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BLOCH CONSTANT AND VARIATION OF BRANCH POINTS

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1. Introduction.

Let F be the set of functions f regular in the unit disc Δ normalized by the conditions $f(0)=0$ and $f'(0)=1$. For $z\in\Delta$, let $B(f)$ denote the least upper bound of the radii of all unramified disc centered at $f(z)$ which is contained in the Riemann surface of f. The Bloch constant B is defined by

$$B=\inf_{f\in F} B(f).$$

Although, the precise value of B is not known, we have the estimate [4] $B>\frac{\sqrt{3}}{4}$. In 1937 Ahlfors-Grunsky [1] obtained an upper bound $B\leqq B(g)=0.47\cdots$ by constructing a function $g\in F$ called the Ahlfors-Grunsky function. Also, they conjectured that $B=B(g)$. The function $g:\Delta\to C$ is obtained as follows. Let S be the interior of the N. E. (non-Euclidian) triangle in Δ with the angles $\frac{\pi}{6}$, $\frac{\pi}{6}$ and $\frac{\pi}{6}$ and the vertices at σ, $\omega\sigma$ and $\omega^2\sigma$ where $\sigma=(\sqrt{3}+1)^{-1/2}$ and $\omega=e^{2\pi i/3}$. Let T be the interior of the regular triangle with the vertices at τ, $\omega\tau$, $\omega^2\tau$ where

$$\tau=\sigma\cdot\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)\Big/\Gamma\left(\frac{1}{4}\right)=0.47\cdots.$$

By Schwarz's reflection the analytic function mapping S conformally onto T with $g(\omega^k\sigma)=\omega^k\tau$ $(k=0, 1, 2)$ is continued analytically to a function $g\in F$ defined on Δ, which is the Ahlfors-Grunsky function. It is well known that g is a normal branched covering of the complex plane C which is simply branched at every point of the regular triangular lattice $\{(n+\omega m)\tau \mid n, m\in \boldsymbol{Z}\}$ but is not branched elsewhere.

A. W. Goodman [3] introduced a variation of branch points for an analytic function in Δ. We denote by $f_{c,\lambda}$ Goodman's branch variation of f.

The aim of this paper is to prove

THEOREM. *For every branch point $c\in\Delta$ and sufficiently small $\lambda\in \boldsymbol{C}$ $(\lambda\neq 0)$, the Ahlfors-Grunsky function g satisfies*

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$$B(g)<B(g_{c,\lambda}/g'_{c,\lambda}(0)).$$

Note that $g_{c,\lambda}/g'_{c,\lambda}(0)\in F$. The above Theorem asserts that the Bloch radius $B(f)$ attains a local minimum at the Ahlfors-Grunsky function g when we vary its branch points slightly. For related results the reader is refered to H. Yanagihara's recent paper [7].

2. Proof of Theorem.

Goodman's branch variation [3] is described as follows. Let $f(z):\Delta\to\boldsymbol{C}$ be regular in Δ with $f'(0)>0$, and map Δ onto a Riemann surface, having a simple branch point at $f(c)$. For $\lambda\in\boldsymbol{C}$ sufficiently small we form a new Riemann surface R^*, from R, by moving the branch point at $f(c)$ to $f(c)+\lambda$, while holding the boundary and all other branch points of R fixed. Since R^* is simply-connected, there is a unique function $f_{c,\lambda}$ mapping Δ conformally onto R^* such that $f'_{c,\lambda}(0)>0$. The function $f_{c,\lambda}$ is called a Goodman's branch variation of f.

LEMMA 1. *For sufficiently small $\lambda\in\boldsymbol{C}$, we have*

$$f_{c,\lambda}(z)=f(z)-\frac{1}{2}zf'(z)\left[A\cdot\frac{c+z}{c-z}+\bar{A}\cdot\frac{1+\bar{c}z}{1-\bar{c}z}\right]+0(\lambda^2),$$

where $A=\dfrac{\lambda}{c^2f''(c)}$ and the estimate is uniform for z in compact subsets of Δ.

Proof. See [3]. For another derivation of the above formula using *q. c.* variation, see [6]. q. e. d.

In particular, for Ahlfors-Grunsky function g, Lemma 1 shows

$$g'_{c,\lambda}(0)=1-Re\frac{\lambda}{c^2f''(c)}+0(\lambda^2). \tag{1.1}$$

Let $r(x, y, z)$ denote the (Euclidian) radius of the circle passing through the points x, y and z $(\in\boldsymbol{C})$. Since, if $\gamma z=az+b$ $(a, b\in\boldsymbol{C})$,

$$r(\gamma x, \gamma y, \gamma z)=|a|r(x, y, z),$$

we have

$$\frac{\partial r}{\partial x}(\gamma x, \gamma y, \gamma z)=\frac{|a|}{a}\frac{\partial r}{\partial x}(x, y, z). \tag{1.2}$$

On the other hand, an elementary calculation gives

$$\frac{\partial r}{\partial x}(1, \omega, \omega^2)=\frac{1}{6}. \tag{1.3}$$

Thus, if we move the branch point $g(c)$ to $g(c)+\lambda$, the maximal radius $\rho(\lambda)$ of all unramified disks in R^* is given by

$$\rho(\lambda)=\tau+\max_{k\in Z} 2\, Re\left[\lambda\cdot\frac{1}{6}e^{k\pi i/3}\right]+0(\lambda^2). \tag{1.4}$$

This follows from the fact that the branch points of g forms a regular triangular lattice and from the identities (1.2) and (1.3).

LEMMA 2. *If the inequality*

$$|c^2 g''(c)|>2\sqrt{3}\tau \tag{1.5}$$

holds, then we have

$$B(g)<B(g_{c,\lambda}/g'_{c,\lambda}(0))$$

for sufficiently small $\lambda\in \boldsymbol{C}$ $(\lambda\neq 0)$.

Proof. Since $B(g_{c,\lambda})=\rho(\lambda)$ and $B(g)=\tau$, the identities (1.1) and (1.4) give

$$\begin{aligned} B(g_{c,\lambda}/g'_{c,\lambda}(0))&=B(g_{c,\lambda})/g'_{c,\lambda}(0)\\ &=\tau+\max_{k\in Z} Re\frac{\lambda}{3}\left[\frac{3\tau}{c^2g''(c)}+e^{k\pi i/3}\right]+0(\lambda^2)\\ &\geqq B(g)+\frac{|\lambda|}{3}\left[\max_{k\in Z} Re\; e^{(k\pi i/3)+\theta}-\left|\frac{3\tau}{c^2g''(c)}\right|\right]+0(\lambda^2) \end{aligned}$$

with $\theta=\arg\lambda$. Since

$$\max_{k\in Z} Re\; e^{(k\pi i/3)+\theta}\geqq\frac{\sqrt{3}}{2}\qquad \text{for any}\quad \theta\in\boldsymbol{R},$$

(1.5) easily implies the desired inequality. *q. e. d.*

Let Γ be the group of $N.E.$ isometries of Δ generated by l_1, l_2 and l_3 where l_i $(i=1, 2, 3)$ are the reflections in each side of $N.E.$ triangle S. We denote by Γ_0 the conformal subgroup of Γ. Poincaré's polygon theorem [5] implies that the group Γ is discontinuous and that the triangle S is a fundamental polygon for Γ.

LEMMA 3. *For any* $c\in\Delta$ *with* $g'(c)=0$, *we have*

$$|c^2g''(c)|\geqq|\sigma^2 g''(\sigma)|. \tag{1.6}$$

Proof. Observing that c is of the form $\gamma(\omega^k\sigma)$ for some $\gamma\in\Gamma_0$ and $k\in\boldsymbol{Z}$, we may assume by symmetry that $c=\gamma\sigma$ $(\gamma\in\Gamma_0)$. Next, note that, for $\gamma\in\Gamma_0$,

$$g(\gamma z)=\omega^k g(z)+\text{const.},\qquad k\in\boldsymbol{Z}.$$

Differentiating the above identity, we have

$$g''(\gamma z)(\gamma' z)^2+g'(\gamma z)\gamma'' z=\omega^k g''(z),$$

so that, by $g'(c)=0$,

$$g''(c)(\gamma'\sigma)^2=\omega^k g''(\sigma).$$

Hence,

$$|c^2 g''(c)|=\frac{|c|^2}{(1-|c|^2)^2}\cdot(1-\sigma^2)^2|g''(\sigma)|.$$

The Lemma will be proved if we show that $|c|\geqq\sigma$. This is easy. Let $z=\gamma_0\sigma$ ($\gamma_0\in\Gamma$) be a point in Δ such that

$$|z|=\min_{\gamma\in\Gamma}|\gamma\sigma|.$$

The existence of such a point z is clear from the discontinuity of the group Γ. Then we have

$$d(z, 0)\leqq d(l_i\gamma_0\sigma, 0), \qquad i=1, 2, 3,$$

where $d(\cdot, \cdot)$ denotes the hyperbolic distance. Hence,

$$d(z, 0)\leqq d(z, l_i(0)), \qquad i=1, 2, 3,$$

so that $z\in\bar{S}$. Since S is a fundamental polygon, we conclude that $z=\sigma$, as desired. *q. e. d.*

LEMMA 4.

$$g''(\sigma)=-(\sqrt{3}+1)^2\cdot 2^{-37/12}\cdot 3^{-5/8}\cdot\pi^{-5/2}\cdot\Gamma^2\left(\frac{1}{4}\right)\Gamma^3\left(\frac{1}{3}\right).$$

Proof. Here, our basic reference is [2]. Let S_1 be the interior of the *N. E.* triangle in Δ with the angles $\frac{\pi}{6}$, $\frac{\pi}{6}$ and $\frac{\pi}{6}$ that has its vertices at 0, σ_1, $e^{\pi i/6}\sigma_1$ with some positive constant σ_1, and let T_1 be the interior of the regular triangle with the vertices at 0, τ, $e^{\pi i/3}\tau$. We denote by g_1 the function mapping S_1 conformally onto T_1 with $g_1(0)=0$, $g_1(\sigma_1)=\tau$ and $g_1(e^{\pi i/6}\sigma_1)=e^{\pi i/3}\tau$. Then it is easy to see that there exists a Möbius transformation φ with $\varphi(0)=\sigma$ satisfying the identity

$$g_1(z)=e^{i\theta}(g\circ\varphi(z)-\tau)$$

with some real constant θ. Differentiation shows

$$g''(\sigma)=-(1-\sigma^2)^{-2}g_1''(0), \tag{1.7}$$

since geometrically it is clear that $g''(\sigma)<0$ and $g_1''(0)>0$. On the other hand, we can express g_1 as a composition $v\circ u^{-1}$ where u (resp. v) is the function mapping the upper half-plane $\{Im\, z>0\}$ onto S_1 (resp. T_1) in such a way that the origin is kept fixed and the other vertices of the triangle correspond to 1 and ∞. By [2, Vol. II, pp. 162-163], $u(z)$ and $v(z)$ have the expansion at the origin,

$$u(z)=C_1 z^{1/6}+\text{higher order terms},$$

and

$$v(z)=C_2 z^{1/3}+\text{higher order terms},$$

where the coefficients C_1 and C_2 are given explicitly by

$$C_1=\frac{\sqrt{2}}{\sqrt{\sqrt{3}+1}}\cdot\frac{\Gamma(5/6)\Gamma(3/4)}{\Gamma(5/12)\Gamma(7/6)} \tag{1.8}$$

and

$$C_2=\frac{\sqrt{3}}{\sqrt{\sqrt{3}+1}}\cdot\frac{\Gamma(2/3)\Gamma(11/12)}{\Gamma(1/4)\Gamma(4/3)} \tag{1.9}$$

Hence,

$$g_1(z)=v\cdot u^{-1}(z)=C_2C_1^{-2}\cdot z^2+\text{higher order terms},$$

so that

$$g_1''(0)=2C_2C_1^{-2}. \tag{1.10}$$

By applying standard formulas for the function $\Gamma(z)$, the identities (1.7)-(1.10) yield the value of $g''(\sigma)$, as desired. *q.e.d.*

Lemmas 2 and 3 imply that to conclude our THEOREM it is only necessary to show

$$|\sigma^2 g''(\sigma)|>2\sqrt{3}\tau.$$

However, a computation using Lemma 4 gives

$$|\sigma^2 g''(\sigma)|=2.34\cdots>2\sqrt{3}\tau=1.63\cdots.$$

This completes the proof of the THEOREM.

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