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KODAI MATH. J.
9 (1986), 385—400

ESTIMATION OF THE CORRELOGRAM FOR A STATIONARY GAUSSIAN PROCESS BY RANDOM CLIPPING

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Abstract

This paper deals with the problem of estimating the correlogram of a stationary Gaussian process with known mean and variance. An unbiased estimate using random clipping by normally distributed random variable with non-zero mean is discussed, and the variance of the estimate is compared with those of competitors. Numerical comparison is performed for AR(2) process, and it indicates that the suggested estimate is preferable in many cases.

1. Introduction.

It is known that in the bivariate normal distribution $N(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2; \rho)$, if the only unknown parameter is the correlation ρ, there are infinitely many unbiased estimates of ρ based on a sufficient statistics because the statistic is not complete (see Iwase [6]). Consequently, various kinds of estimates are proposed and discussed. The same account will be true of the correlogram for a stationary Gaussian process with known mean and variance. (see Huzii [2], [3], and Iwase [4], [5]). In this paper we shall consider an unbiased estimate of the correlogram and compare the variance of the estimate with those of competitors. It will be seen that the suggested estimate has a superiority over the others.

Let $\{X_t\}$ be a real valued stationary Gaussian process with discrete time parameter t such that the mean $E[X_t]$ and the variance $\mathrm{Var}[X_t]$ are known, and for simplicity we assume $E[X_t]=0$ and $\mathrm{Var}[X_t]=1$. Then the correlogram is identical with the covariogramme $E[X_tX_{t+h}]$. Throughout this paper we write $\rho_h=E[X_tX_{t+h}]$ for $h\geqq 0$, and following the prevailing custom, we try to estimate ρ_h from the given series $X_1, X_2, \cdots, X_{N+h}$ for $h\geqq 0$ in the case $N\geqq 3$.

The unbiased estimate

$$\gamma_h=\frac{1}{N}\sum_{t=1}^{N} X_tX_{t+h}$$

is usually applied to the estimating of ρ_h. Another estimate is a simplified estimate of ρ_h which is given by

Received February 6, 1986

$$\gamma_h^{(0)}=\sqrt{\frac{\pi}{2}}\frac{1}{N}\sum_{t=1}^{N}X_t\,\mathrm{sgn}(X_{t+h}),$$

where $\mathrm{sgn}(y)$ means 1, 0 and -1 if $y>0$, $y=0$ and $y<0$ respectively (see Takahasi and Husimi [8]). Hüzii [2], [3] and Iwase [4] numerically compared the variance of $\gamma_h^{(0)}$ with that of γ_h and showed, in some models, that $\gamma_h^{(0)}$ has a smaller variance than γ_h when h is small. On the other hand, when $N=1$, Okamoto and Iwase [7] improved the simplified estimate of ρ_1 by using a function $C_m(X)$ for $m\geqq 0$ which means 1, 0 and -1 if $X>m$, $|X|\leqq m$ and $X<-m$ respectively. They showed that the optimum value of the level m is about 2/3 by employing a criterion of minimum variance when ρ_1 is not equal to one.

In the previous paper [9] we still more improved the estimate using random clipping by normal distributed random variable with zero mean and showed that the variance of the proposed estimate is shrunk by the random clipping and that the variance become smaller than that of the simplified estimate. Here we shall consider another improvement of the estimate.

Suppose that $\{U_t\}$ is a sequence of independent random variables having a normal distribution with mean μ and variance σ^2, denoted by $N(\mu, \sigma^2)$ where $\mu\geqq 0$ and $\sigma\geqq 0$, and also $\{U_t\}$ is independent of $\{X_t\}$. Note that if $U_t=\mu$ for all t, then $\sigma^2=0$. The new estimate of ρ_h is defined as

$$\begin{aligned}\gamma_h^{(1)}&=\gamma_h^{(1)}(\mu, \sigma^2)\\&=\sqrt{\frac{\pi(\sigma^2+1)}{2}}\exp\left[\frac{\mu^2}{2(\sigma^2+1)}\right]\frac{1}{2N}\sum_{t=1}^{N}\{X_tC_{U_{t+h}}(X_{t+h})+X_{t+h}C_{U_t}(X_t)\}.\end{aligned}$$

This estimate is reduced to that of Tanaka and Shimizu [9] when $\mu=0$, and also that of Okamoto and Iwase [7] when $N=1$, $h=1$ and $\sigma^2=0$.

In Section 2 we show the unbiasedness of $\gamma_h^{(1)}$ and give the variance of this estimate in Theorem. Also in Section 3 we numerically investigate the relation between the values of the parameters μ and σ^2 and the variance of $\gamma_h^{(1)}(\mu, \sigma^2)$, and furthermore, the variance is compared with those of the other estimates.

2. Mean and variance of the estimate.

Let a vector of random variables $(X, Y)'$ follow the bivariate normal distribution with mean vector $(0, 0)'$ and covariance matrix $\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$, denoted by $N(0, 0, 1, 1; \rho)$. It is seen from Okamoto and Iwase [7] that

$$E[X\,\mathrm{sgn}(Y-u)]=\sqrt{\frac{2}{\pi}}\rho\exp\left(-\frac{u^2}{2}\right),$$

where the clipping level u is real. If the level is random variable U which follows $N(\mu, \sigma^2)$ and independent of X and Y, then we have

$$E_U E[X \operatorname{sgn}(Y-U)]=\sqrt{\frac{2}{\pi(\sigma^2+1)}}\rho \exp\left[-\frac{\mu^2}{2(\sigma^2+1)}\right].$$

Here E_U means the expectation with respect to U. Since $C_U(X)$ in the estimate $\gamma_h^{(1)}$ can be expressed as

$$C_U(X)=\frac{1}{2}[\operatorname{sgn}(X-U)+\operatorname{sgn}(X+U)],$$

it follows that the estimate $\gamma_h^{(1)}$ is unbiased.

The variance of the estimate is as follows.

THEOREM. *Let $\sigma^2>0$. For $h=1, 2, 3, \cdots$, the variance of $\gamma_h^{(1)}(\mu, \sigma^2)$ is given by*

$$\begin{aligned}
&\operatorname{Var}[\gamma_h^{(1)}(\mu, \sigma^2)] \\
&=\frac{(\sigma^2+1)}{4N^2}\exp\left(\frac{\mu^2}{\sigma^2+1}\right)\Big\{N\Big[\frac{2\rho_h^2}{\sigma^2+1}E(\mu, \sigma^2)+F(\mu, \sigma^2) \\
&\qquad +2\rho_h I_1(\mu, \sigma^2; \rho_h)+\rho_h I_2(\mu, \sigma^2; \rho_h)+(1+\rho_h^2)G(\mu, \sigma^2; \rho_h)\Big] \\
&\quad +(N-h)\Big[2\rho_h(1+\rho_{2h})I_1(\mu, \sigma^2; \rho_h)+2\rho_h I_2(\mu, \sigma^2; \rho_h)+2(\rho_h^2+\rho_{2h})G(\mu, \sigma^2; \rho_h) \\
&\qquad +2\rho_h^2 I_1(\mu, \sigma^2; \rho_{2h})+I_2(\mu, \sigma^2; \rho_{2h})+2\rho_h^2 G(\mu, \sigma^2; \rho_{2h}) \\
&\qquad +\frac{2\rho_h^2}{\sigma^2+1}E(\mu, \sigma^2)+2\rho_{2h}F(\mu, \sigma^2)\Big] \\
&\quad +\sum_{\substack{k=1\\k\neq h}}^{N-1}(N-k)\Big[2\rho_h(\rho_{k-h}+\rho_{k+h})I_1(\mu, \sigma^2; \rho_k)+2\rho_k I_2(\mu, \sigma^2; \rho_k) \\
&\qquad +2(\rho_h^2+\rho_{k-h}\rho_{k+h})G(\mu, \sigma^2; \rho_k)+2\rho_k\rho_h I_1(\mu, \sigma^2; \rho_{k+h}) \\
&\qquad +\rho_{k-h}I_2(\mu, \sigma^2; \rho_{k+h})+(\rho_k^2+\rho_h^2)G(\mu, \sigma^2; \rho_{k+h}) \\
&\qquad +2\rho_h\rho_k I_1(\mu, \sigma^2; \rho_{k-h})+\rho_{k+h}I_2(\mu, \sigma^2; \rho_{k-h}) \\
&\qquad +(\rho_k^2+\rho_h^2)G(\mu, \sigma^2; \rho_{k-h})\Big]\Big\}-\rho_h^2,
\end{aligned}$$

where

$$\begin{aligned}
E(\mu, \sigma^2)&=\sigma \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \\
&\quad +\frac{\mu}{\sqrt{\sigma^2+1}}\exp\left[-\frac{\mu^2}{2(\sigma^2+1)}\right]\int_0^{\mu/\sqrt{\sigma^2(\sigma^2+1)}}\exp\left(-\frac{t^2}{2}\right)dt, \\
F(\mu, \sigma^2)&=\exp\left[-\frac{\mu^2}{2(\sigma^2+1)}\right]\int_0^{1/\sigma}\frac{1}{t^2+1}\exp\left[-\frac{\mu^2 t^2}{2(\sigma^2+1)}\right]dt, \\
G(\mu, \sigma^2; x)&=\frac{1}{[(\sigma^2+1)^2-x^2]^{1/2}}\left[\exp\left(\frac{-\mu^2}{\sigma^2+1+x}\right)+\exp\left(\frac{-\mu^2}{\sigma^2+1-x}\right)\right],
\end{aligned}$$

$$I_1(\mu, \sigma^2; x)=\int_{-x}^{x}\frac{-(4\sigma^2t^2+1+\sigma^2)(\sigma^2+1+t)+\mu^2(2t+1)^2[(\sigma^2+1)^2-t^2]}{(\sigma^2+1+t)[(\sigma^2+1)^2-t^2]^{3/2}}\cdot\exp\left(\frac{-\mu^2}{\sigma^2+1+t}\right)dt,$$

$$I_2(\mu, \sigma^2; x)=\int_{-x}^{x}\frac{1}{[(\sigma^2+1)^2-t^2]^{1/2}}\exp\left(\frac{-\mu^2}{\sigma^2+1+t}\right)dt.$$

Also for $h=0$, we have

$$\begin{aligned}\mathrm{Var}[\gamma_0^{(1)}(\mu, \sigma^2)]=\frac{1}{N^2}\exp\left(\frac{\mu^2}{\sigma^2+1}\right)\Big\{&N[E(\mu, \sigma^2)+(\sigma^2+1)F(\mu, \sigma^2)]\\&+(\sigma^2+1)\sum_{k=1}^{N-1}(N-k)[2\rho_kI_1(\mu, \sigma^2; \rho_k)+\rho_kI_2(\mu, \sigma^2; \rho_k)\\&+(\sigma^2+1)G(\mu, \sigma^2; \rho_k)]\Big\}-1.\end{aligned}$$

The proof of this theorem is given in Appendix later on. On the other hand when $\sigma^2=0$, the variance of $\gamma_h^{(1)}(\mu, 0)$ may be immediately obtained if we put $\sigma^2\to0$ in the above situation. Then it is noted that

$$E(\mu, 0)=\sqrt{\frac{\pi}{2}}\mu\exp\left(-\frac{\mu^2}{2}\right),\quad\text{and}\quad F(\mu, 0)=\sqrt{\frac{\pi}{2}}\int_{\mu}^{\infty}\exp\left(\ \frac{t^2}{2}\right)dt.$$

3. Numerical comparison.

The estimate $\gamma_h^{(1)}(\mu, \sigma^2)$ should be compared with the usual estimate γ_h and the simplified estimate $\gamma_h^{(0)}$. In this section we mainly discuss the results of the case where the process is the second-order autoregressive ($AR(2)$) process which is written

$$X_t=\phi_1X_{t-1}+\phi_2X_{t-2}+a_t\qquad\text{(see Box and Jenkins [1]).}$$

For stationarity, the parameters ϕ_1 and ϕ_2 must lie in the triangular region

$$\phi_1+\phi_2<1,\quad \phi_2-\phi_1<1,\quad -1<\phi_2<1. \tag{3.1}$$

Then the correlogram satisfies the difference equation

$$\rho_h=\phi_1\rho_{h-1}+\phi_2\rho_{h-2},\qquad h\geqq2$$

with starting values $\rho_0=1$ and $\rho_1=\phi_1/(1-\phi_2)$.

Numerical computations were performed for some pairs of parameters (ϕ_1, ϕ_2) in (3.1), and for each $\mu=0.0$ (0.1) 0.9 and $\sigma^2=0.0$ (0.01) 0.9 in the cases $N=50$ and 250. The comparison of γ_h and $\gamma_h^{(0)}$ for Markov process ($\phi_2=0$) was treated by Huzii [2]. Throughout this section, (μ^*, σ^{*2}) denotes a vector (μ, σ^2) which minimises the value of $\mathrm{Var}[\gamma_h^{(1)}(\mu, \sigma^2)]$ for a fixed h in four decimal places.

Table 1 concerns Markov process, and in which the variances of γ_1, $\gamma_1^{(0)}$ and $\gamma_1^{(1)}(\mu, \sigma^2)$ are given for each $|\phi_1|=0.1, 0.2, \cdots, 0.9$, and for selected values of (μ, σ^2). In this case we have $\rho_1=\phi_1$. When ρ_1 is large in absolute value, the variance of $\gamma_1^{(0)}$ is smaller than that of γ_1, and the variance of $\gamma_1^{(1)}(\mu^*, \sigma^{*2})$ is still smaller than that of $\gamma_1^{(0)}$. On the other hand, when ρ_1 is close to zero, the variances of $\gamma_1^{(0)}$ and $\gamma_1^{(1)}$ exceed that of γ_1, but there is little difference in two variances of $\gamma_1^{(1)}(\mu^*, \sigma^{*2})$ and γ_1. This indicates that the estimate $\gamma_1^{(1)}(\mu^*, \sigma^{*2})$ for selected value (μ^*, σ^{*2}) is preferable except only when the coefficient ρ_1 is sufficiently small. Note in Table 1 that when $|\rho_1|$ is 0.8, we have $\sigma^{*2}\neq 0$ in $\gamma_h^{(1)}$, so that the random clipping has an effect on the shrinkage of the variance.

Table 2 and Figure 1 provide the results for $\Phi_1=1.7$ and $\Phi_2=-0.8$. In this case the correlogram ρ_h for small h $(1\leqq h\leqq 3)$ is so large that the variances of $\gamma_h^{(0)}$ and $\gamma_h^{(1)}(\mu^*, \sigma^{*2})$ are considerably smaller than that of γ_h. The variance $\mathrm{Var}[\gamma_1^{(1)}(0.3, 0)]$ is about one-seventh of $\mathrm{Var}[\gamma_1]$, and is also a half of $\mathrm{Var}[\gamma_1^{(0)}]$. When h is large $(h\geqq 15)$, the variances of $\gamma_h^{(0)}$ and $\gamma_h^{(1)}$ exceed that of γ_h, but the difference between $\mathrm{Var}[\gamma_h^{(1)}(0.7, 0)]$ and $\mathrm{Var}[\gamma_h]$ is almost negligible.

Table 3 and Figure 2 present the results for $\Phi_1=0.1$ and $\Phi_2=-0.9$. In this case the correlogram ρ_h is subject to sharp fluctuations. When $h=1$, ρ_1 is so

Table 1. Variances for Markov process; $h=1$, $\phi_1=\rho_1$, $N=50$ $(N=250)$

$\lvert\phi_1\rvert=\lvert\rho_1\rvert$	$\mathrm{Var}[\gamma_1]$	$\mathrm{Var}[\gamma_1^{(0)}]$	$\mathrm{Var}[\gamma_1^{(1)}(\mu^*, \sigma^{*2})]$	(μ^*, σ^{*2})
0.0	0.02 (0.0040)	0.0314 (0.0063)	0.0224 (0.0045)	(0.6, 0)
0.1	0.0210 (0.0042)	0.03161 (0.0063)	0.0229 (0.0046)	(0.6, 0)
0.2	0.0241 (0.0048)	0.0322 (0.0065)	0.0244 (0.0049)	(0.5, 0)
0.3	0.0295 (0.0059)	0.0334 (0.0067)	0.0267 (0.0054)	(0.3, 0)
0.4	0.0381 (0.0077)	0.0352 (0.0071)	0.0298 (0.0060)	(0.3, 0)
0.5	0.0510 (0.0103)	0.0382 (0.0077)	0.0339 (0.0068)	(0.2, 0)
0.6	0.0708 (0.0144)	0.0429 (0.0087)	0.0398 (0.0081)	(0.1, 0)
0.7	0.1037 (0.0212)	0.0511 (0.0104)	0.0489 (0.0100)	(0.1, 0)
0.8	0.1671 (0.0347)	0.0674 (0.0139)	0.0651 (0.0135)	(0, 0.05)
0.9	0.3413 (0.0740)	0.1125 (0.0243)	0.0982 (0.0221)	(0.25, 0)

Table 2. Variances for the case: $\phi_1=1.7$, $\phi_2=-0.8$, $N=50$ ($N=250$)

h	ρ_h	$\mathrm{Var}[\gamma_h]$	$\mathrm{Var}[\gamma_h^{(0)}]$	$\mathrm{Var}[\gamma_h^{(1)}(\mu^*, \sigma^{*2})]$	(μ^*, σ^{*2})
1	0.9444	0.2206 (0.0466)	0.0625 (0.0131)	0.0297 (0.0065)	(0.3, 0)
2	0.8056	0.1751 (0.0368)	0.0571 (0.0117)	0.0319 (0.0067)	(0.3, 0)
3	0.6139	0.1204 (0.0250)	0.0526 (0.0105)	0.0352 (0.0071)	(0.3, 0)
4	0.3992	0.0767 (0.0155)	0.0543 (0.0107)	0.0419 (0.0082)	(0.3, 0)
5	0.1875	0.0569 (0.0113)	0.0647 (0.0128)	0.0530 (0.0105)	(0.4, 0)
6	−0.0006	0.0626 (0.0127)	0.0821 (0.0166)	0.0662 (0.0134)	(0.7, 0)
7	−0.1511	0.0853 (0.0179)	0.1024 (0.0211)	0.0862 (0.0178)	(0.5, 0)
8	−0.2563	0.1144 (0.0243)	0.1204 (0.0251)	0.1034 (0.0217)	(0.4, 0)
9	−0.3148	0.1368 (0.0294)	0.1327 (0.0278)	0.1148 (0.0243)	(0.3, 0)
10	−0.3302	0.1470 (0.0317)	0.1381 (0.0290)	0.1199 (0.0254)	(0.3, 0)
15	−0.0570	0.1080 (0.0225)	0.1326 (0.0274)	0.1117 (0.0232)	(0.7, 0)
20	0.1089	0.1234 (0.0263)	0.1460 (0.0307)	0.1251 (0.0265)	(0.6, 0)
25	0.0172	0.1176 (0.0248)	0.1450 (0.0303)	0.1220 (0.0257)	(0.7, 0)
30	−0.0359	0.1197 (0.0254)	0.1468 (0.0308)	0.1240 (0.0262)	(0.7, 0)

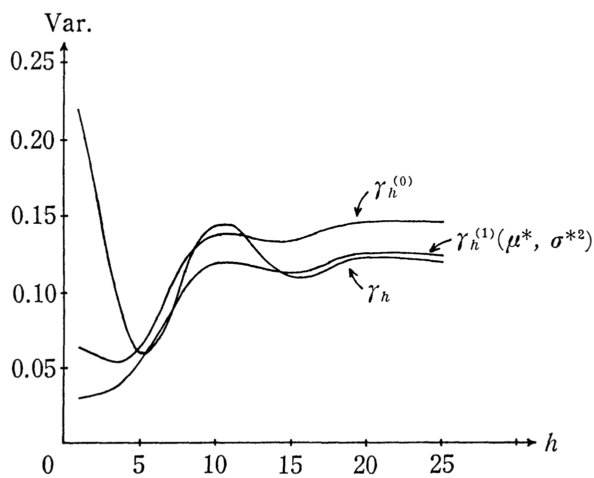

Fig. 1. ($N=50$)

Table 3. Variances for the case: $\phi_1=0.1$, $\phi_2=-0.9$, $N=50$ ($N=250$)

h	ρ_h	Var[r_h]	Var[$r_h^{(0)}$]	Var[$r_h^{(1)}(\mu^*, \sigma^{*2})$]	(μ^*, σ^{*2})
1	0.0526	0.0021 (0.0004)	0.0213 (0.0043)	0.0018 (0.0003)	(0.4, 0)
2	−0.8947	0.3024 (0.0718)	0.0997 (0.0234)	0.0889 (0.0215)	(0.3, 0)
3	−0.1368	0.0153 (0.0033)	0.0302 (0.0061)	0.0116 (0.0024)	(0.4, 0)
4	0.7916	0.2843 (0.0677)	0.1117 (0.0261)	0.1023 (0.0245)	(0.3, 0)
5	0.2023	0.0373 (0.0083)	0.0451 (0.0094)	0.0277 (0.0059)	(0.3, 0)
6	−0.6922	0.2596 (0.0621)	0.1193 (0.0278)	0.1106 (0.0263)	(0.3, 0)
7	−0.2513	0.0637 (0.0144)	0.0633 (0.0135)	0.0470 (0.0102)	(0.3, 0)
8	0.5978	0.2323 (0.0558)	0.1239 (0.0287)	0.1150 (0.0272)	(0.3, 0)
9	0.2860	0.0910 (0.0208)	0.0827 (0.0179)	0.0673 (0.0148)	(0.3, 0)
10	−0.5095	0.2057 (0.0494)	0.1266 (0.0291)	0.1171 (0.0275)	(0.3, 0)
15	−0.3235	0.1559 (0.0369)	0.1337 (0.0301)	0.1198 (0.0274)	(0.3, 0)
20	0.1713	0.1297 (0.0297)	0.1393 (0.0308)	0.1238 (0.0278)	(0.4, 0)
25	0.2596	0.1823 (0.0447)	0.1723 (0.0400)	0.1579 (0.0373)	(0.3, 0)
30	0.0029	0.1351 (0.0305)	0.1605 (0.0358)	0.1391 (0.0313)	(0.8, 0)

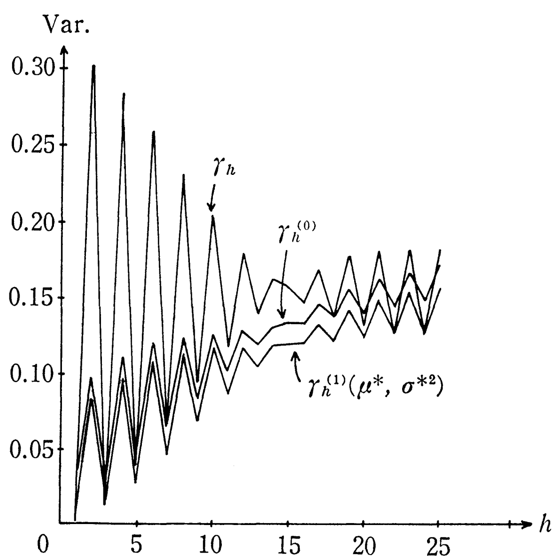

Fig. 2. ($N=50$)

Table 4. Variances for Iwase's model; $N=50$ ($N=250$)

h	ρ_h	$\mathrm{Var}[\gamma_h]$	$\mathrm{Var}[\gamma_h^{(0)}]$	$\mathrm{Var}[\gamma_h^{(1)}(\mu^*, \sigma^{*2})]$	(μ^*, σ^{*2})
1	0.5650	0.0596 (0.0128)	0.0215 (0.0045)	0.0169 (0.0036)	(0, 0.01) or (0.05, 0)
2	−0.2663	0.0336 (0.0072)	0.0272 (0.0057)	0.0213 (0.0045)	(0.2, 0)
3	−0.7230	0.1688 (0.0369)	0.0760 (0.0164)	0.0739 (0.0160)	(0, 0.01) or (0.05, 0)
4	−0.4889	0.0906 (0.0195)	0.0526 (0.0110)	0.0479 (0.0101)	(0.1, 0)
5	0.1139	0.0354 (0.0074)	0.0432 (0.0089)	0.0352 (0.0073)	(0.5, 0)
6	0.5116	0.1395 (0.0306)	0.0881 (0.0189)	0.0837 (0.0181)	(0.1, 0)
7	0.4070	0.1076 (0.0234)	0.0780 (0.0165)	0.0724 (0.0154)	(0.2, 0)
8	−0.0209	0.0492 (0.0102)	0.0621 (0.0128)	0.0516 (0.0110)	(0.7, 0)
9	−0.3537	0.1159 (0.0255)	0.0940 (0.0201)	0.0876 (0.0189)	(0.2, 0)
10	−0.3287	0.1125 (0.0246)	0.0939 (0.0200)	0.0872 (0.0187)	(0.2, 0)
15	−0.1550	0.0940 (0.0204)	0.1006 (0.0215)	0.0909 (0.0195)	(0.4, 0)
20	0.0676	0.0904 (0.0195)	0.1039 (0.0221)	0.0922 (0.0198)	(0.6, 0)
25	0.0803	0.0966 (0.0211)	0.1094 (0.0235)	0.0978 (0.0212)	(0.6, 0)

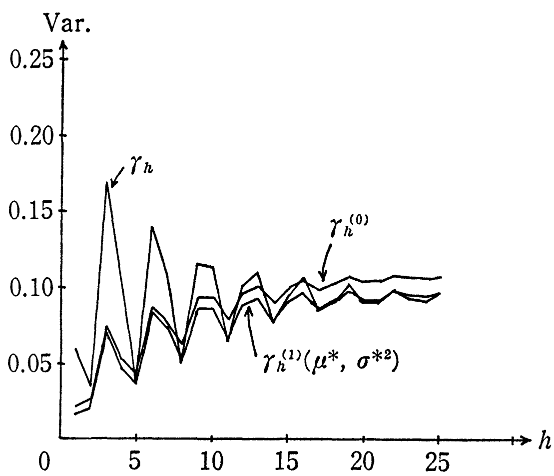

Fig. 3. ($N=50$)

Table 5. Variances for Huzii's model; $N=50$ ($N=250$)

h	ρ_h	$\mathrm{Var}[\gamma_h]$	$\mathrm{Var}[\gamma_h^{(0)}]$	$\mathrm{Var}[\gamma_h^{(1)}(\mu^*, \sigma^{*2})]$	(μ^*, σ^{*2})
1	0.9572	0.2487 (0.0516)	0.0720 (0.0149)	0.0241 (0.0062)	(0.3, 0)
2	0.8536	0.2269 (0.0471)	0.0772 (0.0159)	0.0381 (0.0079)	(0.3, 0)
3	0.7192	0.1985 (0.0411)	0.0850 (0.0173)	0.0381 (0.0113)	(0.3, 0)
4	0.5759	0.1706 (0.0353)	0.0954 (0.0194)	0.0724 (0.0147)	(0.3, 0)
5	0.4384	0.1480 (0.0305)	0.1075 (0.0218)	0.0884 (0.0180)	(0.3, 0)
6	0.3154	0.1327 (0.0273)	0.1199 (0.0244)	0.1022 (0.0208)	(0.3, 0)
7	0.2116	0.1243 (0.0256)	0.1315 (0.0268)	0.1131 (0.0231)	(0.4, 0)
8	0.1283	0.1212 (0.0250)	0.1412 (0.0289)	0.1204 (0.0247)	(0.5, 0.02)
9	0.0647	0.1214 (0.0250)	0.1487 (0.0305)	0.1249 (0.0257)	(0.6, 0.03)
10	0.0189	0.1230 (0.0254)	0.1539 (0.0317)	0.1281 (0.0264)	(0.7, 0.04)
15	−0.0416	0.1289 (0.0268)	0.1608 (0.0332)	0.1338 (0.0277)	(0.7, 0.04)
20	−0.0140	0.1285 (0.0267)	0.1607 (0.0332)	0.1334 (0.0277)	(0.7, 0.04)
25	0.0004	0.1285 (0.0267)	0.1608 (0.0332)	0.1334 (0.0277)	(0.7, 0.04)

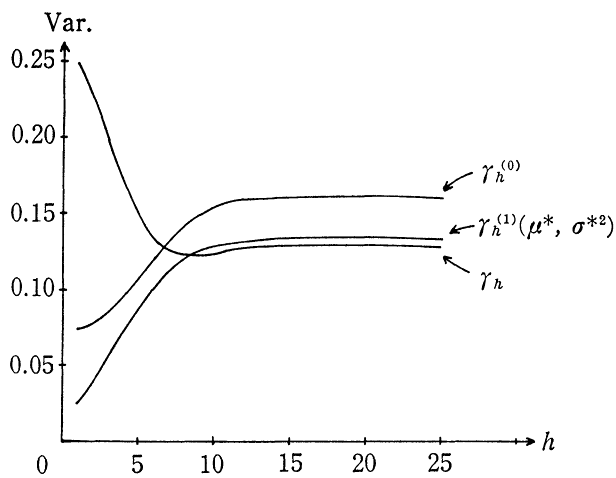

Fig. 4. ($N=50$)

small that the variance of $\gamma_1^{(0)}$ exceeds that of γ_1, and the former is about ten times as large as the latter, but, on the contrary the variance of $\gamma_1^{(1)}(0.4, 0)$ is below the latter. For $h=2$, on the other hand, the value ρ_2 is so large that the variance of $\gamma_2^{(0)}$ is smaller than that of γ_2, but the former is not so small as that of $\gamma_2^{(1)}(0.3, 0)$. This confirms that the estimate $\gamma_h^{(1)}(\mu^*, \sigma^{*2})$ is preferable except for large h.

The following Tables and Figures treat of the other models. Table 4 and Figure 3 present the results for the case where the correlogram is given by

$$\rho_h=\exp\left(-\frac{1}{10}\right)\left(\cos h+\frac{1}{10}\sin h\right)$$

which was treated in Iwase [4], [5]. It is seen that the variance of our estimate $\gamma_h^{(1)}(\mu^*, \sigma^{*2})$ is the smallest of the three except for large h, and that for large h the difference between the two variances of γ_h and $\gamma_h^{(1)}(\mu^*, \sigma^{*2})$ is almost negligible. We remark that for $h=1$ or 3, (μ^*, σ^{*2}) has two values, but there is little difference between $\mathrm{Var}[\gamma_h^{(1)}(\mu, \sigma^2)]$ and $\mathrm{Var}[\gamma_h^{(1)}(\mu^*, \sigma^{*2})]$ for other (μ, σ^2) around them.

Table 5 and Figure 4 are based on the process which has

$$\rho_h=\sqrt{2}(0.8)^h\cos(h\log 0.8+\pi/4) \qquad \text{(see Huzii [3]).}$$

This correlogram is analogous to the one in Table 2. For small h, ρ_h is so large that $\gamma_h^{(0)}$ and $\gamma_h^{(1)}$ have smaller variances than γ_h, and still more the variance of $\gamma_h^{(1)}(0.3, 0)$ is smaller than that of $\gamma_h^{(0)}$. Note that for $h\geqq 8$, we have $\sigma^{*2}\neq 0$. This is a rare phenomenon in our examples.

The performance of the estimate $\gamma_h^{(1)}$ is much better for $AR(2)$ process. It appears that, on the whole, $\gamma_h^{(1)}$ looks like a hybrid between γ_h and $\gamma_h^{(0)}$, and which is superior to them. The choice of the parameters μ^* and σ^{*2} is depend on the parent correlogram ρ_h, but when ρ_h is large in absolute value, we may put $\mu^*=0.2\sim 0.3$ and $\sigma^{*2}=0$, and also when ρ_h is small, $\mu^*=0.6\sim 0.7$ and $\sigma^{*2}=0$ will be preferable. From Tables 4 and 5, we feel that the similar results hold for the other models.

Appendix

Proof of Theorem.

First, let $h=0$. Then

$$\text{(A. 1)} \quad E[(\gamma_0^{(1)})^2]=\frac{\pi(\sigma^2+1)}{2N^2}\exp\left(\frac{\mu^2}{\sigma^2+1}\right)\left\{\sum_{t=1}^{N}E[X_t^2C_{U_t}^2(X_t)] +2\sum_{t<s}E[X_tX_sC_{U_t}(X_t)C_{U_s}(X_s)]\right\}.$$

We evaluate these expectations individually. Now we have

$$E[X_t^2C_{U_t}^2(X_t)]=\frac{1}{2}+\frac{1}{2}E[X_t^2\,\mathrm{sgn}(X_t-U_t)\,\mathrm{sgn}(X_t+U_t)]\,,$$

and observe that

$$\int_0^\infty\left\{\exp\left[-\frac{(u-\mu)^2}{2\sigma^2}\right]+\exp\left[-\frac{(u+\mu)^2}{2\sigma^2}\right]\right\}\int_u^\infty\exp\left(-\frac{y^2}{2\sigma^2}\right)dy\,du$$

$$=2\sigma\exp\left[-\frac{\mu^2}{2(\sigma^2+1)}\right]\int_0^{1/\sigma}\frac{1}{1+x^2}\exp\left[-\frac{\mu^2x^2}{2(\sigma^2+1)}\right]dx\,.$$

Then we have

$$\text{(A. 2)}\quad E[X_t^2C_{U_t}^2(X_t)]=\frac{2\sigma}{\pi(\sigma^2+1)}\exp\left(-\frac{\mu^2}{2\sigma^2}\right)$$
$$+\frac{2\mu}{\pi(\sigma^2+1)^{3/2}}\exp\left[-\frac{\mu^2}{2(\sigma^2+1)}\right]\int_0^{\mu/\sqrt{\sigma^2(\sigma^2+1)}}\exp\left(-\frac{x^2}{2}\right)dx$$
$$+\frac{2}{\pi}\exp\left[-\frac{\mu^2}{2(\sigma^2+1)}\right]\int_0^{1/\sigma}\frac{1}{1+x^2}\exp\left[-\frac{\mu^2x^2}{2(\sigma^2+1)}\right]dx\,.$$

To evaluate the second term of (A.1), we prepare the following lemmas.

LEMMA 1. *Let u ann v be real numbers. If $(X, Y)'$ has $N(0, 0, 1, 1\,;\rho)$, then*

$$E(u, v)=E[XY\,\mathrm{sgn}(X-u)\,\mathrm{sgn}(Y-v)]$$
$$=\frac{\rho}{\pi^2}E_1(u, v\,;\rho)+\frac{\rho}{\pi^2}E_2(u, v\,;\rho)+\frac{1+\rho^2}{\pi^2}E_3(u, v\,;\rho)-\frac{\rho}{\pi^2}E_4(u, v\,;\rho)\,,$$

where

$$E_1(u, v\,;\rho)=-2\pi v\exp\left(-\frac{v^2}{2}\right)\int_0^u\exp\left(-\frac{x^2}{2}\right)dx$$
$$-2\pi(1-v^2)\int_0^\rho\frac{1}{\sqrt{1-x^2}}\exp\left[-\frac{u^2-2uvx+v^2}{2(1-x^2)}\right]dx$$
$$+4\pi v\int_0^\rho\frac{x(u-vx)}{(1-x^2)^{3/2}}\exp\left[-\frac{u^2-2uvx+v^2}{2(1-x^2)}\right]dx$$
$$-2\pi\int_0^\rho\frac{x^2}{(1-x^2)^{3/2}}\left[1-\frac{(u-vx)^2}{1-x^2}\right]\exp\left[-\frac{u^2-2uvx+v^2}{2(1-x^2)}\right]dx\,,$$

$$E_2(u, v\,;\rho)=E_1(v, u\,;\rho)\,,$$

$$E_3(u, v\,;\rho)=\frac{2\pi}{\sqrt{1-\rho^2}}\exp\left[-\frac{u^2-2uv\rho+v^2}{2(1-\rho^2)}\right],$$

$$E_4(u, v\,;\rho)=-2\pi\left[\int_0^u\exp\left(-\frac{x^2}{2}\right)dx\right]\left[\int_0^v\exp\left(-\frac{x^2}{2}\right)dx\right]$$
$$-2\pi\int_0^\rho\frac{1}{\sqrt{1-x^2}}\exp\left[-\frac{u^2-2uvx+v^2}{2(1-x^2)}\right]dx\,.$$

This lemma is due to Okamoto and Iwase [7].

LEMMA 2. *Let U and V be independent random variables having $N(\mu, \sigma^2)$. For real value x such that $|x|<1$,*

(A. 3) $$E\left[\exp\left[-\frac{U^2-2UVx+V^2}{2(1-x^2)}\right]\right]=\exp\left(-\frac{\mu^2}{\sigma^2+1+x}\right)\sqrt{\frac{1-x^2}{(\sigma^2+1)^2-x^2}}$$

(A. 4) $$E\left[U^2\exp\left[-\frac{U^2-2UVx+V^2}{2(1-x^2)}\right]\right]=E\left[V^2\exp\left[-\frac{U^2-2UVx+V^2}{2(1-x^2)}\right]\right]$$
$$=\exp\left(-\frac{\mu^2}{\sigma^2+1+x}\right)\left\{\frac{\sigma^2(\sigma^2+1-x^2)}{(\sigma^2+1)^2-x^2}+\frac{\mu^2(1+x)^2}{(\sigma^2+1+x)^2}\right\}\sqrt{\frac{1-x^2}{(\sigma^2+1)^2-x^2}}$$

(A. 5) $$E\left[UV\exp\left[-\frac{U^2-2UVx+V^2}{2(1-x^2)}\right]\right]$$
$$=\exp\left(-\frac{\mu^2}{\sigma^2+1+x}\right)\left\{\frac{\sigma^4 x}{(\sigma^2+1)^2-x^2}+\frac{\mu^2(1+x)^2}{(\sigma^2+1+x)^2}\right\}\sqrt{\frac{1-x^2}{(\sigma^2+1)^2-x^2}}.$$

Now put $X_t=X$, $X_s=Y$, $U_t=U$ and $U_s=V$ in (A.1) for simplicity. Then the second term is rewritten as

$$E[XYC_U(X)C_V(Y)]=\frac{1}{2}\{E[XY\,\mathrm{sgn}(X-U)\,\mathrm{sgn}(Y-V)]$$
$$+E[XY\,\mathrm{sgn}(X-U)\,\mathrm{sgn}(Y+V)]\}.$$

Here we assume that, for $\sigma^2>0$, U and V are independent random variables having $N(\mu, \sigma^2)$ and also independent of X and Y. It is shown from Lemma 1 and Lemma 2 that

(A. 6) $$E[E_1(U, V; \rho)+E_1(U, -V; \rho)]$$
$$=-2\pi\int_{-\rho}^{\rho}\frac{4\sigma^2x^2+1+\sigma^2}{[(\sigma^2+1)^2-x^2]^{3/2}}\exp\left[-\frac{\mu^2}{\sigma^2+1+x}\right]dx$$
$$+2\pi\int_{-\rho}^{\rho}\frac{\mu^2(2x+1)^2}{(\sigma^2+1+x)\sqrt{(\sigma^2+1)^2-x^2}}\exp\left[-\frac{\mu^2}{\sigma^2+1+x}\right]dx$$
$$=2\pi I_1(\mu, \sigma^2; \rho),$$

(A. 7) $$E[E_2(U, \pm V; \rho)]=E[E_1(U, \pm V; \rho)],$$

(A. 8) $$E[E_3(U, V; \rho)+E_3(U, -V; \rho)]$$
$$=\frac{2\pi}{\sqrt{(\sigma^2+1)^2-\rho^2}}\left[\exp\left(-\frac{\mu^2}{\sigma^2+1+\rho}\right)+\exp\left(-\frac{\mu^2}{\sigma^2+1-\rho}\right)\right]$$
$$=2\pi G(\mu, \sigma^2; \rho),$$

(A. 9) $$E[E_4(U, V; \rho)+E_4(U, -V; \rho)]$$
$$=-2\pi\int_{-\rho}^{\rho}\frac{1}{\sqrt{(\sigma^2+1)^2-x^2}}\exp\left(-\frac{\mu^2}{\sigma^2+1+x}\right)dx$$
$$=-2\pi I_2(\mu, \sigma^2; \rho).$$

From these results we have

(A. 10) $E[XYC_U(X)C_V(Y)]$

$$=\frac{1}{\pi}[2\rho I_1(\mu,\ \sigma^2;\ \rho)+\rho I_2(\mu,\ \sigma^2;\ \rho)+(\rho^2+1)G(\mu,\ \sigma^2;\ \rho)].$$

Therefore from (A.1), (A.2) and (A.10) we have $E[(\gamma_0^{(1)})^2]$ and this gives the variance of $\gamma_h^{(1)}$ for lag $h=0$.

Second, let $h>0$. Then

$$E[(\gamma_h^{(1)})^2]=\frac{(\sigma^2+1)\pi}{8N^2}\exp\Big(\frac{\mu^2}{\sigma^2+1}\Big)\sum_{t=1}^{N}\sum_{s=1}^{N}E[X_tX_sC_{U_{t+h}}(X_{t+h})C_{U_{s+h}}(X_{s+h})$$
$$+X_sX_{t+h}C_{U_{s+h}}(X_{s+h})C_{U_t}(X_t)+X_{s+h}X_tC_{U_s}(X_s)C_{U_{t+h}}(X_{t+h})$$
$$+X_{s+h}X_{t+h}C_{U_s}(X_s)C_{U_t}(X_t)].$$

We divide the summation into three parts

$$\sum_{t=1}^{N}\sum_{s=1}^{N}E[\cdot]=2\sum_{\substack{t+h\neq s\\ t<s}}D_1(t,\ s)+2\sum_{\substack{t+h=s\\ t<s}}D_2(t,\ s)+\sum_{t=s}D_3(t)$$

and evaluate these expectations individually.

(I) Case where $t+h\neq s$ and $t<s$: We use the following lemma.

LEMMA 3. *Let u and v be real numbers. If $(X_1,\ X_2,\ X_3,\ X_4)'$ has a normal distribution with mean vector* $(0,\ 0,\ 0,\ 0)'$ *and covariance matrix*

$$\begin{pmatrix}1 & a_1 & a_2 & a_3\\ a_1 & 1 & a_4 & a_5\\ a_2 & a_4 & 1 & a_6\\ a_3 & a_5 & a_6 & 1\end{pmatrix},$$

then

$$E[X_1X_2\operatorname{sgn}(X_3-u)\operatorname{sgn}(X_4-v)]$$
$$=\frac{1}{\pi^2}\{a_2a_4E_1(u,\ v;\ a_6)+a_3a_5E_2(u,\ v;\ a_6)$$
$$+(a_3a_4+a_2a_5)E_3(u,\ v;\ a_6)-a_1E_4(u,\ v;\ a_6)\},$$

where $E_i(u,\ v;\ \cdot)$, $i=1,\ 2,\ 3,\ 4$, are given in Lemma 1.

From (A.11) we write $D_1(t,\ s)=D_{11}(t,\ s)+D_{12}(t,\ s)+D_{13}(t,\ s)+D_{14}(t,\ s)$. Then we have from Lemma 3

$$D_{11}(t,\ s)=\frac{1}{2}E[X_tX_s\operatorname{sgn}(X_{t+h}-U_{t+h})\operatorname{sgn}(X_{s+h}-U_{s+h})]$$
$$+E[X_tX_s\operatorname{sgn}(X_{t+h}-U_{t+h})\operatorname{sgn}(X_{s+h}+U_{s+h})]$$
$$=\frac{1}{2\pi^2}\{\rho_h\rho_{s-t-h}E[E_1(U,\ V;\ \rho_{s-t})+E_1(U,\ -V;\ \rho_{s-t})]$$

$$+\rho_{s+h-t}\rho_h E[E_2(U, V; \rho_{s-t})+E_2(U, -V; \rho_{s-t})]$$
$$+(\rho_h^2+\rho_{s-t+h}\rho_{s-t-h})E[E_3(U, V; \rho_{s-t})+E_3(U, -V; \rho_{s-t})]$$
$$-\rho_{s-t}E[E_4(U, V; \rho_{s-t})+E_4(U, -V; \rho_{s-t})]\}.$$

Hence it follows from (A.6)-(A.9)

$$D_{11}(t, s)=\frac{1}{\pi}\{\rho_h(\rho_{s-t-h}+\rho_{s-t+h})I_1(\mu, \sigma^2; \rho_{s-t})+\rho_{s-t}I_2(\mu, \sigma^2; \rho_{s-t})$$
$$+(\rho_h^2+\rho_{s-t-h}\rho_{s-t+h})G(\mu, \sigma^2; \rho_{s-t})\}.$$

A similar argument shows that

$$D_{12}(t, s)=\frac{1}{\pi}\{2\rho_h\rho_{s-t}I_1(\mu, \sigma^2; \rho_{s-t+h})+\rho_{s-t-h}I_2(\mu, \sigma^2; \rho_{s-t+h})$$
$$+(\rho_h^2+\rho_{s-t}^2)G(\mu, \sigma^2; \rho_{s-t+h})\},$$
$$D_{13}(t, s)=\frac{1}{\pi}\{2\rho_h\rho_{s-t}I_1(\mu, \sigma^2; \rho_{s-t-h})+\rho_{s-t+h}I_2(\mu, \sigma^2; \rho_{s-t-h})$$
$$+(\rho_h^2+\rho_{s-t}^2)G(\mu, \sigma^2; \rho_{s-t-h})\},$$

and $D_{14}(t, s)=D_{11}(t, s)$. Hence by combining these results $D_1(t, s)$ is obtained.

(II) Case where $t+h=s$: We use the following lemma.

LEMMA 4. *Let u and v be real numbers. If $(X_1, X_2, X_3)'$ has a normal distribution with mean vector $(0, 0, 0)'$ and covariance matrix*

$$\begin{pmatrix} 1 & a_1 & a_2 \\ a_1 & 1 & a_3 \\ a_2 & a_3 & 1 \end{pmatrix},$$

then

(i) $$E[X_1X_2 \operatorname{sgn}(X_2-u)\operatorname{sgn}(X_3-v)]=\frac{1}{\pi^2}\{a_1E_1(u, v; a_3)+a_2a_3E_2(u, v; a_3)$$
$$+(a_2+a_1a_3)E_3(u, v; a_3)-a_1E_4(u, v; a_3)\},$$

(ii) $$E[X_1^2 \operatorname{sgn}(X_2-u)\operatorname{sgn}(X_3-v)]=\frac{1}{\pi^2}\{a_1^2E_1(u, v; a_3)+a_2^2E_2(u, v; a_3)$$
$$+2a_1a_2E_3(u, v; a_3)-E_4(u, v; a_3)\},$$

where $E_i(u, v; \cdot)$, $i=1, 2, 3, 4$, are given in Lemma 1.

Now let $D_2(t, s)=D_{21}(t, s)+D_{22}(t, s)+D_{23}(t, s)+D_{24}(t, s)$, where

$$D_{21}(t, s)=E[X_tX_{t+h}C_{U_{t+h}}(X_{t+h})C_{U_{t+2h}}(X_{t+2h})],$$
$$D_{22}(t, s)=E[X_{t+h}^2C_{U_t}(X_t)C_{U_{t+2h}}(X_{t+2h})],$$
$$D_{23}(t, s)=E[X_{t+h}X_{t+2h}C_{U_{t+h}}(X_{t+h})C_{U_t}(X_t)],$$
$$D_{24}(t, s)=E[X_tX_{t+h}C_{U_{t+h}}^2(X_{t+h})].$$

Then if $\{U_t\}$ is a sequence of independent random variables having $N(\mu, \sigma^2)$, then from Lemma 4 and (A.6)-(A.9) it follows that

$$D_{21}(t, s)=\frac{1}{\pi}\{\rho_h(1+\rho_{2h})I_1(\mu, \sigma^2; \rho_h)+\rho_h I_2(\mu, \sigma^2; \rho_h)$$
$$+(\rho_{2h}^2+\rho_h^2)G(\mu, \sigma^2; \rho_h)\},$$

$$D_{22}(t, s)=\frac{1}{\pi}\{2\rho_h I_1(\mu, \sigma^2; \rho_{2h})+I_2(\mu, \sigma^2; \rho_{2h})+2\rho_h^2 G(\mu, \sigma^2; \rho_{2h})\},$$

$D_{23}(t, s)=D_{21}(t, s)$, and

$$D_{24}(t, s)=\frac{1}{2}\{\rho_{2h}+E[X_t X_{t+2h}\,\mathrm{sgn}(X_{t+h}-U_{t+h})\,\mathrm{sgn}(X_{t+h}+U_{t+h})]\}$$
$$=\frac{1}{\pi(\sigma^2+1)}\{\rho_h^2 E(\mu, \sigma^2)+\rho_{2h}(\sigma^2+1)F(\mu, \sigma)\}.$$

Thus from these results we have $D_2(t, s)$.

(III) Case where $t=s$: Let $D_3(t)=2\{D_{31}(t, h)+D_{32}(t, h)\}$, where

$$D_{31}(t, h)=E(X_t^2 C_{U_{t+h}}^2(X_{t+h})],$$

$$D_{32}(t, h)=E[X_t X_{t+h} C_{U_t}(X_t) C_{U_{t+h}}(X_{t+h})].$$

Then we have

$$D_{31}(t, h)=\frac{2}{\pi(\sigma^2+1)}\{\rho_h^2 E(\mu, \sigma^2)+(\sigma^2+1)F(\mu, \sigma^2)\}.$$

From Lemma 1 and Lemma 2 we get

$$D_{32}(t, h)=\frac{1}{\pi}\{2\rho_h I_1(\mu, \sigma^2; \rho_h)+\rho_h I_2(\mu, \sigma^2; \rho_h)+(1+\rho_h^2)G(\mu, \sigma^2; \rho_h)\}.$$

Thus $D_3(t)$ is evaluated.

Therefore from these results, putting $s-t=k$, we can obtain $E[(\gamma_h^{(1)})^2]$ and the variance of $\gamma_h^{(1)}$ for $h\geqq 1$.

This completes the proof of Theorem.

Acknowledgements·—The author wishes to thank Prof. K. Kunisawa for his helpful advice in the preparation of this paper and is also grateful to Dr. K. Shimizu and Dr. M. Jimbo for their many useful comments and advices.

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