Y. KITADA
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ORIENTATION REVERSING INVOLUTIONS ON BRIESKORN SPHERES

BY YASUHIKO KITADA

§ 1. Introduction and Results.

Free involutions on homotopy spheres have been studied extensively by many topologists, and in particular, when the sphere bounds a parallelizable manifold, interesting examples have been constructed using Brieskorn spheres or plumbing methods ([1], [2], [5], [6], [7]). But as for non-free involutions, especially when the involution reverses the orientation, few results have been known so far.

The purpose of this paper is to classify the orientation reversing involutions on Brieskorn spheres of dimension $4k+1$ which are defined by the conjugation of complex numbers. In this case, the fixed point set is of dimension $2k$ and to classify these examples, we meet with the failure of the "Gap Hypothesis" ([3], [9]). However, with the aid of $\boldsymbol{Z}_2$-surgery theory due to K. H. Dovermann [4], this situation can be handled.

Le us begin with showing our examples. Let $\boldsymbol{C}_c^{n+2}$ be the complex $(n+2)$-space with the conjugate involution, and f_d be a polynomial function

$$f_d(z_0, z_1, \cdots, z_{n+1})=z_0^d+z_1^2+\cdots+z_{n+1}^2$$

where n is even, and $d=2q+1$ is odd.

Denote by S^{2n+3} the unit sphere in $\boldsymbol{C}_c^{n+2}$ and set

$$W_d^{2n+1}=f_d^{-1}(0)\cap S^{2n+3}.$$

The involution on $\boldsymbol{C}_c^{n+2}$ keeps W_d^{2n+1} invariant and defines an involution T_d on W_d^{2n+1}.

The second construction is the equivariant attaching method ([1]). For a unit vector $x\in\boldsymbol{R}^{n+1}$, let θ_x be the reflection with respect to the hyperplane normal to x:

$$\theta_x y=y-2\langle x, y\rangle x\,,$$

where $\langle\,,\rangle$ is the usual inner product in $\boldsymbol{R}^{n+1}$. Let ϕ be the diffeomorphism of $S^n\times S^n$ defined by

$$\phi(x, y)=(\theta_x\theta_y x,\ \theta_x\theta_y y)\,.$$

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Then using the relation $\theta_{Ax}=A\theta_x A^{-1}$ for $A\in O(n+1)$, it is easy to verify that

$$\psi^q(x, y)=((\theta_x\theta_y)^q x, (\theta_x\theta_y)^q y).$$

The diffeomorphism ψ is equivariant with respect to the involution: $(x, y)\mapsto(x, -y)$ of $S^n\times S^n$. Therefore we have an involution τ_q on the manifold

$$\Sigma_q^{2n+1}=S^n\times D^{n+1}\bigcup_{\psi^q} D^{n+1}\times S^n,$$

where $D^{n+1}\times S^n$ is attached to $S^n\times D^{n+1}$ via ψ^q and the involution is given by $\tau_q(x, y)=(x, -y)$. About these two examples, we can prove their equivalence.

THEOREM 1. *$(W_{2q+1}^{2n+1}, T_{2q+1})$ and $(\Sigma_q^{2n+1}, \tau_q)$ are diffeomorphic involutions. In fact they are diffeomorphic as $\boldsymbol{Z}_2\times O(n+1)$-manifolds. (For the $O(n+1)$-structure, see* [2], [6].)

Denote by $S^{t+r-1}(r)$ the unit sphere in $\boldsymbol{R}_+^t\oplus\boldsymbol{R}_-^r$, where $\boldsymbol{R}_+^t$ (resp. $\boldsymbol{R}_-^r$) has the trivial (resp. antipodal) involution. Since the fixed point set of W_{2q+1}^{2n+1} is S^n, we have a $\boldsymbol{Z}_2$-equivariant homotopy equivalence:

$$W_{2q+1}^{2n+1}\longrightarrow S^{2n+1}(n+1).$$

This map can be constructed as follows. Take an equivariant open disk neighborhood U of a fixed point in $W=W_{2q+1}^{2n+1}$, and consider the natural collapsing map

$$f: W\longrightarrow W/W-U.$$

Then $W/W-U$ is $\boldsymbol{Z}_2$-homeomorphic to $S^{2n+1}(n+1)$, and f is of degree$=1$, and at the fixed point set, $f^{\boldsymbol{Z}_2}$ is also a degree 1 map. By the result of Matumoto [8], f is a $\boldsymbol{Z}_2$-homotopy equivalence.

For the $\boldsymbol{Z}_2$-equivariant normal cobordism class of our example, we have

THEOREM 2. *W_{2q+1}^{2n+1} is $\boldsymbol{Z}_2$-equivariantly normally cobordant to the linear involution $S^{2n+1}(n+1)$.*

Finally, by the $\boldsymbol{Z}_2$-surgery theory, we can classify W_{2q+1}^{2n+1} for all q.

THEOREM 3. *$(W_{2q+1}^{2n+1}, T_{2q+1})$ and $(W_{2q'+1}^{2n+1}, T_{2q'+1})$ are diffeomorphic as involutions if $q\equiv q'$ (mod. 4) or $q+q'\equiv 3$ (mod. 4) holds.*

Thus, when $n+2$ is not a power of 2, W_{2q+1}^{2n+1} and $W_{2q'+1}^{2n+1}$ are $\boldsymbol{Z}_2$-equivariantly diffeomorphic if and only if they are diffeomorphic forgetting the involution.

§2. $\boldsymbol{Z}_2\times O(n+1)$-action on W_{2q+1}^{2n+1}.

Let the action of $O(n+1)$ be defined by $A(z_0, z_1, \cdots, z_{n+1})=(z_0, A(z_1, \cdots, z_{n+1}))$ on W_{2q+1}^{2n+1} and on Σ_q^{2n+1},

$$A(x, y)=(Ax, Ay), \qquad \text{for } A\in O(n+1).$$

The orbit space is the 2-disk D^2 in either case and the projection maps are given by

$$p: W_{2q+1}^{2n+1} \longrightarrow D^2, \quad p(z_0, z_1, \cdots, z_{n+1})=-z_0/r_0$$

where $r_0>0$ satisfies $r_0^2+r_0^{2q+1}=1$ and

$$\pi: \Sigma_q^{2n+1} \longrightarrow D^2, \quad \pi(x, y)=\frac{|x|^2-|y|^2+2i\langle x, y\rangle}{|x|^2+|y|^2}.$$

If we give the orbit space D^2 the complex conjugate involution, these projections are equivariant.

From the theory of $O(n+1)$-manifolds, these two examples are diffeomorphic as $O(n+1)$-manifolds ([2], [6]). In our case, the involution commutes with the $O(n+1)$-action and the fixed point set of the subgroup $O(n-1)$ (*i.e.* W_{2q+1}^3 or Σ_q^3) meets every $O(n+1)$-orbit. Therefore, to prove the uniqueness of the involution which is commutative with the $O(n+1)$-action, it is enough to prove the uniqueness of the involution on W_{2q+1}^3 or on Σ_q^3.

The involution $\tau=\tau_q$ on Σ_q^3 satisfies the following properties:

(a) τ commutes with the $O(2)$-action.
(b) τ commutes with the projection π onto the orbit space, where the orbit space D^2 is given the conjugation involution.
(c) The fixed point set of τ is S^1 and lies over $1\in D^2$.

LEMMA. *The involution on Σ_q^3 which satisfies the three conditions above is unique up to $O(2)$-isotopy.*

Proof. Express the points of $\Sigma_q^3=S^1\times D^2\bigcup_{\phi^q} D^2\times S^1$ by pairs of complex numbers (x, y). The projection $\pi:\Sigma_q^3\to D^2$ is given by

$$\pi(x, y)=\frac{(x+iy)(\bar{x}+i\bar{y})}{|x|^2+|y|^2}$$

and the attaching map can be written as

$$\phi^q(x, y)=((x\bar{y})^{2q}x, (x\bar{y})^{2q}y).$$

Let $D_1=\operatorname{int} D^2$, then the portion over D_1, $\pi^{-1}(D_1)$ is an $O(2)$-bundle which is usually called the regular bundle, and the singular bundle $\pi^{-1}(\partial D^2)$ consists of the points with orbit type $(O(2)/K)$ where

$$K=\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}=O(1)\subset O(2).$$

On $(\Sigma_q^3)^K$, the $N(K)/K$ $(\cong \boldsymbol{Z}_2)$-action is given by $(x, y)\mapsto(-x, -y)$. Consider the total space of the singular bundle:

$$\Sigma^{(K)}=O(2)/K\times_{N(K)/K}\Sigma^{K}.$$

Identify $O(2)/K$ with S^1 via the identification:

$$\alpha:\begin{pmatrix} a & -\varepsilon b \\ b & \varepsilon a \end{pmatrix}K\longmapsto a+ib\,, \qquad \varepsilon=\pm 1\,.$$

The $N(K)/K$-action corresponds to $a+ib\mapsto -(a+ib)$. The subset $\Sigma^{(K)}$ of points having the isotropy type (K) can then be written as $\Sigma^{(K)}=S^1\times_{Z_2}S^1$, where $(z, w)\sim(-z, -w)$. The point $[z, w]\in\Sigma^{(K)}$ can be re-parametrized by the point $(u, v)\in S^1\times S^1$ such that $(u, v)=(zw, w^2)$. Then the $O(2)$-action on $\Sigma^{(K)}$ can be expressed by

$$A(u, v)=(\phi(A)u, v) \qquad \text{if} \quad A\in SO(2) \qquad \text{and}$$
$$A(u, v)=(\phi(A)\bar{u}v, v) \qquad \text{if} \quad A\in O(2)-SO(2)\,,$$

where $\phi: O(2)\to O(2)/K\xrightarrow{\alpha}S^1\subset \boldsymbol{C}$. The projection $\pi:\Sigma^{(K)}\to\partial D^2$ is just the second projection map $(u, v)\mapsto v$. We shall consider the involution τ on the singular bundle $\Sigma^{(K)}$. Given $u\in S^1$, choose $A\in SO(2)$ such that $\phi(A)=u$. Since τ commutes with A, we have

$$\begin{aligned}\tau(u, v)&=\tau(\phi(A)\cdot 1, v)=\tau A(1, v)=A\tau(1, v)\\ &=A(\sigma_v, \bar{v})=(u\sigma_v, \bar{v})\,,\end{aligned}$$

where σ_v is an element of S^1 such that $\tau(1, v)=(\sigma_v, \bar{v})$. Let $B\in O(2)-SO(2)$ satisfy $\phi(B)=1$. Then we have

$$\tau B(u, v)=\tau(\bar{u}v, v)=(\bar{u}v\sigma_v, \bar{v}) \qquad \text{and}$$
$$B\tau(u, v)=B(u\sigma_v, \bar{v})=(\bar{u}\bar{\sigma}_v\bar{v}, \bar{v})\,.$$

Hence $v\sigma_v=\bar{v}\bar{\sigma}_v$ must be real, *i.e.* $\sigma_v=\pm\bar{v}$. However at the fixed point set of the involution, $v=1$ and σ_v must be 1. This shows that $\tau(u, v)=(u\bar{v}, \bar{v})$ on $\Sigma^{(K)}$.

Next we shall look at the involution on the regular bundle $\Sigma_q^3-\Sigma^{(K)}\to D_1$. Similarly as before, we may write

$$\tau(A, v)=(A\rho_v, \bar{v}) \qquad \text{for} \quad (A, v)\in O(2)\times D_1\,.$$

From the relation $\tau^2=\mathrm{id}$, we have $\rho_v\rho_{\bar{v}}=I$ (the identity matrix). Suppose that $\rho_v\in SO(2)$, then for real v, we have $(\rho_v)^2=I$, and $\rho_v=\pm I$. But since τ has no fixed points on the regular bundle, ρ_v cannot be I. On the other, as v approaches 1, the $\boldsymbol{Z}_2$-equivariant map $O(2)\to O(2)/K$ must generate the fixed point set. Therefore ρ_v cannot be $-I$. This shows that $\rho_v\notin SO(2)$, and $(\rho_v)^2=I$ is always satisfied. Since the $\boldsymbol{Z}_2$-equivariant map

$$\rho_v: D_1\longrightarrow O(2)-SO(2)$$

is unique up to homotopy, the involution on Σ_q^3 is unique up to $O(2)$-isotopy. This proves Lemma.

§ 3. Z_2-Normal Cobordism and Surgery.

We shall next consider the normal cobordism class of the Z_2-homotopy equivalence for $W_{2q+1}^{2n+1} \to S^{2n+1}(n+1)$. By using equivariant deformation or directly by Theorem 1, we may replace W_{2q+1}^{2n+1} by a $Z_2 \times O(n+1)$-diffeomorphic manifold

$$W_{2q+1,\varepsilon}^{2n+1} = \{(z_0, z_1, \cdots, z_{n+1}) \in S^{2n+3} \mid f_d(z_0, z_1, \cdots, z_{n+1}) = \varepsilon\},$$

(abbreviated W_ε) where ε is a small positive real number and $d = 2q+1$. W_ε is the boundary of the manifold

$$F_\varepsilon^{2n+2} = \{(z_0, z_1, \cdots, z_{n+1}) \in D^{2n+4} \mid f_d(z_0, z_1, \cdots, z_{n+1}) = \varepsilon\},$$

which also has the $Z_2 \times O(n+1)$-action. Define

$$g : F_\varepsilon^{2n+2} \longrightarrow D^{2n+2}(n+1)$$

by $g(z_0, z_1, \cdots, z_{n+1}) = (w_1, w_2, \cdots, w_{n+1})$, where $w_k = z_k/\sqrt{1-|z_0|^2}$, $(k=1, 2, \cdots, n+1)$. Then degree$(g) = d$ and degree$(g^{Z_2}) = 1$. The normal bundle ν_{F_ε} of the embedding of F_ε in C_c^{n+2} is Z_2-isomorphic to $C_c \times F_\varepsilon$. Here, C_c is the one dimensional complex vector space with the conjugation as the Z_2-action. The Z_2-real vector bundle isomorphism $\tau_{F_\varepsilon} \times C_c \cong \tau(C_c^{n+2}) \mid F_\varepsilon$ (τ_{F_ε}: tangent bundle of F_ε) shows that

$$\begin{array}{ccc} \tau_{F_\varepsilon} \times C_c & \longrightarrow & C_c^{n+2} \times D^{2n+2}(n+1) \\ \downarrow & & \downarrow \\ F_\varepsilon & \xrightarrow{g} & D^{2n+2}(n+1) \end{array}$$

is a Z_2-normal map of degree$=d=2q+1$. We shall convert this normal map to the one with degree$=1$.

Let $\omega = \exp(2\pi i/d)$ be the primitive d-th root of unity and $p_m = (\varepsilon^{1/d}\omega^m, 0, \cdots, 0)$ $(m=0, 1, \cdots, 2q=d-1)$ be the d points in F_ε which constitute the inverse image $g^{-1}(0, \cdots, 0)$. Then the involution fixes p_0 and maps p_m to p_{d-m} for $m=1, \cdots, 2q$. Let D_m $(m=0, 1, \cdots, 2q)$ be a $(2n+2)$-disk which is mapped diffeomorphically to

$$D_\delta = \left\{(w_1, \cdots, w_{n+1}) \in D^{2n+2}(n+1) \mid \sum_{k=1}^{n+1} |w_k|^2 \leqq \delta\right\},$$

where δ is a small positive real number. Then put

$$F' = F_\varepsilon - \bigcup_{m=0}^{2q} \operatorname{int} D_m .$$

Since its boundary $\partial F'$ is the disjoint union of ∂F_ε and ∂D_m $(m=0, 1, \cdots, 2q)$, F' is a cobordism between

$$\partial_- F' = \partial D_0 \quad \text{and}$$

$$\partial_+ F' = (-\partial D_1) \cup \cdots \cup (-\partial D_{2q}) \cup \partial F_\varepsilon .$$

Next, remove $2q$ 1-handles from F' to obtain a cobordism F'' between $\partial_- F''=\partial_- F'=\partial D_0$ and $\partial_+ F''=\bigcup_{m=1}^{2q}(-\partial D_m)\#\partial F_\varepsilon$. To be precise, for $m=1, \cdots, 2q$ take $2q$ curves

$$L_m=\{(t\varepsilon^{1/d}\omega^{\eta m},\ \eta i(\varepsilon(t^d-1))^{1/2}, 0, \cdots, 0) \mid t_\delta \leqq t \leqq t_1\},$$

where $\eta=1$ $(m=1, \cdots, q)$ or -1 $(m=q+1, \cdots, 2q)$, and t_δ, t_1 are real numbers >1 satisfying

$$\varepsilon(t_\delta{}^d-1)=\delta(1-t_\delta{}^2\varepsilon^{2/d}) \quad \text{and}$$
$$\varepsilon(t_1{}^d-1)=1-t_1{}^2\varepsilon^{2/d}.$$

Again the involution maps L_m to L_{d-m}. L_m connects ∂D_m to ∂F_ε. Remove $2q$ 1-handles with core L_m equivariantly and we obtain the cobordism F''. By construction it is easy to see that $\partial_+ F''$ is diffeomorphic to $\partial F_\varepsilon = W^{2n+1}_{2q+1,\varepsilon}$. As for the map, g restricted to F' maps F' into $D^{2n+2}(n+1)-\operatorname{int} D_\delta = S^{2n+1}(n+1)\times I$, where we identify ∂D_δ with $S^{2n+1}(n+1)\times 0$ and $\partial D^{2n+2}(n+1)$ with $S^{2n+1}(n+1)\times 1$. g maps $\partial_- F'$ to $S^{2n+1}(n+1)\times 0$, but does not map $\partial D_m \cap \partial_+ F'$ to $S^{2n+1}(n+1)\times 1$ for $m \geqq 1$.

To make g a correct normal map, we can move g by homotopy relative to $(\partial F_\varepsilon \cap \partial_+ F'')\cup \partial_- F''$ to get a map (denoted by g') with $g'(\partial_+ F'')\subset S^{2n+1}(n+1)\times 1$. Moreover since the original map g maps each core L_m of the 1-handle to a line in $D^{2n+2}(n+1)$ and hence to a point in $S^{2n+1}(n+1)$, this homotopy can be chosen to preserve this property. Thus we obtain a $\boldsymbol{Z}_2$-normal map g' of degree$=1$:

$$g' : F'' \longrightarrow S^{2n+1}(n+1)\times I\,,$$

between $\partial_- F''=S^{2n+1}(n+1)$ and $\partial_+ F''=W^{2n+1}_{2q+1,\varepsilon}$.

Proof of Theorem 3.

We begin with the case $q'=0$. To apply the $\boldsymbol{Z}_2$-surgery theory of Dovermann [4], we must check three invariants for the normal cobordism F'' constructed above.

(1) At the fixed point set, the cobordism is already a product cobordism. Hence the $\boldsymbol{Z}_2$ homology obstruction $\sigma_{\boldsymbol{Z}_2}$ vanishes.

(2) $r=\operatorname{rank}_{\boldsymbol{Z}[\boldsymbol{Z}_2]} K_{2n+1}(F'', \boldsymbol{Z})$ (mod. 2)
$=\operatorname{rank}_{\boldsymbol{Z}} K_{2n+1}(F'', \boldsymbol{Z})/2$ (mod. 2)
$=(d-1)/2$ (mod. 2)$=q$ (mod. 2)

(3) The abstract Kervaire obstruction c (forgetting the $\boldsymbol{Z}_2$-action) is well known:

$$c=\begin{cases} 0 & \text{if } 2q+1\equiv \pm 1 \pmod{8} \\ 1 & \text{if } 2q+1\equiv \pm 3 \pmod{8}. \end{cases}$$

From (2) and (3), if $q\equiv 0$ (mod. 4), then all these obstructions vanish. On the other, when $q\equiv 3$ (mod. 4), only r does not vanish. However, this case can be handled in the following manner. Consider the involution $(x, y)\mapsto (y, x)$ on

$S^{n+1}\times S^{n+1}$. Form a connected sum of F'' with $S^{n+1}\times S^{n+1}$ around a fixed point in the interior of F''. This procedure does not change (1) and (3), but changes r by one. Therefore surgery is possible when the abstract Kervaire obstruction c vanishes. Thus by the equivariant s-cobordism theorem of Rothenberg [10], we have proved Theorem 3 when $q'=0$. For the general case, consider the $\boldsymbol{Z}_2$-normal cobordism between $S^{2n+1}(n+1)$ and $W^{2n+1}_{2q'+1,\varepsilon}$ and glue it to F'' along $S^{2n+1}(n+1)$ to obtain the $\boldsymbol{Z}_2$-normal cobordism between $W^{2n+1}_{2q+1,\varepsilon}$ and $W^{2n+1}_{2q'+1,\varepsilon}$. Then apply the same argument.

There is an alternative proof of Theorem 3 not using the result of Theorem 2:

For two values of q, say q and q', take F_ε for each q denoted by $F(q)$ and $F(q')$ respectively. Then $F(q)$ and $F(q')$ are $\boldsymbol{Z}_2$-quivariantly parallelizable and hence their connected sum along the boundary around a fixed point $F(q)\# F(q')$ is also equivariantly parallelizable. Consider the equivariant surgery problem of killing the homotopy groups of $F(q)\# F(q')$ relative to the boundary. As before, we must examine the three invariants. The first obstruction σ^{Z_2} vanishes since the connected sum of the fixed point sets, each diffeomorphic to S^n, is also diffeomorphic to S^n. If q and q' satisfies the condition of Theorem 3, then the third obstruction (abstract Kervaire obstruction) also vanishes. For the obstruction r, the argument goes similarly as before.

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Department of Applied Mathematics
Faculty of Engineering
Yokohama National University
Tokiwadai, Hodogaya-ku, Yokohama
240, Japan