0.$$

The first inequality in the theorem yields

$$\frac{h_{\lambda+\delta}(r; a)-h_{\lambda}(r; a)}{\delta}\geqq\frac{\Phi(\delta, a)}{\delta}(1-h_{\lambda}(r; a)),$$

whence follows, as $\delta$ tends to $+0$, the inequality

$$\frac{\partial}{\partial\lambda}h_{\lambda}(r; a)\geqq\Phi'(0, a)(1-h_{\lambda}(r; a)).$$

This linear differential inequality can be brought readily into finite form. In fact, by rewriting it in the form

$$\frac{\partial}{\partial\lambda}(e^{\Phi'(0,a)\lambda}h_{\lambda}(r; a))\geqq\Phi'(0, a)e^{\Phi'(0,a)\lambda}$$

and then integrating with respect to $\lambda$ over the interval $(\lambda, \lambda+\delta)$, we obtain the desired estimation for $h$. Similar argument applies also for $H$. Next, we have in view of the expression for $\Phi(\delta, a)$ given in the theorem

$$\frac{\partial}{\partial\delta}\Phi(\delta, a)=2a^{\delta}\sum_{\nu=2}^{\infty}(-1)^{\nu}\frac{\log(\nu+a-1)-\log a}{(\nu+a-1)^{\delta}},$$

whence follows the desired expression for $\Phi'(0, a)$. Finally, if $a=k$ is a positive integer, we see that

$$\frac{\partial}{\partial\delta}\Phi(\delta, k)=2(-1)^{k-1}k^{\delta}\log k\Big((1-2^{1-\delta})\zeta(\delta)-\sum_{\kappa=1}^{k}\frac{(-1)^{\kappa-1}}{\kappa^{\delta}}\Big)$$

$$+2(-1)^{k-1}k^{\delta}\Big((1-2^{1-\delta})\zeta'(\delta)+2^{1-\delta}\log 2\cdot\zeta(\delta)+\sum_{\kappa=2}^{k}\frac{(-1)^{\kappa-1}}{\kappa^{\delta}}\log\kappa\Big).$$

In view of $\zeta(0)=-1/2$ and $\zeta'(0)=-(1/2)\log 2\pi$, we get

$$\Phi'(0, k)=2(-1)^{k-1}\log k\Big(\frac{1}{2}-\sum_{\kappa=1}^{k}(-1)^{\kappa-1}\Big)$$

$$+2(-1)^{k-1}\Big(\frac{1}{2}\log 2\pi-\log 2+\sum_{\kappa=2}^{k}(-1)^{\kappa-1}\log\kappa\Big),$$

which becomes the desired form, by remembering the elementary relations

$$\frac{1}{2}-\sum_{\kappa=1}^{k}(-1)^{\kappa-1}=\frac{(-1)^{k}}{2},\qquad \sum_{\kappa=2}^{k}(-1)^{\kappa-1}\log\kappa=\log\frac{k}{(2^{[k/2]}\cdot[k/2]\,!)^{2}}.$$

In the following lines, we shall supplement some properties of the quantities $\Phi(\delta, a)$ and $\Phi'(0, a)$ contained in Theorem 10 and its Corollary.

For lower values of $\delta$, $a$ we see that

$$\Phi(1, 1)=2\log 2,\qquad \Phi(2, 1)=\frac{\pi^{2}}{6},\qquad \Phi(1, 2)=3-4\log 2$$

and hence, in particular, $\Phi(2, 1)>\Phi(1, 1)>\Phi(1, 2)$. Now, we shall indicate that $\Phi(\delta, a)$ shows such monotoneity in general.

THEOREM 11. *For any fixed $a>0$ we have*

$$\Phi(+0, a)=0 \quad \textit{and} \quad \Phi(\infty, a)=1.$$

*When $\delta$ increases from* 0 *to* $\infty$, *$\Phi(\delta, a)$ increases strictly from* 0 *to* 1.

*Proof*. The relation $\Phi(+0, a)=0$ has been shown on the way of proving the Corollary of Theorem 10. Next, we have

$$1-\Phi(\delta, a)=\frac{a^{\delta}}{\Gamma(\delta)}\int_{I}\frac{2t}{1+t}t^{a-1}\Big(\log\frac{1}{t}\Big)^{\delta-1}dt>0.$$

Let any small positive number $\varepsilon$ be given. Then, $2t/(1+t)<\varepsilon/2$ as $t<\varepsilon/4$ and hence

$$1-\Phi(\delta, a)<\frac{a^{\delta}}{\Gamma(\delta)}\Big(\frac{\varepsilon}{2}\int_{0}^{\varepsilon/4}+\int_{\varepsilon/4}^{1}\Big)t^{a-1}\Big(\log\frac{1}{t}\Big)^{\delta-1}dt$$

$$<\frac{a^{\delta}}{\Gamma(\delta)}\Big(\frac{\varepsilon}{2}\int_{I}t^{a-1}\Big(\log\frac{1}{t}\Big)^{\delta-1}dt+\Big(\log\frac{4}{\varepsilon}\Big)^{\delta-1}\int_{I}t^{a-1}dt\Big)$$

$$=\frac{\varepsilon}{2}+\frac{1}{\Gamma(\delta)}\left(a\log\frac{4}{\varepsilon}\right)^{\delta-1}.$$

In view of Stirling formula applied to $\Gamma(\delta)$, we see that

$$\frac{1}{\Gamma(\delta)}\left(a\log\frac{4}{\varepsilon}\right)^{\delta-1}\sim\frac{1}{\sqrt{2\pi\delta}\,e}\left(\frac{ea}{\delta}\log\frac{4}{\varepsilon}\right)^{\delta-1}\to 0 \qquad \text{as } \delta\to\infty$$

and hence there exists $\Delta(\varepsilon)$ such that $1-\Phi(\delta, a)<\varepsilon$ as $\delta>\Delta(\varepsilon)$. This shows $\Phi(\infty, a)=1$. Finally, let $0<\delta<\delta'$. Then

$$\Phi(\delta', a)-\Phi(\delta, a)=\int_I\frac{1-t}{1+t}t^{a-1}\left(\frac{a^{\delta'}}{\Gamma(\delta')}\left(\log\frac{1}{t}\right)^{\delta'-1}-\frac{a^{\delta}}{\Gamma(\delta)}\left(\log\frac{1}{t}\right)^{\delta-1}\right)dt.$$

Put $T=\exp(-(1/a)(\Gamma(\delta')/\Gamma(\delta))^{1/(\delta'-\delta)})$. Then we see that as $t\lessgtr T$

$$\frac{a^{\delta'}}{\Gamma(\delta')}\left(\log\frac{1}{t}\right)^{\delta'-1}\gtrless\frac{a^{\delta}}{\Gamma(\delta)}\left(\log\frac{1}{t}\right)^{\delta-1} \quad\text{and}\quad \frac{1-t}{1+t}\gtrless\frac{1-T}{1+T}$$

and hence

$$\Phi(\delta', a)-\Phi(\delta, a)$$
$$>\frac{1-T}{1+T}\int_I t^{a-1}\left(\frac{a^{\delta'}}{\Gamma(\delta')}\left(\log\frac{1}{t}\right)^{\delta'-1}-\frac{a^{\delta}}{\Gamma(\delta)}\left(\log\frac{1}{t}\right)^{\delta-1}\right)dt=0.$$

THEOREM 12. *For any fixed $\delta>0$ we have*

$$\Phi(\delta, +0)=1 \qquad \textit{and} \qquad \Phi(\delta, \infty)=0.$$

*When $a$ increases from 0 to $\infty$, $\Phi(\delta, a)$ decreases strictly from 1 to 0.*

*Proof.* We see that

$$\Phi(\delta, a)=\frac{a^{\delta}}{\Gamma(\delta)}\int_I\left(t^{a-1}-\frac{2t^a}{1+t}\right)\left(\log\frac{1}{t}\right)^{\delta-1}dt$$
$$=1-\frac{2a^{\delta}}{\Gamma(\delta)}\int_I\frac{t^a}{1+t}\left(\log\frac{1}{t}\right)^{\delta-1}dt\to 1 \qquad \text{as } a\to+0,$$

since the last integral remains finite for $\delta>0$. Or, the result could be derived more simply by means of the series form of $\Phi$. Next, let any small positive number $\varepsilon$ be given. Then, $(1-t)/(1+t)<\varepsilon/2$ as $1>t>\eta\equiv(2-\varepsilon)/(2+\varepsilon)$ and hence

$$0<\Phi(\delta, a)<\frac{a^{\delta}}{\Gamma(\delta)}\left(\int_0^{\eta}+\frac{\varepsilon}{2}\int_{\eta}^{1}\right)t^{a-1}\left(\log\frac{1}{t}\right)^{\delta-1}dt$$
$$<\frac{a^{\delta}}{\Gamma(\delta)}\int_0^{\eta}t^{a-1}\left(\log\frac{1}{t}\right)^{\delta-1}dt+\frac{\varepsilon}{2}.$$

Since $a^{\delta}t^{a-1}\to 0$ as $a\to\infty$ uniformly for $t\in[0, \eta]$, there exists $A(\varepsilon)$ such that $0<\Phi(\delta, a)<\varepsilon$ as $a>A(\varepsilon)$. Finally, let $0<a<a'<1$. Then

$$\Phi(\delta,\ a')-\Phi(\delta,\ a)=\frac{1}{\Gamma(\delta)}\int_I \frac{1-t}{1+t}(a'^{\delta}t^{a'-1}-a^{\delta}t^{a-1})\left(\log\frac{1}{t}\right)^{\delta-1}dt\,.$$

We see that as $t\lessgtr(a/a')^{\delta/(a'-a)}$

$$\frac{1-t}{1+t}\gtrless\frac{1-(a/a')^{\delta/(a'-a)}}{1+(a/a')^{\delta/(a'-a)}}\qquad\text{and}\qquad a'^{\delta}t^{a'-1}\lessgtr a^{\delta}t^{a-1}$$

and hence

$$\Phi(\delta,\ a')-\Phi(\delta,\ a)$$
$$<\frac{1-(a/a')^{\delta/(a'-a)}}{1+(a/a')^{\delta/(a'-a)}}\frac{1}{\Gamma(\delta)}\int_I (a'^{\delta}t^{a'-1}-a^{\delta}t^{a-1})\left(\log\frac{1}{t}\right)^{\delta-1}dt=0\,.$$

REMARK. If $\phi(t)$ is a measurable function bounded on $I$ and left-continuous at 1, a similar argument as above for deriving $\Phi(\delta,\ \infty)=0$ in which $(1-t)/(1+t)$ is replaced by $\phi(t)-\phi(1)$ yields

$$\int_I (\phi(t)-\phi(1))\rho_\delta(t\,;\ a)dt\to 0,\quad \text{i. e.,}\ \int_I \phi(t)\rho_\delta(t\,;\ a)dt\to\phi(1)\quad \text{as}\ a\to\infty.$$

This relation corresponds to the fact that the probability density

$$\rho_\delta(t\,;\ a)=\frac{a^{\delta}}{\Gamma(\delta)}t^{a-1}\left(\log\frac{1}{t}\right)^{\delta-1}$$

is a kernel of singular integral tending to concentrate at $t=1$ as $a\to\infty$.

Now, we shall denote $\Phi'(0,\ a)$ briefly by $\Psi(a)$, namely

$$\Psi(a)=\Phi'(0,\ a)=\left[\frac{\partial}{\partial\delta}\Phi(\delta,\ a)\right]^{\delta=+0}.$$

As shown in the Corollary of Theorem 10, the quantity $\Psi(k)$ with a positive integer $k$ is represented in terms of elementary expressions; in particular, we have

$$\Psi(1)=\log\frac{\pi}{2},\qquad \Psi(2)=\log\frac{4}{\pi},\qquad \Psi(3)=\log\frac{3\pi}{8},$$
$$\Psi(4)=\log\frac{32}{9\pi},\qquad \Psi(5)=\log\frac{45\pi}{128},\qquad \Psi(6)=\log\frac{256}{75\pi},\qquad \text{etc.}$$

We supplement here the monotoneity of $\Psi(a)$.

THEOREM 13. *We have*

$$\Psi(+0)=\infty\qquad \textit{and}\qquad \Psi(\infty)=0\,.$$

*When $a$ increases from* 0 *to* $\infty$, $\Psi(a)$ *decreases strictly from* $\infty$ *to* 0.

*Proof.* The expression for $\Phi(\delta, a)$ obtained in the proof of the Corollary of Theorem 10 yields, after differentiation with respect to $\delta$,

$$\frac{\partial}{\partial\delta}\Phi(\delta, a)=a^{\delta}\Big(\frac{\log a}{\Gamma(\delta+1)}-\frac{\Gamma'(\delta+1)}{\Gamma(\delta+1)^2}\Big)\int_I \frac{d}{dt}\Big(\frac{1-t}{1+t}t^a\Big)\cdot\Big(\log\frac{1}{t}\Big)^{\delta}dt$$

$$+\frac{a^{\delta}}{\Gamma(\delta+1)}\int_I \frac{d}{dt}\Big(\frac{1-t}{1+t}t^a\Big)\cdot\Big(\log\frac{1}{t}\Big)^{\delta}\log\log\frac{1}{t}\,dt,$$

whence follows after integration by parts

$$\Psi(a)=\int_I \frac{d}{dt}\Big(\frac{1-t}{1+t}t^a\Big)\cdot\log\log\frac{1}{t}\,dt=\int_I \frac{1-t}{1+t}t^{a-1}\Big(\log\frac{1}{t}\Big)^{-1}dt.$$

The decresing property of $\Psi(a)$ is evident in view of the last expression. Now, for any $\varepsilon\in(0, 1/2)$ we get

$$\Psi(a)>\frac{1}{3}\varepsilon^a\int_{\varepsilon}^{1/2} t^{-1}\Big(\log\frac{1}{t}\Big)^{-1}dt=\frac{1}{3}\varepsilon^a\Big[-\log\log\frac{1}{t}\Big]_{\varepsilon}^{1/2},$$

whence follows

$$\liminf_{a\to+0}\Psi(a)\geqq\frac{1}{3}\Big[-\log\log\frac{1}{t}\Big]_{\varepsilon}^{1/2}.$$

Since $\varepsilon\in(0, 1/2)$ is arbitrary, we conclude $\Psi(+0)=\infty$. Next, since the integrand of the above integral expressing $\Psi(a)$ is uniformly bounded on $I$ for $a\geqq 2$ and tends to 0 as $a\to\infty$, it follows that $\Psi(\infty)=0$.

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