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ON SOME SUBMANIFOLDS OF A LOCALLY PRODUCT MANIFOLD

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An investigation of properties of submanifolds of the almost product or locally product Riemannian manifolds has been started in the last years, many interesting results being obtained. So, Okumura [8], Adati and Miyazawa [1], Miyazawa [7], studied the hypersurfaces of such manifolds, Adati [2], defined and studied the invariant, anti-invariant and non-invariant submanifolds, while Bejancu [4], analyzed the semi-invariant submanifolds which are corresponding to CR-submanifolds of a Kaehler manifold [3].

The purpose of this paper is to give some properties of the anti-invariant and semi-invariant submanifolds, by using cohomology groups.

In § **1** we recall the definition of these submanifolds and some known results, already. An example of semi-invariant submanifold is given.

In § **2** we associate to a semi-invariant submanifold a de Rham cohomology class (as in [5] for CR-submanifolds) and we obtain a connection between the properties of the invariant and anti-invariant distributions and the cohomology of the submanifold (theorem 2.2).

The stability of some anti-invariant submanifolds of a locally product Riemannian manifold is studied in § **3** and we give algebraic conditions for stability.

§ 1. Anti-invariant and semi-invariant submanifolds of a locally product Riemannian manifold. Let $(\tilde{M}, g, F)$ be a C^∞-differentiable almost product Riemannian manifold, where g is a Riemannian metric and F is a non-trivial tensor field of type (1.1). Moreover g and F satisfy the following conditions

$$F^2=I\,,\quad (F\neq\pm I)\,;\quad g(FX, FY)=g(X, Y)\,,\quad X, Y\in\mathfrak{X}(\tilde{M}) \tag{1.1}$$

where I is the identity and $\mathfrak{X}(\tilde{M})$ is the Lie algebra of vector fields on $\tilde{M}$.

We denote by $\tilde{\nabla}$ the Levi-Civita connection on $\tilde{M}$ with respect to g and furthermore we assume that $\tilde{M}$ is locally product, that is

$$\tilde{\nabla}_X F=0 \qquad X\in\mathfrak{X}(\tilde{M})\,. \tag{1.2}$$

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ and denote

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by the same symbol g the Riemannian metric induced on M.

If on M there exist two complementary and orthogonals distributions D and $D^{\perp}$, satisfying the following conditions:

$$F(D_x)=D_x\,;\quad F(D_x^{\perp})\subset T_xM^{\perp}\qquad \text{for each}\quad x\in M$$

then M is called a *semi-invariant submanifold* of the locally product Riemannian manifold $\tilde{M}$ [4].

Particularly if $\dim D_x=\dim T_xM$ ($\dim D_x=0$) for each $x\in M$ then M is an *invariant submanifold* (*anti-invariant submanifold*) of $\tilde{M}$ [2].

We have the following results

THEOREM 1.1. [2] *In a 2n-dimensional locally product Riemannian manifold every anti-invariant submanifold of dimension n is totally geodesic.*

THEOREM 1.2. [4] *For a semi-invariant submanifold of a locally product Riemannian manifold the following assertions hold:*
a) *the distribution $D^{\perp}$ is involutive if and only if*

$$g(h(Y,Z),FX)=0\qquad X,Y\in D^{\perp},\quad Z\in D$$

where h denote the second fundamental form of M;
b) *M is D-geodesic (i.e. $h(X,Y)=0$ for $X,Y\in D$) if and only if D is involutive and each leaf of D is totally geodesic immersed in $\tilde{M}$;*
c) *the distribution D is involutive if and only if*

$$h(X,FY)=h(FX,Y)\qquad X,Y\in D.$$

Example. Let M be a normal metric almost paracontact manifold [9], [10], with the structure (φ,ξ,η,g) and denote by D^* the distribution $\{X\in TM:\eta(X)=0\}$. It is well-known that by putting

$$FX=\varphi(X),\quad X\in D^*;\quad F\xi=\frac{d}{dt};\quad F\left(\frac{d}{dt}\right)=\xi \tag{1.3}$$

F defines an almost product metric structure on $M\times R$. The product metric on $M\times R$ satisfies condition (1.1) and as M is normal, it follows that $M\times R$ with this structure is local product. M is a closed submanifold of $M\times R$ and from (1.3) follows that $F\xi\in TM^{\perp}$, $FD^*=D^*$ and then M is semi-invariant in $M\times R$.

§2. **Cohomology of semi-invariant submanifolds.** Furthermore we assume M as a compact without boundary manifold.

If $\dim D^{\perp}=q$ and $\dim D=p$ then we denote by $\mathcal{B}_{D^{\perp}}=\{X_1,\cdots X_q\}$, $\mathcal{B}_D=\{X_{q+1},\cdots,X_{q+p}\}$ two orthonormal local bases in $D^{\perp}$, resp. in D.

PROPOSITION 2.1. *If the distribution $D^{\perp}$ is involutive, then each leaf of $D^{\perp}$ is minimal in M.*

Proof. The mean-curvature vector of $D^{\perp}$ is

$$H_{D^{\perp}}=\frac{1}{q}\sum_{i=1}^{q}(\nabla_{X_i}X_i)^{\perp}$$

where $(\nabla_X X)^{\perp}$ is the component of $\nabla_X X$ in D.

By applying the Gauss formula

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X, Y) \qquad X, Y\in\mathcal{X}(M) \tag{2.1}$$

where ∇ is the Riemannian connection on M, induced by $\tilde{\nabla}$, from the conditions (1.1) and (1.2) we have

$$g(Y, \nabla_X X)=g(FY, \tilde{\nabla}_X(FX)), \qquad X\in\mathcal{B}_{D^{\perp}}, \quad Y\in D \tag{2.2}$$

Now, by using the Weingarten formula

$$\tilde{\nabla}_X N=-A_N X+\nabla_X^{\perp}N \qquad N\in TM^{\perp} \tag{2.3}$$

from (2.2) we can deduct

$$g(Y, \nabla_X X)=-g(FY, A_{FX}X) \tag{2.4}$$

and taking into account the known equality

$$g(h(X, Y), N)=g(A_N X, Y) \tag{2.5}$$

we can write

$$g(Y, \nabla_X X)=-g(h(X, FY), FX) \tag{2.6}$$

From (2.6) and the theorem 1.2, a) follows $g(Y, \nabla_X X)=0$ and then $H_{D^{\perp}}=0$.
Q. E. D.

PROPOSITION 2.2. *For a D-geodesic semi-invariant submanifold the distribution D is minimal.*

Proof. If $X\in\mathcal{B}_D$ then there exists $\bar{X}\in D$ such that $F\bar{X}=X$ and from (1.1), (1.2), (2.1), (2.3), (2.5) we obtain

$$g(Y, \nabla_X X)=g(FY, h(F\bar{X}, \bar{X})), \qquad Y\in D^{\perp}. \tag{2.7}$$

But M is D-geodesic and then from (2.7) follows $g(Y, \nabla_X X)=0$, hence D is minimal. Q. E. D.

We define on M the 1-forms $\omega^1, \cdots, \omega^q$, satisfying the following conditions

$$\omega^i(Z)=0, \quad Z\in D; \quad \omega^i(X_j)=\delta^i_j, \quad X_j\in\mathcal{B}_{D^{\perp}}; \quad i, j\in\overline{1, q}. \tag{2.8}$$

Then we give the q-form $\omega=\omega^1\wedge\cdots\wedge\omega^q$, globally defined on M and we have the following

PROPOSITION 2.3. *If the distribution D is involutive and $D^{\perp}$ is minimal then the q-form ω is closed.*

Proof. It is enough to prove that

$$d\omega(Y, X_1, \cdots, X_q)=0 \quad \text{for} \quad Y\in D, \quad X_1, \cdots, X_q\in \mathcal{B}_{D^{\perp}} \tag{2.9}$$

$$d\omega(Y_1, Y_2, X_1, \cdots, X_{q-1})=0 \quad \text{for} \quad Y_1, Y_2\in D, \quad X_1, \cdots, X_{q-1}\in \mathcal{B}_{D^{\perp}}. \tag{2.10}$$

From the definition of the forms ω^i it follows

$$d\omega(Y, X_1, \cdots, X_q)=\sum_{i=1}^{q} g([Y, X_i], X_i) \tag{2.11}$$

Now, the connection $\tilde{\nabla}$ is Riemannian and then

$$g(X_i, \tilde{\nabla}_Y X_i)=0 \qquad X_i\in \mathcal{B}_{D^{\perp}}, \quad Y\in D. \tag{2.12}$$

Taking into account (2.12), (2.1) follows

$$g([Y, X_i], X_i)=g(\nabla_{X_i} X_i, Y) \tag{2.13}$$

But $D^{\perp}$ is minimal and then from (2.11) and (2.13) we obtain (2.9).

The distribution D being involutive, (2.10) follows from the equality

$$d\omega(Y_1, Y_2, X_1, \cdots, X_{q-1})=-\omega([Y_1, Y_2], X_1, \cdots, X_{q-1}). \tag{2.14}$$

In the same manner as above we can define on M the 1-forms $\theta^{q+1}, \cdots, \theta^{q+p}$ by

$$\theta^{q+i}(Z)=0, \quad Z\in D^{\perp}; \quad \theta^{q+i}(X_{q+j})=\delta^i_j, \quad X_{q+j}\in \mathcal{B}_D, \quad i, j\in \overline{1, p}. \tag{2.15}$$

By using a similar computation as in the proof of the proposition 2.3 we can state the following

PROPOSITION 2.4. *If D is minimal and if $D^{\perp}$ is involutive then the p-form $\theta=\theta^{q+1}\wedge\cdots\wedge\theta^{q+p}$ is closed on M.*

Now from Propositions 2.3, 2.4 we have $\theta=*\omega$ and by applying the Hodge-de Rham theorem we obtain the

THEOREM 2.1. *For any compact semi-invariant submanifold M, of a Riemannian locally product manifold, having the distribution D involutive and $D^{\perp}$ minimal, a cohomology de Rham class $[\omega]\in H^q(M, R)$ is well-defined. This class is non trivial if D is minimal and $D^{\perp}$ is involutive.*

From Proposition 2.1 and Theorem 2.1 follows

THEOREM 2.2. *Let M be a compact semi-invariant submanifold of a locally product Riemannian manifold. If the distributions D, $D^{\perp}$ are involutives and D is minimal then*

$$H^{q}(M, R)\neq 0 \qquad q=\dim D^{\perp}.$$

From Theorems 1.2, 2.2 and Proposition 2.2 we deduce the following

PROPOSITION 2.5. *For every compact and totally geodesic semi-invariant submanifold of a locally product Riemannian manifold the Betti number b_q, $q=\dim D^{\perp}$, not vanish.*

Now taking into account the Theorem 2.2 and the above example we obtain the following

PROPOSITION 2.6. *Let M be a compact normal metric almost paracontact manifold. If the distribution D^* is involutive and minimal then*

$$H^{1}(M, R)\neq 0$$

Remark. It is well-known that if M is a SP-Sasakian manifold then the distribution D^* is involutive [9], [10].

Next we can make some comments on the obtained results.

A. Let S^{2n+1} be the unit sphere in R^{2n+2}, $n\geqq 2$, endowed with the standard Sasakian structure (f, ξ, η). It is known that the tensor field F given by

$$g(FX, Y)=(\nabla_X \eta)Y$$

defines a SP-Sasakian structure on S^{2n+1}. Moreover $H^{1}(S^{2n+1}, R)=0$ for $n\geqq 1$ and then the sphere S^{2n+1}, $n\geqq 2$, is a semi-invariant submanifold of $S^{2n+1}\times R$, so that the distribution D^* is not minimal.

B. Suppose M is a compact SP-Sasakian manifold totally geodesic immersed in the Riemannian locally product manifold $M\times R$. From Theorem 1.2 and Proposition 2.2 it follows that the distribution D^* is involutive and minimal. Hence the first Betti number of M not vanish.

§ 3. Stability of anti-invariant submanifolds.

Let M be a compact n-dimensional anti-invariant submanifold of the $2n$-dimensional locally product Riemannian manifold $\tilde{M}$.

By applying (2.1), (2.3) and Theorem 1.1 we have the

LEMMA 3.1. *For every $X, Y\in TM$ the next equalities holds*

$$\nabla^{\perp}_{Y}(FX)=F(\nabla_Y X) \qquad A_{FX}Y=0.$$

Let $\{X_1, \cdots, X_n\}$ be a orthonormal local basis in TM and lets denote by S and $\tilde{S}$ the Ricci tensors associated to the manifolds M and $\tilde{M}$.

LEMMA 3.2. *For every $X \in TM$ we have*

$$\sum_{i=1}^{n} \tilde{R}(X_i, FX, FX, X_i) = \tilde{S}(X, X) - S(X, X).$$

Proof. From (1.1), (1.2) it follows that

$$\tilde{R}(X_i, FX, FX, X_i) = \tilde{R}(FX_i, X, X, FX_i). \tag{3.1}$$

Now the required equality is a consequence of (3.1) and of the Gauss equation

$$\begin{aligned} R(U, V, W, T) = \tilde{R}(U, V, W, T) &+ g(h(U, T), h(V, W)) \\ &- g(h(U, W), h(V, T)). \end{aligned} \tag{3.2}$$

Let N be a normal vector field and denote by $\mathcal{V}''(N)$ the second normal variation of M induced by N. Then we have ([6], chap. I)

$$\mathcal{V}''(N) = \int_M \left\{ \|\nabla^{\perp} N\|^2 - \sum_{i=1}^{n} \tilde{R}(X_i, N, N, X_i) - \|A_N\|^2 \right\} dV \tag{3.3}$$

where dV is the volume form of M.

On the other hand if η is the 1-form associated to the vector field $X \in TM$ then we have the well-known formula ([6], chap. V)

$$\int_M \left\{ S(X, X) + \|\nabla X\|^2 - \frac{1}{2}\|d\eta\|^2 - (\delta\eta)^2 \right\} dV = 0. \tag{3.4}$$

Taking into account the Lemma 3.1 and the Theorem 1.1, from (3.3), (3.4) follow the

PROPOSITION 3.1. *The normal variation induced by the normal vector field $N = FX$ of the compact anti-invariant submanifold M in a locally product Riemannian manifold is given by*

$$\mathcal{V}''(N) = \int_M \left\{ \frac{1}{2}\|d\eta\|^2 + (\delta\eta)^2 - \tilde{S}(N, N) \right\} dV.$$

Now we can state the following

THEOREM 3.1. *Let M be a compact anti-invariant submanifold of the locally product Riemannian manifold $\tilde{M}$.*
a) *If $\tilde{S}$ is negative definite then M is stable.*
b) *If $H^1(M, R) \neq 0$ and $\tilde{S}$ is positive definite then M is unstable.*

Proof. a) is an immediate consequence of the Proposition 3.1 because we obtain $\mathcal{V}''(N) > 0$ for every $N \in TM^{\perp}$.
b) Since $H^1(M, R) \neq 0$ there exists an harmonic 1-form η on M and if X is it associated vector field, we have $d\eta = \delta\eta = 0$ and then $\mathcal{V}''(FX) < 0$. Consequently M is unstable in $\tilde{M}$.

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