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KODAI MATH. J.
9 (1986), 421—424

ON THE STABILITY OF MINIMAL SURFACES IN S^3

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§1. Introduction.

Let $f: M \to R^3$ be a minimal immersion of a 2-dimensional orientable smooth manifold into the 3-dimensional Euclidean space, and let D be a conpact domain in M with the boundary consisting of a finite union of piecewise smooth curves. J. L. Barbosa and M. doCarmo proved the following stability theorem.

THEOREM 1 (Barbosa and doCarmo [1]). *If the area of the image of the Gauss map g of D is smaller than 2π, then D is stable.*

For the proof of this theorem, they defined on M a metric induced by g from S^2 and connected stability with the first eigenvalue in $g(D)$ for the Laplacian with respect to the new metric. In this paper we study the stability of minimal surfaces in S^3. For this we study the associated immersion into S^5 defined by Lawson [4] and discuss about the first eigenvalue. After that we use [5] for the estimate of the first eigenvalue. Our result is stated in Theorem 3.4 below.

§2. Preparations.

2.1. Let M be a 2-dimensional orientable smooth manifold and $f: M \to S^3$ be a minimal immersion into the 3-dimensional unit sphere in R^4. In the following we follow some definitions stated in [4].

Let $z = x + iy$ be a local coordinate on M and set $\partial = (1/2)(\partial/\partial x - i\partial/\partial y)$. Then the metric induced by f from S^3 is of the form

$$ds^2 = 2F|dz|^2.$$

The Gauss map $g: M \to S^3$ can be represented in the local coordinate as $g = (1/iF) f \Lambda \partial f \Lambda \bar{\partial} f$ and it is a branched minimal immersion. Here Λ represents the exterior product and we identify $\Lambda^3 R^4$ with R^4.

Define $h: M \to S^5$ by $h = f \Lambda g$. Here we identify $\Lambda^2 R^4$ with R^6. Let K be the Gauss curvature of M. K satisfies $K \leqq 1$. Then the metric induced by h from S^5 has the form

Received March 6, 1986

$$ds_h^2=2(2-K)F|dz|^2=(2-K)ds^2$$

and h satisfies $\partial\bar{\partial}h=-(2-K)Fh$. Therefore h is a minimal immersion into S^5.

2.2. Let D be a compact domain in M and its boundary ∂D is a finite union of piecewise smooth curves. Let N be a unit normal field along $f(M)$. Given a piecewise smooth function $u: D\to R$ with $u\equiv 0$ on ∂D, the second derivative of the area for the variation whose deformation vector field is given by $V=uN$ is

$$I(V, V)=\int_D u(-\Delta u-2(2-K)u)dM.$$

§ 3. The stability theorem.

LEMMA 3.1. *Let M, D and h be as above. Suppose D is unstable and $h|_D$: $D\to S^5$ is an embedding, then $\lambda_1(h(D))\leqq 2$.*

Proof. As D is unstable, there exists a piecewise smooth function $u: D\to R$ with $u\equiv 0$ on ∂D such that

$$\int_D u(-\Delta u-2(2-K)u)dM\leqq 0.$$

By using Stokes theorem we compute the left-hand side.

$$\begin{aligned}&\int_D u(-\Delta u-2(2-K)u)dM\\&=\int_D\|\mathrm{grad}\,(u)\|_M^2dM-\int_{\partial D}\langle u\mathrm{grad}\,(u),\ n\rangle_M ds-2\int_D(2-K)u^2dM\\&=\int_{h(D)}((2-K)\|\mathrm{grad}\,(u(h^{-1}))\|_{M_h}^2)(1/(2-K))dM_h\\&\quad-2\int_{h(D)}(u(h^{-1}))^2dM_h\\&=\int_{h(D)}\|\mathrm{grad}\,(u(h^{-1}))\|_{M_h}^2dM_h-2\int_{h(D)}(u(h^{-1}))^2dM_h,\end{aligned}$$

where n is a unit normal vector to ∂D and ds is its element of arc. Thus we obtain

$$\int_{h(D)}\|\mathrm{grad}\,(u(h^{-1}))\|_{M_h}^2dM_h\leqq 2\int_{h(D)}(u(h^{-1}))^2dM_h,$$

and $u(h^{-1})\equiv 0$ on $\partial(h(D))$, hence $\lambda_1(h(D))\leqq 2$. q. e. d.

Remark 3.2. By using the method in [1] we may obtain some resul a little different from Lemma 3.1.

We state the following theorem in [5] to estimate $\lambda_1(h(D))$. Let $M\to\overline{M}$ be an isometric immersion of a Riemannian manifold M of dimension m into a Riemannian manifold $\overline{M}$ of dimesion $\overline{m}$. We use the following notation.

$\overline{K}$=sectional curvature of $\overline{M}$.

H=mean curvature vector field of the immersion.

$R(\overline{M}, M)$-injectivity radius of $\overline{M}$ restricted to M.

w_m=volume of the unit ball in m-dimensional Euclidean space.

b=a positive real number.

THEOREM 3.3 (Tanno [5]). *Let M be a submanifold of $\overline{M}$ satisfying $\overline{K}\leqq b^2$. Let D be a compact domain of M. Assume the following.*

$$m|H|\leqq\kappa,\ \kappa^m \mathrm{Vol}(D)\leqq c_3(m,\bar{\alpha})^{-m},\ b\theta(\bar{\alpha})=1/\gamma\leqq 1,\ 2\rho_0(\bar{\alpha})\leqq R(\overline{M}, M),$$

where let $\alpha_2(0<\alpha_2<1)$ be the real number which minimizes

$$[(m-\alpha)2^{m-1}-(1-\alpha)]/\alpha(1-\alpha)^{1/m}$$

and for a real number $\bar{\alpha}(0<\bar{\alpha}\leqq\alpha_2<1)$

$$\theta(\bar{\alpha})=[\mathrm{Vol}(D)/(1-\bar{\alpha})w_m]^{1/m},$$

$$\rho_0(\bar{\alpha})=b^{-1}\sin^{-1}[b\theta(\bar{\alpha})],$$

$$c_3(m,\bar{\alpha})=\gamma\sin^{-1}(1/\gamma)\frac{(m-\bar{\alpha})2^{m-1}-(1-\bar{\alpha})}{(m-1)\bar{\alpha}(1-\bar{\alpha})^{1/m}}(m/m-1)w_m^{-1/m}.$$

Then $\lambda_1(D)\geqq[c_3(m,\bar{\alpha})^{-1}(\mathrm{Vol}(D))^{-1/m}-\kappa]^2/4$.

By using the theorem for the minimal isometric immersion $h: M\to S^5$, if

$$\mathrm{Vol}(D)=\int_D(2-K)dM<a=\max_{0<\bar{\alpha}\leqq\alpha_2}\pi(1-\bar{\alpha})\left(\sin\frac{\bar{\alpha}}{4\sqrt{2}(3-\bar{\alpha})}\right)^2$$

then $\lambda_1(D)>2$.

Here we have for any function $p: D\to R$ with $p\equiv 0$ on ∂D,

$$\int_D\|\mathrm{grad}(p)\|_{M_h}^2 dM_h>2\int_D p^2 dM_h.$$

And if D is unstable and $h|_D: D\to S^5$ is an embedding, then for any function $q: h(D)\to R$ with $q\equiv 0$ on $\partial h(D)$

$$\int_{h(D)}\|\mathrm{grad}(q)\|_{M_h}^2 dM_h>2\int_{h(D)}q^2 dM_h.$$

Thus we have $\lambda_1(h(D))>2$ and this contradicts Lemma 3.1. Now we have the following theorem.

THEOREM 3.4. *Let M, D, h and a be as above. Suppose $h|_D: D\to S^5$ is an embedding and*

$$\int_D (2-K)dM < a,$$

then D is stable in S^3.

We introduce another theorem in [5] to compare with our theorem above.

THEOREM 3.5. (Tanno [5]). *Let M be a minimal surface of a unit sphere* S^n *and D be a a compact domain of M. If*

$$\int_D (2-K)^2 dM < 1/8c_3(2, \bar{\alpha})^2,$$

then D is stable in S^n.

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