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A CERTAIN PROPERTY OF GEODESICS OF THE FAMILY OF RIEMANNIAN MANIFOLDS O_n^2 (VIII)

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§ 0. Introduction.

This is exactly a continuation of Part (VII) ([18]) with the same title written by the present author which proved the following conjecture is true for $5 \leqq n \leqq 9.7$. We shall show that this conjecture is also true for $4.5 \leqq n \leqq 5$ in the present paper by the same method as in Part (VII) which is a little revised one and may be valid also for $3 \leqq n \leqq 4.5$ (see the final remark of this paper), but perhaps useless for $2 \leqq n \leqq 3$, because the constant b_n, having carried out an important role in the proof of the conjecture, is defined for $2.5 \leqq n < \infty$ and monotone decreasing in $3 \leqq n < \infty$. As the previous ones we shall use the numerical data obtained by means of computors in the verification. We shall also use the same notation in the previous ones Part (I)~(VII).

The period T of any non-trivial solution $x(t)$ of the non-linear differential equation of order 2:

$$\text{(E)} \qquad nx(1-x^2)\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+(1-x^2)(nx^2-1)=0$$

with a constant $n>1$ such that $x^2+x'^2<1$ is given by the integral:

$$\text{(0.1)} \qquad T=\sqrt{nc}\int_{x_1}^{x_2}\frac{dx}{x\sqrt{(n-x)\{x(n-x)^{n-1}-c\}}},$$

where $x_0=n\{\min x(t)\}^2$, $x_1=n\{\max x(t)\}^2$, $0<x_0<1<x_1<n$ and $c=x_0(n-x_0)^{n-1}=x_1(n-x_1)^{n-1}$.

CONJECTURE C. The period T as a function of $\tau=(x_1-1)/(n-1)$ and n is monotone decreasing with respect to n (>2) for any fixed τ ($0<\tau<1$).

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§ 1. Preliminaries.

Setting $T=\Omega(\tau, n)$, we have the formulas

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$$\frac{\partial\Omega(\tau, n)}{\partial n}=-\frac{\sqrt{c}}{2b^2n\sqrt{n}}\int_{x_0}^{x_1}\frac{(1-x)\sqrt{x(n-x)^{n-1}-c}}{x^2(n-x)^n}V(x, x_1)dx \tag{1.1}$$

((7.4) and Proposition 3 in (III)), where $b=\sqrt{B-c}$, $B=(n-1)^{n-1}$ and $V(x, x_1)$ are defined as follows:

$$V(x, x_1):=\frac{x^2N(x, x_1)}{(1-x)^5\sqrt{n-x}}+\frac{X^2N(X, x_1)}{(X-1)^5\sqrt{n-X}}, \tag{1.2}$$

where

$$f_0(z):=(2n-1-z)B-(n-z)^{n-1}\{n-z+(n-1)z^2\}, \tag{1.3}$$

$$f_1(z):=\{4n-1-(2n+1)z\}B-(n-z)^{n-1}\{n+(2n-1)z-(n+1)z^2\}, \tag{1.4}$$

$$F_2(z):=-P_2(z)B+(n-z)^{n-1}P_3(z), \tag{1.5}$$

$$P_2(z):=(2n+1)z^2-2(2n^2+5n-4)z+16n^2-16n+3, \tag{1.6}$$

$$P_3(z):=-(n-1)z^3+(2n^2-7n+8)z^2+(n-3)(4n-1)z+3n(2n-1), \tag{1.7}$$

$$\lambda(z):=\log(n-z)+\frac{n-1}{n-z},\quad \psi(z):=z(n-z)^{n-1}, \tag{1.8}$$

$$\tilde{\lambda}(z):=\lambda(z)-\frac{n}{n-1}\,\frac{(z-1)^2}{z(n-z)}=\log(n-z)+\frac{nz-1}{(n-1)z}, \tag{1.9}$$

$$\begin{aligned} N(z, x_1):=&(n-z)F_2(z)\{\lambda(z)-\tilde{\lambda}(x_1)\}\\ &+3(z-1)^2f_0(z)-2n(z-1)^3\{B-z(n-z)^{n-1}\}, \end{aligned} \tag{1.10}$$

and $X=X_n(x)$, $0<x<1<X<n$, defined by $\psi(x)=\psi(X)$.

$V(x, x_1)$ is increasing with respect to x_1 in $X_n(x)\leqq x_1<n$ for $0<x<1$ and $V(x, X_n(x))$ is negative near $x=0$ when $2<n<\frac{5+\sqrt{13}}{4}=2.15138\cdots$ and near $x=1$ when $2<n<\frac{1+\sqrt{13}}{2}=2.30277\cdots$, by Lemma 8.1 in (III). We shall show that $V(x, X_n(x))>0$ when $3\leqq n\leqq 5$, which implies $\partial\Omega(\tau, n)/\partial n<0$ by (1.1). We have the formula (8.1) in (III):

$$V(x, X(x))=U_0(x)-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x), \tag{1.11}$$

where $X(x)=X_n(x)$ and

$$U_0(x):=\frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X(x))\},\quad U_1(x):=\frac{3x^2f_0(x)}{(x-1)^3\sqrt{n-x}},$$

$$U_2(x):=\frac{2nx^2\{B-\psi(x)\}}{(1-x)^2\sqrt{n-x}},\quad U_3(x):=\frac{nxF_2(x)}{(n-1)(x-1)^3\sqrt{n-x}},$$

$$U_4(x):=U_3(X(x)),\quad U_5(x):=U_1(X(x)),\quad U_6(x):=U_2(X(x)),$$

and the inequality (2.4) in (VI):

$$V(x,\ X(x)) > \frac{2n(n-x)^{2(n-1)}\{B-\phi(x)\}\sqrt{B}}{\{B-(n-x)^{n-1}\}^2\sqrt{nB-(n-x)^{n-1}}} + \frac{2nX\sqrt{n-X}B}{(n-1)(X-1)^3}H(X) + \frac{x^2\sqrt{n-x}F_2(x)\{\lambda(x)-\tilde{\lambda}(X)\}}{(1-x)^5} \quad \text{for} \quad 0<x<1, \tag{1.13}$$

where

$$H(X) := \left(\frac{n-X}{n-1}\right)^{n-1}\{(n-1)X^2+(2n-5)X+3n\} - \{8n-5-(2n+1)X\}. \tag{1.14}$$

All the factors except $H(X)$ in the right hand side of (1.13) are positive. For $n \geqq 2.5$, we defined the constant b_n by $H(b)=0$, $1<b\leqq n$. Regarding b_n, we have the following

LEMMA 1.1. *b_n has the following properties:*

i) $b_n > \dfrac{2n+10}{2n+1}$ *for* $n>2.5$;

ii) $b_{2.5}=b_3=2.5$, $2.5<b_n<n$ *for* $2.5<n<3$, $2.25<b_n<2.5$ *for* $3<n<4$ *and* $b_n<2.268$ *for* $n\geqq 4$;

iii) *b_n is decreasing in* $3<n<\infty$.

(See Lemma 2.2, Lemma 2.3, Lemma 2.4 and Theorem 1 in [20]).

Since we have

$$H(X)\geqq 0 \quad \text{for} \quad b_n \leqq X < n \quad \text{and} \quad H(X)<0 \quad \text{for} \quad 1<X<b_n,$$

it is sufficient to prove $V(x, X_n(x))>0$ for $X_n^{-1}(b_n)<x<1$ for our purpose.

Regarding the evaluation of $X_n^{-1}(X)$, we cite some results. We set

$$w = w(n,\ X) := \left(\frac{n-X}{n-1}\right)^{n-1} \quad \text{for } 1<X<n. \tag{1.15}$$

LEMMA 1.2. *$w(n, X)$ is increasing with respect to n (>1) with $1<X<n$ and decreasing with respect to X in $1<X<n$. When $n>2$, we have*

$$w(n,\ X) < x = X_n^{-1}(X) \quad \text{for} \quad 1<X<n. \tag{1.16}$$

(Lemma 4.1 in (VII)).

We set for $n\geqq 2$, $1<C<n$,

$$k_n(C) := \frac{n+1+(n-1)C-\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}{2C}, \tag{1.17}$$

and obtain

$$n-k_n(C) = \frac{(n+1)(C-1)+\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}{2C}.$$

LEMMA 1.3. *i) $k_n(C)$ is increasing with respect to C in $1<C\leqq\frac{5n-1}{2(n+1)}$, with $n>2$, and*

$$k_n(1)=1 \quad and \quad k_n\left(\frac{5n-1}{2(n+1)}\right)=\frac{(n+1)^2}{5n-1}.$$

ii) $k_n(C)$ is increasing with respect to n ($\geqq 2.5$) and $n-k_n(C)$ is increasing with respect to n (>2), with $1<C\leqq\frac{5n-1}{2(n+1)}$.

Proof. i) and the first part of ii) are proved in Lemma 2.3 in (VII). Regarding the second part, it is evident from

$$(1.18)\quad \frac{\partial}{\partial n}(n-k_n(C))$$
$$=\frac{(n-1)C^2-2nC+5n-3+(C-1)\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}{2C\sqrt{(n-1)\{(n-1)C^2-2(n+1)C+5n-1\}}}$$

and $(n-1)C^2-2nC+5n-3>0$ and $(n-1)C^2-2(n+1)C+5n-1>0$ for any C and $n>2$. Q. E. D.

LEMMA 1.4. *When $n>2$, for $1<C\leqq\frac{5n-1}{2(n+1)}$, $k_n(C)\leqq k<n-1$ and $1<X\leqq C$ we have*

$$(1.19)\qquad x=X_n^{-1}(X)<\left(\frac{n-X}{n-1}\right)^{n-k}.$$

(Lemma 2.2 in (VII)).

Noticing

$$n-k_n\left(\frac{5n-1}{2(n+1)}\right)=\frac{(n-1)(4n+1)}{5n-1},$$

we set

$$(1.20)\qquad y=y(n,X):=\left(\frac{n-X}{n-1}\right)^{\frac{(n-1)(4n+1)}{5n-1}} \quad \text{for} \quad 1<X<n.$$

LEMMA 1.5. *$y(n, X)$ is increasing with respect to n (>1), with $1<X<n$, and decreasing with respect to X in $1<X<n$, with $n>1$.*
(Lemma 4.3 in (VII)).

§ 2. Evaluation of $V(x, X_n(x))$ near $x=1$ for $3\leqq n<5$.

We shall make an analogous formula on the evaluation of $V(x, X_n(x))$ near $x=1$ for $3\leqq n\leqq 5$ to those in Proposition 4 in § 5 of (VII).

First, we shall investigate the case of $3\leqq n<4$.

LEMMA 2.1. *We have*

$$(2.1)\quad \frac{U_0(x)}{\sqrt{n-1}} > \frac{n^2(n^2-n+1)}{6}(n-1)^{n-4}(X-1)$$
$$-\frac{n(n^2-n+1)\{3n(2n-13)p_n+18n^2-28n-4\}}{108}(n-1)^{n-5}(X-1)^2$$
$$+\frac{n(n^2-n+1)(4n-5)p_n(6np_n+18n^2-28_n-4)}{216}(n-1)^{n-6}(X-1)^3$$

for $1<X\leqq C\leqq 3/2,\ 3\leqq n\leqq 4$,

where

$$(2.2)\qquad p_n=\frac{(n-1)^{n-k}}{(n-k)(n-C)^{n-k-1}},\qquad k=k_n(C)$$

and

$$(2.3)\qquad \frac{X-1}{1-x}<p_n.$$

Proof. By using partially the argument in the proof of Lemma 5.1 in (VII) and assuming $3\leqq n\leqq 4$, we have

$$\frac{d}{dX}\frac{(2-X)(n-X)^{n-3}}{(n-x)^n}=-\frac{(n-2)(3-X)}{(n-x)^n(n-X)^{4-n}}-\frac{n(X-1)(2-X)(n-X)^{2n-5}}{(1-x)(n-x)^{2n-1}}$$
$$>-\frac{(n-2)\cdot 2}{(n-1)^n(n-C)^{4-n}}-\frac{np_n}{(n-1)^4}$$
$$=-\frac{1}{(n-1)^4}\left\{2(n-2)\left(\frac{n-1}{n-C}\right)^{4-n}+np_n\right\}$$

for $1<X\leqq C\leqq 3/2$,

since we have

$$\frac{3}{2}<\frac{5n-1}{2(n+1)}$$

and (2.3) holds by Lemma 2.5 in (VII). On the other hand we have

$$\left(\frac{n-1}{n-C}\right)^{4-n}\leqq\left(\frac{n-1}{n-1.5}\right)^{4-n}\leqq\left(\frac{4}{3}\right)^{4-n}\leqq\frac{4}{3},$$

and hence

$$\frac{d}{dX}\frac{(2-X)(n-X)^{n-3}}{(n-x)^n}>-\frac{1}{(n-1)^4}\left\{\frac{8(n-2)}{3}+np_n\right\},$$

and so

$$\frac{(n-1)(2-X)(n-X)^{n-3}}{(n-x)^n}>\frac{1}{(n-1)^2}-\frac{1}{(n-1)^3}\left\{\frac{8(n-2)}{3}+np_n\right\}(X-1).$$

Thus, we obtain

$$\frac{d^2}{dX^2}\{\lambda(x)-\tilde{\lambda}(X)\}$$

$$=\frac{(n-1)(2-X)(n-X)^{n-3}}{(n-x)^n}-\frac{n(X-1)^2(n-X)^{2n-4}}{(1-x)(n-x)^{2n-1}}+\frac{2}{(n-1)X^3}+\frac{1}{(n-X)^2}$$

$$>\frac{1}{(n-1)^2}-\frac{1}{(n-1)^3}\left\{\frac{8(n-2)}{3}+np_n\right\}(X-1)-\frac{np_n}{(n-1)^3}(X-1)+\frac{2}{n-1}$$

$$-\frac{6}{n-1}(X-1)+\frac{1}{(n-1)^2}+\frac{2}{(n-1)^3}(X-1)$$

$$=\frac{2n}{(n-1)^2}-\frac{X-1}{3(n-1)^3}(6np_n+18n^2-28n-4)$$

and hence

$$\lambda(x)-\tilde{\lambda}(X)>\frac{n}{(n-1)^2}(X-1)^2-\frac{6np_n+18n^2-28n-4}{18(n-1)^3}(X-1)^3.$$

Finally we obtain

$$U_0(x)=\frac{x^2\sqrt{n-x}}{(1-x)^5}F_2(x)\{\lambda(x)-\tilde{\lambda}(X)\}$$

$$>\frac{n^2(n^2-n+1)(n-1)^{n-3.5}}{6}\left\{1-\frac{(4n-5)p_n}{2(n-1)}(X-1)\right\}$$

$$\times\left\{1-\frac{6np_n+18n^2-28n-4}{18n(n-1)^3}(X-1)\right\}(X-1),$$

which can be written as the inequality of the statement of this lemma.

Q. E. D.

LEMMA 2.2. *We have*

$$\frac{U_5(x)}{\sqrt{n-1}}>\frac{n(2n-1)}{2}(n-1)^{n-3}+\frac{n(5n^2-3n+1)}{4}(n-1)^{n-4}(X-1) \tag{2.4}$$

$$+\left\{\frac{n(n-2)(n-3)q_n(C)}{40(n-1)}-\frac{n(n-2)(3n-1)(4n-3)}{8}\right\}(n-1)^{n-5}(X-1)^2$$

$$+\frac{n(n-2)(n-3)(4n-3)q_n(C)}{80(n-1)}(n-1)^{n-6}(X-1)^3$$

for $1<X\leqq C\leqq 2,\ n\geqq 3,$

where

$$q_n(C)=(n^2-1)C^2-(10n^2-9n-4)C+3n(7n-8). \tag{2.5}$$

(Lemma 5.2 in (VII)).

LEMMA 2.3. *We have*

$$(2.6)\qquad -\frac{U_1(x)}{\sqrt{n-1}}>-\frac{n(2n-1)}{2}(n-1)^{n-3}+\frac{n(5n^2-3n+1)}{4}(n-1)^{n-4}(1-x)$$
$$+\frac{n(16n^3-124n^2+46n-9)}{80}(n-1)^{n-5}(1-x)^2$$
$$-\frac{n(n-2)(24n^3+224n^2-221n+39)}{160}(n-1)^{n-6}(1-x)^3$$
$$-\frac{3n(n-2)(n-3)(4n-1)(8n^2-8n+3)}{320}(n-1)^{n-7}(1-x)^4$$
$$\textit{for}\quad 0<x<1,\ 3\leqq n<4.$$

Proof. By (5.4) in (VII) we have

$$f_o^{(6)}(x)=-n(n-1)(n-2)(n-3)(n-4)$$
$$\times\{(n^2-1)x^2-(12n^2-11n-5)x+n(31n-35)\}(n-x)^{n-7}.$$

Since we have

$$\frac{12n^2-11n-5}{2(n^2-1)}>1\qquad\text{for}\quad n\geqq 2$$

and

$$[(n^2-1)x^2-(12n^2-11n-5)x+n(31n-35)]_{x=1}=4(n-1)(5n-1)>0,$$

so

$$f_o^{(6)}(x)\geqq 0\qquad\text{for}\quad 0\leqq x\leqq 1,\quad\text{with}\quad 3\leqq n<4,$$

which implies

$$f_o^{(5)}(x)\leqq f_o^{(5)}(1)=3n(n-2)(n-3)(4n-1)(n-1)^{n-4}.$$

Hence, by means of the argument used in the proof of Lemma 5.3 in (VII), we obtain the inequality:

$$f_o(x)>-\frac{n(2n-1)B}{6(n-1)}(1-x)^3-\frac{n(n-2)(3n-1)B}{12(n-1)^2}(1-x)^4$$
$$-\frac{n(n-2)(n-3)(4n-1)B}{40(n-1)^3}(1-x)^5\qquad\text{for}\quad 0<x<1,$$

Using the inequality (5.9) in (VII) as follows:

$$\frac{x^2}{\sqrt{n-x}}<\frac{1}{\sqrt{n-1}}-\frac{4n-3}{2(n-1)^{1.5}}(1-x)+\frac{8n^2-8n+3}{8(n-1)^{2.5}}(1-x)^2\qquad\text{for}\quad 0<x<1.$$

we obtain

$$-U_1(x)=\frac{3x^2f_o(x)}{(1-x)^3\sqrt{n-x}}>-\frac{1}{\sqrt{n-1}}\left\{1-\frac{4n-3}{2(n-1)}(1-x)+\frac{8n^2-8n+3}{8(n-1)^2}(1-x)^2\right\}$$
$$\times\frac{nB}{2(n-1)}\left\{2n-1+\frac{(n-2)(3n-1)}{2(n-1)}(1-x)+\frac{3(n-2)(n-3)(4n-1)}{20(n-1)^2}(1-x)^2\right\}$$

which can be written as (2.6), by removing the braces and arranging in order of powers of $1-x$. Q. E. D.

LEMMA 2.4. *We have*

$$\frac{U_2(x)}{\sqrt{n-1}} > n^2(n-1)^{n-3} - \frac{n^2(8n-1)}{6}(n-1)^{n-4}(1-x) - \frac{n^2(n-2)(4n-3)}{3}(n-1)^{n-5}(1-x)^2 \quad \text{for} \quad 0<x<1,\ n>3. \tag{2.7}$$

(Lemma 5.4 in (VII)).

LEMMA 2.5. *We have*

$$-\frac{U_6(x)}{\sqrt{n-1}} > -n^2(n-1)^{n-3} - \frac{n^2(8n-1)}{6}(n-1)^{n-4}(X-1) + \frac{n^2\{(20C^2-115C+88)n^2-(55C^2-377C+395)n+3(C^2-50C+50)\}}{15(3-C)^2}(n-1)^{n-5}(X-1)^2 - \frac{(4-C)n^2(n-2)(n-3)(4n-3)}{6(3-C)^2}(n-1)^{n-6}(X-1)^3 \quad \text{for} \quad 1<X\leqq C\leqq \frac{3}{2},\ 3\leqq n<4. \tag{2.8}$$

Proof. We shall use the argument of the proof of Lemma 5.5 in (VII). For X $(1<X\leqq C)$ there exists X_1 $(1<X_1<X)$ such that

$$B-X(n-X)^{n-1} = \frac{n(n-1)^{n-2}}{2}(X-1)^2 - \frac{n(n-2)(n-1)^{n-3}}{3}(X-1)^3 + \frac{n(n-1)(n-2)(n-3)(4-X_1)(n-X_1)^{n-5}}{24}(X-1)^4.$$

Since we have

$$\frac{\partial}{\partial x}(4-x)(n-x)^{n-5} = (4-n)(5-x)(n-x)^{n-6} > 0 \quad \text{for} \quad 0<x<n<4,$$

which implies

$$(4-X_1)(n-X_1)^{n-5} < (4-C)(n-C)^{n-5},$$

we obtain

$$\frac{B-X(n-X)^{n-1}}{(X-1)^2} < \frac{n(n-1)^{n-2}}{2} \times\left\{1-\frac{2(n-2)}{3(n-1)}(X-1)+\frac{(n-2)(n-3)(4-C)}{12(n-1)^2}\left(\frac{n-1}{n-C}\right)^{5-n}(X-1)^2\right\}.$$

Furthermore

$$\frac{d}{dn}\log\left(\frac{n-1}{n-C}\right)^{5-n} = -\log\frac{n-1}{n-C} - (5-n)\frac{C-1}{(n-1)(n-C)} < 0 \quad \text{for} \quad 3\leqq n<4$$

and

$$\left(\frac{n-1}{n-C}\right)^{5-n} \leqq \left(\frac{2}{3-C}\right)^2 = \frac{4}{(3-C)^2}.$$

Hence we obtain

$$\frac{B-X(n-X)^{n-1}}{(X-1)^2} < \frac{n(n-1)^{n-2}}{2} \times\left\{1-\frac{2(n-2)}{3(n-1)}(X-1)+\frac{(n-2)(n-3)}{3(n-1)^2}\cdot\frac{(4-C)}{(3-C)^2}(X-1)^2\right\} \tag{2.9}$$

$$\text{for } 1<X\leqq C\leqq 3/2,\ 3\leqq n<4.$$

Next, we have (5.14) in (VII) as

$$\frac{X^2}{\sqrt{n-X}} < \frac{1}{\sqrt{n-1}}\left\{1+\frac{4n-3}{2(n-1)}(X-1)+\frac{8n^2-8nC+3C^2}{8(n-C)^2}\cdot\sqrt{\frac{n-1}{n-C}}(X-1)^2\right\}.$$

For simplicity, taking into consideration of $3\leqq n<4$ and $C\leqq 3/2$, we have

$$\frac{1}{8(n-C)^2}\sqrt{\frac{n-1}{n-C}} = \frac{1}{8(n-1)^2}\left(\frac{n-1}{n-C}\right)^{5/2} \leqq \frac{1}{8(n-1)^2}\cdot\frac{4\sqrt{2}}{(3-C)^2\sqrt{3-C}}$$

$$\leqq \frac{\sqrt{2}}{2(n-1)^2(3-C)^2\sqrt{3-C}} \leqq \frac{1}{(n-1)^2(3-C)^2\sqrt{3}} < \frac{3}{5(n-1)^2(3-C)^2}.$$

Hence we obtain

$$\frac{X^2}{\sqrt{n-X}} < \frac{1}{\sqrt{n-1}}\left\{1+\frac{4n-3}{2(n-1)}(X-1)+\frac{3(8n^2-8nC+3C^2)}{5(n-1)^2(3-C)^2}(X-1)^2\right\} \tag{2.10}$$

$$\text{for } 1<X\leqq C\leqq 3/2,\ 3\leqq n<4.$$

From (2.9) and (2.10) we obtain

$$-U_6(x) = -\frac{2nX^2\{B-X(n-X)^{n-1}\}}{(X-1)^2\sqrt{n-X}}$$

$$> -n^2(n-1)^{n-5/2}\left\{1+\frac{4n-3}{2(n-1)}(X-1)+\frac{3(8n^2-8nC+3C^2)}{5(n-1)^2(3-C)^2}(X-1)^2\right\}$$

$$\times\left\{1-\frac{2(n-2)}{3(n-1)}(X-1)+\frac{(n-2)(n-3)(4-C)}{3(n-1)^2(3-C)^2}(X-1)^2\right\}$$

and, arranging in order of powers of $X-1$,

$$-\frac{U_6(x)}{\sqrt{n-1}} > -n^2(n-1)^{n-3} - \frac{n^2(8n-1)}{6(n-1)}(n-1)^{n-4}(X-1)$$

$$+\frac{n^2\{(20C^2-115C+88)n^2-(55C^2-377C+395)n+3(C^2-50C+50)\}}{15(3-C)^2}(n-1)^{n-5}(X-1)^2$$

$$-\frac{(4-C)n^2(n-2)(n-3)(4n-3)}{6(3-C)^2}(n-1)^{n-6}(X-1)^3$$

$$+\frac{n^2(n-2)(8n^2-8nC+3C^2)}{5(3-C)^4}(n-1)^{n-7}$$
$$\times[2(n-1)(3-C)^2-(n-3)(4-C)(X-1)](X-1)^3.$$

Regarding the last expression, we see

$$8n^2-8nC+3C^2=8(n-C/2)^2+C^2>0$$

and

$$\begin{aligned}&2(n-1)(3-C)^2-(n-3)(4-C)(X-1)\\ &\geqq 2(n-1)(3-C)^2-(n-3)(4-C)(C-1)\\ &\geqq 2(n-1)9/4-(n-3)(4-3/2)(3/2-1)\\ &=(13n-3)/4>0 \qquad \text{for} \quad 1<X\leqq C\leqq 3/2,\ 3\leqq n<4.\end{aligned}$$

Therefore, omitting this expression from the above inequality, we can obtain (2.8). Q. E. D.

LEMMA 2.6. *We have*

$$(2.11)\qquad \frac{U_4(x)}{\sqrt{n-1}}>\frac{n^2(n^2-n+1)}{6}(n-1)^{n-4}(X-1)$$
$$-\frac{n^2(2n^3-4n^2+6n-3)}{12}(n-1)^{n-5}(X-1)^2$$
$$-\frac{n^2(2n-1)(4n^3-7n^2+9n-4)}{24}(n-1)^{n-6}(X-1)^3$$
$$\textit{for} \quad 1<X\leqq C\leqq 3/2,\ 3\leqq n<5.$$

Proof. Using the argument in the proof of Lemma 5.6 in (VII), we have

$$F_2(x)>\frac{n(n-1)^3(n^2-n+1)}{6}(X-1)^4\left\{1-\frac{4n^3-7n^2+9n-4}{2(n-1)(n^2-n+1)}(X-1)\right\}(n-X_1)^{n-5},$$

where X_1 is a value such that $1<X_1<X\leqq C\leqq 3/2$. Since

$$(n-X_1)^{n-5}>(n-1)^{n-5}$$

and

$$\frac{X}{\sqrt{n-X}}>\frac{1}{\sqrt{n-1}}\left\{1+\frac{2n-1}{2(n-1)}(X-1)\right\} \qquad \text{for} \quad 1<X<n.$$

Therefore, we obtain from these

$$U_4(x)=\frac{nXF_2(X)}{(n-1)(X-1)^3\sqrt{n-X}}>\frac{n^2(n^2-n+1)(n-1)^{n-3.5}}{6}(X-1)$$
$$\times\left\{1-\frac{4n^3-7n^2+9n-4}{2(n-1)(n^2-n+1)}(X-1)\right\}\left\{1+\frac{2n-1}{2(n-1)}(X-1)\right\}$$

$$=\frac{n^2(n^2-n+1)(n-1)^{n-3.5}}{6}(X-1)-\frac{n^2(2n^3-4n^2+6n-3)}{12}(n-1)^{n-4.5}(X-1)^2$$
$$-\frac{n^2(2n-1)(4n^3-7n^2+9n-4)}{24}(n-1)^{n-5.5}(X-1)^3.$$

Q. E. D.

PROPOSITION 1. *We have*

$$\frac{V(x, X_n(x))}{n(n-1)^{n-3.5}(X-1)}>\frac{(2n-1)(n^2-n-3)}{6}-\frac{X-1}{n-1} \tag{2.12}$$
$$\times\Big[\frac{40n^4+552n^3-1498n^2+1152n-153}{240}$$
$$+\frac{(n^2-n+1)\{3n(12n-13)p_n+18n^2-28n-4\}}{108}-\frac{(n-2)(n-3)}{40(n-1)}q_n(C)$$
$$-\frac{n\{(20n^2-55n+3)C^2-(115n^2-377n+150)C+88n^2-395n+150\}}{15(3-C)^2}\Big]$$
$$-\Big(\frac{X-1}{n-1}\Big)^2\Big[\frac{160n^5-288n^4+1028n^3-2347n^2+1523n-234}{480}$$
$$-\frac{(4n-5)(n^2-n+1)p_n(6np_n+18n^2-28n-4)}{216}-(n-2)(n-3)(4n-3)$$
$$\times\Big\{\frac{1}{80(n-1)}q_n(C)-\frac{(4-C)n}{6(3-C)^2}\Big\}\Big]$$
$$-\Big(\frac{X-1}{n-1}\Big)^3\frac{3(n-2)(n-3)(4n-1)(8n^2-8n+3)}{320}\quad \text{for}\quad 1<X\leqq C\leqq 3/2,\ 3\leqq n<4,$$

where p_n is given by (2.2) *and $q_n(C)$ is given by* (2.5).

Proof. Using Lemma 2.1—Lemma 2.6 in the expression of $V(x, X_n(x))$:

$$V(x, X_n(x))=U_0(x)-U_1(x)+U_2(x)+U_4(x)+U_5(x)-U_6(x)$$
$$=U_0(x)+U_5(x)-U_1(x)+U_2(x)-U_6(x)+U_4(x)$$

and noticing the facts as

the sum of the coefficients of $(n-1)^{n-4}(1-x)$:

$$\frac{n(5n^2-3n+1)}{4}-\frac{n^2(8n-1)}{6}=-\frac{n(n^2+7n-3)}{12}<0,$$

the sum of the coefficients of $(n-1)^{n-5}(1-x)^2$:

$$\frac{n(16n^3-124n^2+46n-9)}{80}-\frac{n^2(n-2)(4n-3)}{3}=-\frac{n(272n^3-508n^2+342n+27)}{240}<0,$$

and the coefficients of $(n-1)^{n-6}(1-x)^3$ and $(n-1)^{n-7}(1-x)^4$ are also negative for

$3 \leqq n<4$, and $X_n(x)-1>1-x$ for $0<x<1$ (by Lemma 6.2 in (IV), which says $x>2-X_n(x)$ for $0<x<1$ and $n>2$), we may replace $1-x$ by $X-1$ in these evaluating expressions. Then we compute the coefficients of $n(n-1)^{n-3-m}(X-1)^m$, $m=0, 1, 2, 3, 4$. The coefficient corresponding to $m=0$ vanishies;

$$n(n-1)^{n-4}(X-1): \frac{n(n^2-n+1)}{6}+\frac{5n^2-3n+1}{4}+\frac{5n^2-3n+1}{4}$$

$$-\frac{n(8n-1)}{6}-\frac{n(8n-1)}{6}+\frac{n(n^2-n+1)}{6}$$

$$=\frac{2n^3-3n^2-5n+3}{6}=\frac{(2n-1)(n^2-n-3)}{6};$$

$$n(n-1)^{n-5}(X-1)^2: -\frac{(n^2-n+1)\{3n(12n-13)p_n+18n^2-18n-4\}}{108}+(n-2)$$

$$\times\left\{\frac{(n-3)q_n(C)}{40(n-1)}-\frac{(3n-1)(4n-3)}{8}\right\}-\frac{272n^3-508n^2+342n+27}{240}$$

$$+\frac{n\{(20C^2-115C+88)n^2-(55C^2-377C+395)n+3(C^2-50C+50)\}}{15(3-C)^2}$$

$$-\frac{n(2n^3-4n^2+6n-3)}{12}$$

$$=-\frac{40n^4+552n^3-1498n^2+1152n-153}{240}$$

$$-\frac{(n^2-n+1)\{3n(12n-13)p_n+18n^2-28n-4\}}{108}+\frac{(n-2)(n-3)q_n(C)}{40(n-1)}$$

$$+\frac{n\{(20n^2-55n+3)C^2-(115n^2-377n+150)C+88n^2-395n+150\}}{15(3-C)^2};$$

$$n(n-1)^{n-6}(X-1)^3: \frac{(n^2-n+1)(4n-5)p_n(6np_n+18n^2-28n-4)}{216}$$

$$+\frac{(n-2)(n-3)(4n-3)q_n(C)}{80(n-1)}-\frac{(n-2)(24n^3+224n^2-221n+39)}{160}$$

$$-\frac{n(n-2)(n-3)(4n-3)(4-C)}{6(3-C)^2}-\frac{n(2n-1)(4n^3-7n^2+9n-4)}{24}$$

$$=-\frac{160n^5-288n^4+1028n^3-2347n^2+1523n-234}{480}$$

$$+\frac{(4n-5)(n^2-n+1)p_n(6np_n+18n^2-28_n-4)}{216}+(n-2)(n-3)(4n-3)$$

$$\times\left\{\frac{1}{80(n-1)}q_n(C)-\frac{(4-C)n}{6(3-C)}\right\};$$

$$n(n-1)^{n-7}(X-1)^4: \ -\frac{3(n-2)(n-3)(4n-1)(8n^2-8n+3)}{320}$$

Observing these computations, we can obtain easily the inequality (2.12).

Q. E. D.

Second, we shall investigate the case $4\leqq n<5$.

LEMMA 2.1′. *We have*

$$(2.1') \qquad \frac{U_0(x)}{\sqrt{n-1}}>\frac{n^2(n^2-n+1)}{6}(n-1)^{n-4}(X-1)$$
$$-\frac{n^2(n^2-n+1)\{(12n-13)p_n+6n-10\}}{36}(n-1)^{n-5}(X-1)^2$$
$$+\frac{n^2(4n-5)(n^2-n+1)p_n(p_n+3n-5)}{36}(n-1)^{n-6}(X-1)^3$$
$$\textit{for} \quad 1<X\leqq C,\ 4\leqq n<5,$$

where $C=1.225$, $p_n=\left(\frac{n-1}{n-1.225}\right)^{n-1.169}$, *when* $4\leqq n<4.5$; $C=1.163$, $p_n=\left(\frac{n-1}{n-1.2}\right)^{n-1.165}$, *when* $4.5\leqq n<5$, *and* $X=X_n(x)$.

Proof. We obtain this from Lemma 5.1 in (VII). Q. E. D.

LEMMA 2.3′. *We have*

$$(2.6') \qquad -\frac{U_1(x)}{\sqrt{n-1}}>-\frac{n(2n-1)}{2}(n-1)^{n-3}+\frac{n(5n^2-3n+1)}{4}(n-1)^{n-4}(1-x)$$
$$+\left\{\frac{n(8n^3-50n^2+44n-9)}{16}-\frac{3(n-2)(n-3)(7n-8)}{40}\,\frac{(n-1)^4}{n^3}e_{n-1}\right\}(n-1)^{n-5}(1-x)^2$$
$$-\frac{n(n-2)(3n-1)(8n^2-n+3)}{32}(n-1)^{n-6}(1-x)^3+\frac{3(n-2)(n-3)(7n-8)}{320n^3}e_{n-1}(n-1)^{n-3}$$
$$\times[4(n-1)(4n-3)-(8n^2-8n+3)(1-x)](1-x)^3 \qquad \text{for} \quad 0<x<1,\ 4\leqq n<5,$$

where $e_{n-1}=\left(1+\frac{1}{n-1}\right)^{n-1}$.

Proof. We obtain this from the proof of Lemma 5.3 in (VII), regarding the last expression in the brackets. Q. E. D.

LEMMA 2.5′. *We have*

$$(2.8') \qquad -\frac{U_6(x)}{\sqrt{n-1}}>-n^2(n-1)^{n-3}-\frac{n^2(8n-1)}{6}(n-1)^{n-4}(X-1)$$

$$+\left\{\frac{n^2(n-2)(13n-3)}{12}-\frac{n^2(8n^2-8Cn+3C^2)}{8}\left(\frac{n-1}{n-C}\right)^{2.5}\right\}(n-1)^{n-5}(X-1)^2$$

$$-\frac{n^2(n-2)(n-3)(4n-3)}{8}(n-1)^{n-6}(X-1)^3$$

$$+n^2(n-1)^{n-3}\cdot\frac{8n^2-8nC+3C^2}{8(n-C)^2}\cdot\sqrt{\frac{n-1}{n-C}}\cdot\frac{n-2}{n-1}\cdot\left[\frac{2}{3}-\frac{n-3}{4(n-1)}(X-1)\right](X-1)^3,$$

for $1<X\leqq C\leqq \min(2, b_n)$, $n\geqq 4$.

Proof. We obtain this from the proof of Lemma 5.5 in (VII), regarding the last expression in the brackets.

PROPOSITION 2. *We have*

$$\frac{V(x, X_n(x))}{(n-1)^{n-3.5}(X-1)}>\frac{n(2n-1)(n^2-n-3)}{6} \tag{2.13}$$

$$-\frac{X-1}{n-1}\left[\frac{16n^5+151n^4-666n^3+1329n^2-1188n+432}{96}\right.$$

$$+\frac{n^2(n^2-n+1)\{(12n-13)p_n+6n-10\}}{36}-\frac{n(n-2)(n-3)}{40(n-1)}q_n(C)$$

$$\left.+\frac{n^2(8n^2-8nC+3C^2)}{8}\left(\frac{n-1}{n-C}\right)^{5/2}\right]$$

$$-\left(\frac{X-1}{n-1}\right)^2\left[\frac{160n^6-264n^5+1898n^4-8485n^3+14555n^2-10386n+2592}{480}\right.$$

$$-\frac{n^2(4n-5)(n^2-n+1)p_n(p_n+3n-5)}{36}-\frac{n(n-2)(n-3)(4n-3)}{80(n-1)}q_n(C)$$

$$\left.-\frac{n^2(n-2)(8n^2-8nC+3C^2)}{12}\left(\frac{n-1}{n-C}\right)^{5/2}\right]$$

$$-\left(\frac{X-1}{n-1}\right)^3\frac{(n-2)(n-3)}{32}\left[\frac{3(n-1)(7n-8)(8n^2-8n+3)}{8}\right.$$

$$\left.+n^2(8n^2-8nC+3C^2)\left(\frac{n-1}{n-C}\right)^{5/2}\right]$$

for $1<X\leqq C$, $4\leqq n<5$,

where $X=X_n(x)$, $C=1.225$, $p_n=\left(\frac{n-1}{n-1.225}\right)^{n-1.169}$ for $4\leqq n<4.5$; $C=1.163$, $p_n=\left(\frac{n-1}{n-1.2}\right)^{n-1.165}$ *for* $4.5\leqq n<5$.

Proof. Using Lemma 2.1′, Lemma 2.2, Lemma 2.3′, Lemma 2.4, Lemma

2.5′ and Lemma 2.6 in the expression of $V(x, X_n(x))$:

$$V(x, X(x))=U_0(x)+U_5(x)-U_1(x)+U_2(x)-U_6(x)+U_4(x)$$

as in the case of $3\leqq n<4$ and noticing the facts as

the sum of the coefficients of $(n-1)^{n-4}(1-x)$:

$$\frac{n(5n^2-3n+1)}{4}-\frac{n^2(8n-1)}{6}=-\frac{n(n^2+7n-3)}{12}<0,$$

the sum of the coefficients of $(n-1)^{n-5}(1-x)^2$:

$$\frac{n(8n^3-50n^2+44n-9)}{16}-\frac{3(n-2)(n-3)(7n-8)}{40}\frac{(n-1)^4}{n^3}e_{n-1}-\frac{n^2(n-2)(4n-3)}{3}$$

$$=-\left\{\frac{n(40n^3-26n^2-36n+27)}{48}+\frac{3(n-2)(n-3)(7n-8)}{40}\frac{(n-1)^4}{n^3}e_{n-1}\right\}<0,$$

the sum of the coefficients of $(n-1)^{n-6}(1-x)^3$:

$$-\frac{n(n-2)(3n-1)(8n^2-8n+3)}{32}+\frac{3(n-2)(n-3)(7n-8)(4n-3)(n-1)^4}{80n^3}e_{n-1}$$

$$=-\frac{n-2}{320}\left\{10n(3n-1)(8n^2-8n+3)-12e_{n-1}\left(1-\frac{1}{n}\right)^3\cdot(n-1)(n-3)(4n-3)(7n-8)\right\}$$

(using $e_{n-1}(1-1/n)^3\leqq e_4(1-1/5)^3=5/4$ for $4\leqq n<5$)

$$\leqq-\frac{n-2}{320}\{240n^4-320n^3+170n^2-30n-15(28n^4-165n^3+320n^2-255n+72)\}$$

$$=-\frac{n-2}{320}(-180n^4+2155n^3-4630n^2+3795n-1080)<0$$

(in fact $-180n^4+2155n^3-4630n^2+3795n-1080\geqq 1110n^2+3795n-1080>0$ for $4\leqq n<5$), and the coefficient of $(n-1)^{n-7}(1-x)^4$ is also negative, we may replace $1-x$ by $X-1$ in these evaluating expressions. Then we compute the coefficients of $(n-1)^{n-3-m}(X-1)^m$, $m=0, 1, 2, 3, 4$. The coefficient corresponding to $m=0$ vanishes;

$$(n-1)^{n-4}(X-1):\ \frac{n(2n-1)(n^2-n-3)}{6};$$

$$(n-1)^{n-5}(X-1)^2:\ -\frac{n^2(n^2-n+1)\{(12n-13)p_n+6n-10\}}{36}+\frac{n(n-2)(n-3)q_n(C)}{40(n-1)}$$

$$-\frac{n(n-2)(3n-1)(4n-3)}{8}-\frac{n(40n^3-26n^2-36n+27)}{48}$$

$$-\frac{3(n-2)(n-3)(7n-8)(n-1)^4}{40n^3}e_{n-1}+\frac{n^2(n-2)(13n-3)}{12}$$

$$-\frac{n^2(8n^2-8nC+3C^2)}{8}\left(\frac{n-1}{n-C}\right)^{5/2}-\frac{n^2(2n^3-4n^2+6n-3)}{12}$$

$$=-\frac{n(8n^4+44n^3-108n^2+102n-9)}{48}-\frac{n(n^2-n+1)\{(12n-13)p_n+6n-10\}}{36}$$

$$+\frac{n(n-2)(n-3)q_n(C)}{40(n-1)}-\frac{n^2(8n^2-8nC+3C^2)}{8}\left(\frac{n-1}{n-C}\right)^{5/2}$$

$$-\frac{3(n-2)(n-3)(7n-8)}{40}\frac{(n-1)^4e_{n-1}}{n^3},$$

regarding the last term, using the inequality $\frac{(n-1)^3e_{n-1}}{n^3}<\frac{5}{4}$ for $4\leqq n<5$, we have

$$\frac{n(8n^4+44n^3-108n^2+102n-9)}{48}+\frac{3(n-2)(n-3)(7n-8)(n-1)^4e_{n-1}}{40n^3}$$

$$<\frac{n(8n^4+44n^3-108n^2+102n-9)}{48}+\frac{3}{32}\cdot(n-1)(n-2)(n-3)(7n-8)$$

$$=\frac{16n^5+88n^4-216n^3+204n^2-18n+9(7n^4-50n^3+125n^2-130n+48)}{96}$$

$$=\frac{16n^5+151n^4-666n^3+1329n^2-1188n+432}{96},$$

hence we can replace the last expression by the following smaller one as follows:

$$-\frac{16n^5+151n^4-666n^3+1329n^2-1188n+432}{96}-\frac{n^2(n^2-n+1)\{(12n-13)p_n+6n-10\}}{36}$$

$$+\frac{n(n-2)(n-3)q_n(C)}{40(n-1)}-\frac{n^2(8n^2-8nC+3C^2)}{8}\left(\frac{n-1}{n-C}\right)^{5/2};$$

$$(n-1)^{n-6}(X-1)^3:\ \frac{n^2(4n-5)(n^2-n+1)p_n(p_n+3n-5)}{36}+\frac{n(n-2)(n-3)(4n-3)q_n(C)}{80(n-1)}$$

$$-\frac{n(n-2)(3n-1)(8n^2-8n+3)}{32}+\frac{3(n-1)(n-2)(n-3)(4n-3)(7n-8)}{80}\frac{(n-1)^3e_{n-1}}{n^3}$$

$$-\frac{n^2(n-2)(n-3)(4n-3)}{8}+\frac{2}{3}\frac{n^2(n-1)^3(8n^2-8nC+3C^2)}{8(n-C)^2}\cdot\sqrt{\frac{n-1}{n-C}}\cdot\frac{n-2}{n-1}$$

$$-\frac{n^2(2n-1)(4n^3-7n^2+9n-4)}{24}$$

$$=-\frac{n(32n^5+48n^4-416n^3+643n^2-311n+18)}{96}+\frac{3(n-1)(n-2)(n-3)(4n-3)(7n-8)}{80}$$

$$\times\frac{(n-1)^3e_{n-1}}{n^3}$$

$$+\frac{n^2(4n-5)(n^2-n+1)p_n(p_n+3n-5)}{36}+\frac{n(n-2)(n-3)(4n-3)q_n(C)}{80(n-1)}$$

$$+\frac{n^2(n-2)(8n^2-8nC+3C^2)}{12}\left(\frac{n-1}{n-C}\right)^{5/2}$$

(using $(n-1)^3e_{n-1}/n^3>(3/4)^3e_3=1$ for $4\leqq n<5$, the sum of the first two terms can be replaced by

$$>-\frac{n(32n^5+48n^4-416n^3+643n^2-311n+18)}{96}+\frac{3(n-1)(n-2)(n-3)(4n-3)(7n-8)}{80}$$

$$=-\frac{5(32n^6+48n^5-416n^4+643n^3-311n^2+18n)-18(28n^5-221n^4+650n^3-895n^2+582n-144)}{480}$$

$$=-\frac{160n^6-264n^5+1898n^4-8485n^3+14555n^2-10386n+2592}{480});$$

$$(n-1)^{n-7}(X-1)^4:\ -\frac{3(n-2)(n-3)(7n-8)(8n^2-8n+3)}{320}\ \frac{(n-1)^4e_{n-1}}{n^3}$$

$$-\frac{n^2(n-2)(n-3)(8n^2-8nC+3C^2)}{32}\left(\frac{n-1}{n-C}\right)^{5/2}$$

(using $(n-1)^3e_{n-1}/n^3<5/4$ for $4\leqq n<5$)

$$>-\frac{3}{256}(n-1)(n-2)(n-3)(7n-8)(8n^2-8n+3)$$

$$-\frac{1}{32}n^2(n-2)(n-3)(8n-8nC+3C)\left(\frac{n-1}{n-C}\right)^{5/2}.$$

Using the above computed results, we obtain the inequality in the statement.

Q. E. D.

§ 3. Concrete evaluation of $V(x, X_n(x))$ near $x=1$ in case $4\leqq n<5$.

In this section, we shall discuss more concretely the content of Proposition 2. We consider the case, by dividing into the two cases $4.5\leqq n<5$ and $4\leqq n<4.5$. First of all, we shall prove the following lemma in order to evaluate p_n.

LEMMA 3.1. *Let* $1<a<b<2a-1$, *then* $\left(\frac{n-1}{n-b}\right)^{n-a}$ *is increasing with respect to n in* $\max\left\{b, \frac{ab+a-2b}{2a-b-1}\right\}<n<+\infty$.

Proof. Differentiating logarithmically the function of n, we obtain

$$\log\frac{n-1}{n-b}-\frac{(b-1)(n-a)}{(n-1)(n-b)}\qquad(\to 0 \text{ as } n\to+\infty),$$

whose derivative is

$$-\frac{b-1}{(n-1)(n-b)}-\frac{(b-1)\{(n-1)(n-b)-(n-a)(2n-b-1)\}}{(n-1)^2(n-b)^2}$$

$$=-\frac{(b-1)\{(2a-b-1)n-(ab+a-2b)\}}{(n-1)^2(n-b)^2},$$

which is negative for $n>(ab+a-2b)/(2a-b-1)$. From these we obtain the statement. We see easily that $ab+a-2b>0$. Q. E. D.

Now, we shall consider the case $4.5\leqq n<5$. Let $C=1.163$ and $p_n=\left(\frac{n-1}{n-1.2}\right)^{n-1.165}$. Then, we see that, setting $a=1.165$, $b=1.2$ in Lemma 3.1, they satisfy the condition: $1<a<b<2a-1$, and we have

$$2a-b-1=0.13 \quad \text{and} \quad ab+a-2b=0.163$$

and so p_n is increasing in $0.163/0.13\doteqdot 1.2538<n<+\infty$. Hence, we obtain for $4.5\leqq n<5$

$$p_n<(4/3.8)^{3.835}=1.2173906\cdots<1.2174,$$

$$p_n\geqq(3.5/3.3)^{3.335}=1.2168104\cdots>1.2168,$$

$$p_n^2\geqq(3.5/3.3)^{6.67}=1.4806277\cdots>1.4806.$$

We obtain from (2.5)

$$q_n(C)=n^2(C^2-10C+21)+n(9C-24)-C^2+4C$$

$$=n^2\times 10.722569-n\times 13.533+3.299431$$

$$>10.7225n^2-13.533n+3.2994.$$

We have also

$$\left(\frac{n-1}{n-C}\right)^{5/2}=\left(\frac{n-1}{n-1.163}\right)^{5/2}\begin{cases}\leqq(3.5/3.337)^{5/2}=1.1266255\cdots<1.1267\\ >(4/3.837)^{5/2}=1.1096103\cdots>1.1096,\end{cases}$$

and

$$8n^2-8nC+3C^2=8n^2-9.304n+4.057707\begin{cases}>8n^2-9.304n+4.0577\\ <8n^2-9.304n+4.0578.\end{cases}$$

Using these evaluations in the right hand side of the inequality (2.13) and taking account of the signs of the expressions in it, we obtain

$$(3.1)\qquad \frac{V(x,\, X_n(x))}{(n-1)^{n-3.5}(X-1)}>\frac{n(2n-1)(n^2-n-3)}{6}$$

$$-\frac{X-1}{n-1}\Big[\frac{16n^5+151n^4-666n^3+1329n^2-1188n+432}{96}$$

$$+\frac{n^2(n^2-n+1)(20.6088n-25.8262)}{36}$$

$$-\frac{n(n-2)(n-3)(10.7225n^2-13.533n+3.2994)}{40(n-1)}$$

$$+\frac{n^2(8n^2-9.304n+4.0578)\times 1.1267}{8}\Bigg]$$

$$-\left(\frac{X-1}{n-1}\right)^2\Bigg[\frac{160n^6-264n^5+1898n^4-8485n^3+14555n^2-10386n+2592}{480}$$

$$-\frac{n^2(4n-5)(n^2-n+1)(3.6504n-4.6034)}{36}$$

$$-\frac{n(n-2)(n-3)(4n-3)(10.7225n^2-13.533n+3.2994)}{80(n-1)}$$

$$-\frac{n^2(n-2)(8n^2-9.304n+4.0577)\times 1.1096}{12}\Bigg]$$

$$-\left(\frac{X-1}{n-1}\right)^3\frac{(n-2)(n-3)(240.1088n^4-611.8625n^3+651.5754n^2-327n+72)}{256}$$

$$\text{for}\quad 1<X\leqq 1.163.$$

We can see the properties of the expressions in (2.13), which are necessary in the process of replacements regarding the above evaluations, in the proof of Theorem 2 in [20]. Here, we show only the evaluation of the coefficient of $\left(\frac{X-1}{n-1}\right)^3$. In fact, we have

$$\begin{aligned}
&3(n-1)(7n-8)(8n^2-8n+3)+8n^2(8n^2-8nC+3C^2)\left(\frac{n-1}{n-C}\right)^{5/2}\\
&<3(n-1)(7n-8)(8n^2-8n+3)+8n^2(8n^2-9.304n+4.0578)\times 1.1267\\
&=3(56n^4-176n^3+205n^2-109n+24)+9.0136(8n^4-9.304n^3+4.0578n^2)\\
&=240.1088n^4-611.8625344n^3+651.5753861n^2-327n+72\\
&<240.1088n^4-611.8625n^3+651.5754n^2-327n+72\,.
\end{aligned}$$

In order to proceed the argument, we evaluate the value of $t=\frac{X-1}{n-1}$, which makes the right hand side of (3.1) as a cubic polynomial of t positive at $n=5$ and $n=4.5$.

Observation 1 ($n=5$). We have

$$\frac{n(2n-1)(n^2-n-3)}{6}=\frac{5\times 9\times 17}{6}=127.5,$$

$$\text{the coefficient of } -t=\frac{88842}{96}+\frac{25\times 21\times 77.2178}{36}-\frac{5\times 3\times 2\times 203.6969}{10\times 4}$$

$$+\frac{25\times 157.5378\times 1.1267}{8}\doteqdot 925.4375+1126.092917-38.19316875+554.6807477$$

$$\doteqdot 2568.017996,$$

$$\text{the coefficient of } -t^2=\frac{2115162}{480}-\frac{25\times 15\times 21\times 13.6486}{36}-\frac{5\times 3\times 2\times 17\times 203.6969}{80\times 4}$$

$$-\frac{25\times 3\times 157.5377\times 1.1096}{12}\doteqdot 4406.5875-2985.63125-324.6419344-1092.52395$$

$$\doteqdot 3.7903656,$$

$$\text{the coefficient of } -t^3=\frac{3\times 2\times 88311.5725}{256}=2069.80248,$$

and so

the right hand side of (3.1) with $n=5$

$$>127.5-2568.018\,t-3.7904\,t^2-2069.8025\,t^3,$$

which becomes 0.12278 ⋯ at $t=0.0495$ and -0.13558 ⋯ at $t=0.0496$.

Observation 2 ($n=4.5$). We have

$$\frac{n(2n-1)(n^2-n-3)}{6}=\frac{4.5\times 8\times 12.75}{6}=76.5,$$

$$\text{the coefficient of } -t=\frac{52752.9375}{96}+\frac{4.5\times 4.5\times 16.75\times 66.9134}{36}$$

$$-\frac{4.5\times 2.5\times 1.5\times 159.531525}{40\times 3.5}+\frac{4.5\times 4.5\times 124.1898\times 1.1267}{8}\doteqdot 549.5097656$$

$$+630.4496906-19.22924632+354.1842644\doteqdot 1514.914474,$$

$$\text{the coefficient of } -t^2=\frac{1097145}{480}-\frac{4.5\times 4.5\times 13\times 16.75\times 11.8234}{36}$$

$$-\frac{4.5\times 2.5\times 1.5\times 15\times 159.531525}{80\times 3.5}-\frac{4.5\times 4.5\times 2.5\times 124.1897\times 1.1096}{12}\doteqdot 2285.71875$$

$$-1448.181759-144.2193474-581.3475094\doteqdot 111.9701342,$$

$$\text{the coefficient of } -t^3=\frac{2.5\times 1.5\times 54498.54634}{256}\doteqdot 798.3185499,$$

and so

the right hand side of (3.1) with $n=4.5$

$$>76.5-1514.9145\,t-111.9702\,t^2-798.3186\,t^3,$$

which becomes 0.06813 ⋯ at $t=0.0502$ and -0.08509 ⋯ at $t=0.0503$.

Connecting linearly the two points (4.5, 0.0502) and (5, 0.0495) on the

(n, t)-plane by

$$t=(565-14n)/10000, \tag{3.2}$$

we have the following proposition which is proved in another paper [20].

PROPOSITION 3. *When* $4.5\leqq n<5$, *the cubic polynomial of* t:

$$\begin{aligned}
&\frac{n(2n-1)(n^2-n-3)}{6}\\
&-t\Big[\frac{16n^5+151n^4-666n^3+1329n^2-1188n+432}{96}\\
&\quad+\frac{n^2(n^2-n+1)(20.6088n-25.8262)}{36}\\
&\quad-\frac{n(n-2)(n-3)(10.7225n^2-13.533n+3.2994)}{40(n-1)}\\
&\quad+\frac{n^2(8n^2-9.304n+4.0578)\times 1.1267}{8}\Big]\\
&-t^2\Big[\frac{160n^6-264n^5+1898n^4-8485n^3+14555n^2-10386n+2592}{480}\\
&\quad-\frac{n^2(4n-5)(n^2-n+1)(3.6504n-4.6034)}{36}\\
&\quad-\frac{n(n-2)(n-3)(4n-3)(10.7225n^2-13.533n+3.2994)}{80(n-1)}\\
&\quad-\frac{n^2(n-2)(8n^2-9.304n+4.0577)\times 1.1096}{12}\Big]\\
&-t^3\frac{(n-2)(n-3)(240.1088n^4-611.8625n^3+651.5754n^2-327n+72)}{256}
\end{aligned}$$

is positive for $0\leqq t\leqq(565-14n)/10000$.

Next, we shall consider the case $4\leqq n<4.5$. Let $C=1.225$ and $p_n=\left(\frac{n-1}{n-1.225}\right)^{n-1.169}$. Then, we see that setting $a=1.169$, $b=1.225$ in Lemma 3.1, they satisfy the condition $1<a<b<2a-1$, and we have

$$2a-b-1=0.113 \quad \text{and} \quad ab+a-2b=0.151025$$

and so p_n is increasing in $0.151025/0.113\doteqdot 1.3365<n<+\infty$. Hence we obtain for $4\leqq n<4.5$

$$p_n < \left(\frac{3.5}{3.275}\right)^{3.331} = 1.2477334 \cdots < 1.2478\,,$$

$$p_n \geqq \left(\frac{3}{2.775}\right)^{2.831} = 1.2469606 \cdots > 1.2469\,,$$

$$p_n^2 \geqq \left(\frac{3}{2.775}\right)^{5.662} = 1.5549109 \cdots > 1.5549\,.$$

We obtain from (2.5)

$$\begin{aligned} q_n(C) &= n^2(C^2-10C+21)+n(9C-24)-C^2+4C \\ &= n^2\times 10.250625 - n\times 12.975 + 3.399375 \\ &> 10.2506n^2 - 12.975n + 3.3993\,. \end{aligned}$$

We have also

$$\left(\frac{n-1}{n-C}\right)^{5/2} = \left(\frac{n-1}{n-1.225}\right)^{5/2} \begin{cases} \leqq (3/2.775)^{5/2} = 1.2151941 \cdots < 1.2152 \\ > (3.5/3.275)^{5/2} = 1.1807062 \cdots > 1.1807 \end{cases}$$

and

$$8n^2-8nC+3C^2 = 8n^2-9.8n+4.501875 \begin{cases} > 8n^2-9.8n+4.5018 \\ < 8n^2-9.8n+4.5019\,. \end{cases}$$

Using these evaluations in the right hand side of the inequality (2.13) and taking account of the signs of the expressions in it, we obtain

$$\begin{aligned} (3.3)\qquad \frac{V(x, X_n(x))}{(n-1)^{n-3.5}(X-1)} &> \frac{n(2n-1)(n^2-n-3)}{6} \\ &- \frac{X-1}{n-1}\left[\frac{16n^5+151n^4-666n^3+1329n^2-1188n+432}{96}\right. \\ &\quad + \frac{n^2(n^2-n+1)(20.9736n-26.2214)}{36} \\ &\quad - \frac{n(n-2)(n-3)(10.2506n^2-12.975n+3.3993)}{40(n-1)} \\ &\quad \left. + \frac{n^2(8n^2-9.8n+4.5019)\times 1.2152}{8}\right] \\ &- \left(\frac{X-1}{n-1}\right)^2\left[\frac{160n^6-264n^5+1898n^4-8485n^3+14555n^2-10386n+2592}{480}\right. \\ &\quad - \frac{n^2(4n-5)(n^2-n+1)(3.7408n-4.6799)}{36} \\ &\quad - \frac{n(n-2)(n-3)(4n-3)(10.2506n^2-12.975n+3.3993)}{80(n-1)} \end{aligned}$$

$$-\frac{n^2(n-2)(8n^2-9.8n+4.5018)\times 1.1807}{12}\Bigg]$$

$$-\left(\frac{X-1}{n-1}\right)^3\frac{(n-2)(n-3)(245.7728n^4-628.2716n^3+658.7657n^2-327n+72)}{256}$$

for $1<X\leqq 1.225$.

In the above procession, we have also used the facts proved in the proof of Theorem 3 in [20]. Here, we show only the evaluation of $p_n(p_n+3n-5)$ as follows:

$$\begin{aligned} p_n(p_n+3n-5)&=p_n^2+p_n(3n-5)\\ &>1.5549+1.24696(3n-5)>3.7408n-4.6799\,. \end{aligned}$$

In order to proceed with the argument, we evaluate the value of $t=(X-1)/(n-1)$, which makes the right hand side of (3.3) as a cubic polynomial of t positive at $n=4.5$ and $n=4$.

Observation 3 ($n=4.5$). Using partially the computations in Observation 2, we have

$$\frac{n(2n-1)(n^2-n-3)}{6}=76.5\,,$$

the coefficient of $-t=\dfrac{52752.9375}{96}+\dfrac{4.5\times 4.5\times 16.75\times 68.1598}{36}$

$$-\frac{4.5\times 2.5\times 1.5\times 152.58645}{40\times 3.5}+\frac{4.5\times 4.5\times 122.4019\times 1.2152}{8}\doteqdot 549.5097656$$

$+642.1931156-18.39211674+376.5051844\doteqdot 1549.815949$,

the coefficient of $-t^2=\dfrac{1097145}{480}-\dfrac{4.5\times 4.5\times 13\times 16.75\times 12.1537}{36}$

$$-\frac{4.5\times 2.5\times 1.5\times 15\times 152.58645}{80\times 3.5}-\frac{4.5\times 4.5\times 2.5\times 122.4018\times 1.1807}{12}\doteqdot 2285.71875$$

$-1488.638348-137.9408756-609.6929284\doteqdot 49.446598$,

the coefficient of $-t^3=\dfrac{2.5\times 1.5\times 55471.46468}{256}\doteqdot 812.5702834$,

and so

the right hand side of (3.3) with $n=4.5$

$$>76.5-1549.816\,t-49.4466\,t^2-812.5703\,t^3,$$

which becomes $0.032586\cdots$ at $t=0.0492$ and $-0.123473\cdots$ at $t=0.0493$.

Observation 4 ($n=4$). We have

$$\frac{n(2n-1)(n^2-n-3)}{6}=\frac{4\times7\times9}{6}=42,$$

$$\text{the coefficient of } -t=\frac{29360}{96}+\frac{16\times13\times57.673}{36}-\frac{4\times2\times1\times115.5089}{40\times3}$$

$$+\frac{16\times93.3019\times1.2152}{8}\doteqdot 305.8333333+333.2217778-7.0059333+226.7609378$$

$$\doteqdot 858.8101156,$$

$$\text{the coefficient of } -t^2=\frac{521800}{480}-\frac{16\times11\times13\times10.2833}{36}-\frac{4\times2\times13\times115.5089}{80\times3}$$

$$-\frac{16\times2\times93.3018\times1.1807}{12}=1087.0833333-653.5608444-50.0538566-293.7638274$$

$$\doteqdot 89.7048046,$$

$$\text{the coefficient of } -t^3=\frac{2\times1\times32012.7056}{256}=250.0992625$$

and so

the right hand side of (3.3) with $n=4$

$$>42-858.8102\,t-89.7049\,t^2-250.0993\,t^3,$$

which becomes $0.021235\cdots$ at $t=0.0486$ and $-0.065695\cdots$ at $t=0.0487$.

Connecting linearly the two points (4, 0.0486) and (4.5, 0.0492) on the (n, t)-plane by

$$t=(12n+438)/10000=6(2n+73)/10000, \tag{3.4}$$

we have the following proposition which is proved also in [20].

PROPOSITION 4. *When $4\leqq n<4.5$, the cubic polynomial of t:*

$$\frac{n(2n-1)(n^2-n-3)}{6}$$

$$-t\Big[\frac{16n^5+151n^4-666n^3+1329n^2-1188n+432}{96}$$

$$+\frac{n^2(n^2-n+1)(20.9736n-26.2214)}{36}$$

$$-\frac{n(n-2)(n-3)(10.2506n^2-12.975n+3.3993)}{40(n-1)}$$

$$+\frac{n^2(8n^2-9.8n+4.5019)\times1.2152}{8}\Big]$$

$$-t^2\Big[\frac{160n^6-264n^5+1898n^4-8485n^3+14555n^2-10386n+2592}{480}$$

$$-\frac{n^2(4n-5)(n^2-n+1)(3.7408n-4.6799)}{36}$$

$$-\frac{n(n-2)(n-3)(4n-3)(10.2506n^2-12.975n+3.3993)}{80(n-1)}$$

$$-\frac{n^2(n-2)(8n^2-9.8n+4.5018)\times 1.1807}{12}\Big]$$

$$-t^3\frac{(n-2)(n-3)(245.7728n^4 \quad 628.2716n^3+658.7657n^2-327n+72)}{256}$$

is positive for $0\leqq t\leqq 6(2n+73)/10000$.

§4. Concrete evaluation of $V(x, X_n(x))$ near $x=1$ in case $3\leqq n<4$.

In this section, we shall discuss more concretely the content of Proposition 1. For simplicity we set

$$A:=\frac{(2n-1)(n^2-n-3)}{6},\quad K:=\frac{40n^4+552n^3-1498n^2+1152n-153}{240},$$

$$F:=\frac{(n^2-n+1)\{3n(12n-13)p_n+18n^2-28n-4\}}{108},\quad G:=\frac{(n-2)(n-3)q_n(C)}{40(n-1)},$$

$$H:=\frac{n\{(20n^2-55n+13)C^2-(115n^2-377n+150)C+88n^2-395n+150\}}{15(3-C)^2}$$

$$\text{(4.1)}\qquad L:=\frac{160n^5-288n^4+1028n^3-2347n^2+1523n-234}{480},$$

$$B:=\frac{(4n-5)(n^2-n+1)p_n(6np_n+18n^2-28n-4)}{216},$$

$$J:=\frac{(n-2)(n-3)(4n-3)q_n(C)}{80(n-1)},\quad D:=\frac{(4-C)n(n-2)(n-3)(4n-3)}{6(3-C)^2},$$

$$E:=\frac{3(n-2)(n-3)(4n-1)(8n^2-8n+3)}{320},$$

which are only effective in this section.

Taking account of Observation 4, we assume $C=1.1$ for $n=3$. Then, we have by (1.17), (2.2) and (2.5)

$$k_3(C)=\frac{2+C-\sqrt{C^2-4C+7}}{C},\quad k_2(1.1)=\frac{3.1-\sqrt{3.81}}{1.1}=1.0437071\cdots,$$

$$p_3=2^{3-k}/(3-k)(3-1.1)^{2-k}=(2/(3-1.0437))\times(20/19)^{2-1.0437}$$
$$=(2/1.9563)\cdot(20/19)^{0.9563}=1.0737358\cdots,$$

and

$$q_3(C)=8C^2-59C+117, \quad q_3(1.1)=61.78.$$

Furthermore, using the above evaluation of p_3, we obtain

$$3n(12n-13)p_n+18n^2-28n-4=207p_3+74\div 296.2633226,$$
$$p_n(6np_n+18n^2-28n-4)=p_3(18p_3+74)\div 100.208803.$$

Using these evaluations, we obtain

$$A=2.5, \quad K=7965/240=33.1875, \quad F=7\times 296.2633226=19.2022523,$$
$$G=0, \quad H=\frac{3(18C^2-54C-243)}{15(3-C)^2}=-\frac{3\times 280.62}{15\times 1.9\times 1.9}\div -15.5468144,$$
$$L=26520/480=55.25, \quad B=7\times 7\times 100.208803/216\doteqdot 22.7325525,$$
$$J=0, \quad D=0, \quad E=0,$$

and

$$K+F-G-H\div 33.1875+19.20225+15.54681=67.93656,$$
$$L-B-J+D\div 55.25-22.73255=32.51745.$$

Using these calculated values, we consider the quadratic polynomial of t:

$$2.5-67.9366\,t-32.5175\,t^2,$$

which becomes $0.0051116\cdots$ at $t=0.0361$ and $-0.0019171\cdots$ at $t=0.0362$.

Taking account of these facts for $n=3$ and $n=4$, we may assume

$$C=1+\frac{n-1}{20}=\frac{n+19}{20} \qquad \text{for} \quad 3\leqq n<4. \tag{4.2}$$

Then, we obtain from (2.5) and (1.17)

$$q_n(C)=\frac{1}{400}(n^4-162n^3+5140n^2-6138n+1159),$$
$$(n-1)C^2-2(n+1)C+5n-1=\frac{1}{400}(n^3-3n^2+1523n-1521)$$
$$=\frac{(n-1)(n^2-2n+1521)}{400},$$

$$k_n(C)=\frac{n^2+38n+1-(n-1)\sqrt{n^2-2n+1521}}{2(n+19)}:=k_n \tag{4.3}$$

(4.4) $$n-k_n(C)=\frac{(n-1)(n+1+\sqrt{n^2-2n+1521})}{2(n+19)}:=s_n\,.$$

Next from (2.2) we obtain

$$p_n=\frac{(n-1)^{n-k}}{(n-k)(n-C)^{n-k-1}}=\frac{n-1}{n-k}\left(\frac{n-1}{n-\frac{n+19}{20}}\right)^{n-k-1},$$

i. e.

(4.5) $$p_n=\frac{n-1}{s_n}\left(\frac{20}{19}\right)^{s_n-1}$$

Since we have

$$(20n^2-55n+3)C^2-(115n^2-377n+150)C+88n^2-395n+150$$

$$=(20n^2-55n+3)\left(\frac{(n-1)^2}{400}+\frac{n-1}{10}+1\right)-(115n^2-377n+150)\left(\frac{n-1}{20}+1\right)$$

$$+88n^2-395n+150=\frac{20n^4-1595n^3+4173n^2-37481n+4083}{400}$$

and $3-C=(41-n)/20$, we obtain

(4.6) $$H=\frac{n\{(20n^2-55n+3)C^2-(115n^2-377n+150)C+88n^2-395n+150\}}{15(3-C)^2}$$

$$=\frac{n(20n^4-1595n^3+4173n^2-37481n+4083)}{15(41-n)^2}.$$

We have also

$$\frac{4-C}{6(3-C)^2}=\frac{400}{6(41-n)^2}\left(3-\frac{n-1}{20}\right)=\frac{10(61-n)}{3(41-n)^2}.$$

Using these facts, G, M and D can be written as

(4.7) $$G=\frac{(n-2)(n-3)(n^4-162n^3+5140n^2-6138n+1159)}{16000(n-1)},$$

(4.8) $$J=\frac{(n-2)(n-3)(4n-3)(n^4-162n^3+5140n^2-6138n+1159)}{32000(n-1)},$$

(4.9) $$D=\frac{10n(n-2)(n-3)(4n-3)(61-n)}{3(41-n)^2}.$$

Observation 5 ($n=4$). We shall compute the values of $K\sim E$ in (4.1) for $n=4$. We have

$$s_4=\frac{3(5+\sqrt{1529})}{2\times 23}=2.87624540\cdots,$$

$$p_4=\frac{3}{2.8762454}\left(\frac{20}{19}\right)^{1.8762454}\fallingdotseq 1.1483950,$$

$$A=7\times 9/6=10.5,\qquad K=26055/240=108.5625,$$

$$F=\frac{13\times(12\times 35\times p_4+172)}{108}=\frac{13\times(105p_4+43)}{27}\fallingdotseq 78.7614509,$$

$$G=\frac{2\times 1\times 48735}{3\times 16000}=2.030625,\qquad H=\frac{4\times(-176033)}{15\times 37\times 37}\fallingdotseq -34.2893596,$$

$$L=124210/480=258.7708333,$$

$$B=\frac{11\times 13\times p_4\times(24p_4+172)}{216}=\frac{143p_4(6p_4+43)}{54}\fallingdotseq 151.7226093,$$

$$J=\frac{2\times 1\times 13\times 48735}{3\times 32000}=13.1990625,\qquad D=\frac{10\times 4\times 2\times 13\times 57}{3\times 37\times 37}\fallingdotseq 14.4338933,$$

$$E=\frac{3\times 2\times 1\times 15\times 99}{320}=27.84375$$

and

$$K+F-G-H\fallingdotseq 108.5625+78.7614509-2.030625+34.2893596=219.5826855,$$

$$L-B-J+D\fallingdotseq 258.7708333-151.7226093-13.1990625+14.4338933=108.2830548.$$

Using these calculated values, we consider the cubic polynomial of t corresponding to the right hand side of (2.12) as follows

$$10.5-219.58269t-108.28306t^2-27.84375t^3,$$

which becomes $0.0064991\cdots$ at $t=0.0467$ and $-0.0164898\cdots$ at $t=0.0468$.

Now, connecting linearly the two points (3, 0.0361) and (4, 0.0467) on the (n, t)-plane by

$$t=(106n+43)/10000. \tag{4.10}$$

We have the following proposition which is proved also in [20].

PROPOSITION 5. *When $3\leqq n<4$, the cubic polynomial of t:*

$$\frac{(2n-1)(n^2-n-3)}{6}-t\left[\frac{40n^4+552n^3-1498n^2+1152n-153}{240}\right.$$
$$+\frac{(n^2-n+1)\{3n(12n-13)p_n+18n^2-28n-4\}}{108}$$
$$-\frac{(n-2)(n-3)(n^4-162n^3+5140n^2-6138n+1159)}{16000(n-1)}$$

$$-\frac{n(20n^4-1595n^3+4173n^2-37481n+4083)}{15(41-n)^2}\Bigg]$$

$$-t^2\Bigg[\frac{160n^5-288n^4+1028n^3-2347n^2+1523n-234}{480}$$

$$-\frac{(4n-5)(n^2-n+1)p_n(6np_n+18n^2-28n-4)}{216}$$

$$-\frac{(n-2)(n-3)(4n-3)(n^4-162n^3+5140n^2-6138n+1159)}{32000(n-1)}$$

$$+\frac{10n(n-2)(n-3)(4n-3)(61-n)}{3(41-n)^2}\Bigg]$$

$$-t^3\frac{3(n-2)(n-3)(4n-1)(8n^2-8n+3)}{320}$$

is positive for $0\leqq t\leqq(106n+43)/10000$, *where* p_n *is given by* (4.5).

§ 5. Proof of $V(x, X_n(x))>0$ for $X_n^{-1}(b_n)<x<1$ with $3\leqq n\leqq 5$.

When $3\leqq n\leqq 4$, we have

$$V(x, X_n(x))>0 \quad \text{for} \quad X_n^{-1}\Big(1+(n-1)\frac{106n+43}{10000}\Big)\leqq x<1 \tag{5.1}$$

by means of Proposition 1 and Proposition 5. When $4\leqq n<4.5$, we have

$$V(x, X_n(x))>0 \quad \text{for} \quad X_n^{-1}\Big(1+(n-1)\frac{6(2n+73)}{10000}\Big)\leqq x<1 \tag{5.2}$$

by means of Proposition 2 and Proposition 4. And, when $4.5\leqq n<5$, we have

$$V(x, X_n(x))>0 \quad \text{for} \quad X_n^{-1}\Big(1+(n-1)\frac{565-14n}{10000}\Big)\leqq x<1 \tag{5.3}$$

by means of Proposition 2 and Proposition 3. In this section, we set

$$Z=\begin{cases} 1+(n-1)\dfrac{106n+43}{10000} & \text{for } 3\leqq n<4,\\ 1+(n-1)\dfrac{6(2n+73)}{10000} & \text{for } 4\leqq n<4.5,\\ 1+(n-1)\dfrac{565-14n}{10000} & \text{for } 4.5\leqq n<5 \end{cases} \tag{5.4}$$

and we see easily that Z is increasing with respect to n in the above three ranges of n.

Now, noticing (5.1)~(5.3), it is sufficient to prove that

$$V(x,\ X_n(x))>0 \qquad \text{for} \quad X_n^{-1}(b_n)<x<X_n^{-1}(Z(n)) \tag{5.5}$$

for our purpose. We shall also depend on the method used in §6, §7 of (VII). First we cite Proposition 1 and Proposition 3 in (VII).

PROPOSITION 6. *When $n>2$, we have*

$$V(x,\ X_n(x))>(n-1)^{n-1}K(n,\ X_n(x)) \qquad \text{for} \quad 0<x<1, \tag{5.6}$$

where

$$\begin{aligned} K(n,\ X):=&\frac{w^2\sqrt{n-w}}{(1-w)^5}\Big[\Big\{-P_2(y)+\Big(\frac{n-y}{n-1}\Big)^{n-1}P_3(y)\Big\} \\ &\times\Big\{\log\frac{n-y}{n-X}+\frac{nXy-(2n-1)X+n-y}{(n-1)X(n-y)}\Big\}+\frac{2n(1-w)^3(1-Xw)}{n-w}\Big] \\ &+\frac{3y^2}{(1-y)^3\sqrt{n-y}}\Big[2n-1-w-\Big(\frac{n-w}{n-1}\Big)^{n-1}\{n-w+(n-1)w^2\}\Big] \\ &-\frac{X}{(n-1)(X-1)^3\sqrt{n-X}}[Q_2(X)+wQ_3(X)], \end{aligned} \tag{5.7}$$

where $X=X_n(x)$,

$$Q_2(z):=(4n^2+2n-3)z^3-(4n^3+18n^2-19n+3)z+n(4n-1)(4n-3), \tag{5.8}$$

$$Q_3(z):=(n-1)(2n-3)z^3-(2n^3-9n^2+13n-3)z^2-2n(2n^2-8n+3)z-3n^2(2n-1) \tag{5.9}$$

and $w=w(n,\ X)$, $y=y(n,\ X)$ by (1.16), (1.20).

Remark. In Proposition 6, we can replace y by

$$\tilde{y}=\tilde{y}(n,\ X;C):=\Big(\frac{n-X}{n-1}\Big)^{n-k_n(C)} \quad \text{for } 1<X\leqq C, \quad \text{with } 1<C\leqq\frac{5n-1}{2(n+1)}, \tag{5.10}$$

by means of Lemma 1.4. With this replacement, we denote this function by $\tilde{K}(n,\ X;C)$ and call the corresponding statement Proposition 6′ in the following.

PROPOSITION 7. *$K(n,\ X)$ satisfies the following inequality: Let $n\geqq 3$ and $1<X<b_n$, and $n_1,\ n_2,\ X_1,\ X_2$ be such that*

$$3\leqq n_1\leqq n\leqq n_2, \qquad a_n\leqq X_1\leqq X\leqq X_2<b_n,$$

then

$$K(n,\ X)\geqq K(n_1,\ n_2;X_1,\ X_2), \tag{5.11}$$

where

$$K(n_1,\ n_2;X_1,\ X_2)=\frac{(w(n_1,\ X_2))^2\sqrt{n_1-w(n_1,\ X_2)}}{(1-w(n_1,\ X_2))^5} \tag{5.12}$$

$$\times\Big[\Big\{-P_2(n_2,\ y(n_2,\ X_1))+\Big(\frac{n_1-y(n_2,\ X_1)}{n_1-1}\Big)^{n_1-1}P_3(n_1,\ y(n_2,\ X_1))\Big\}$$

$$\times\Big\{\log\frac{n_2-y(n_2,\ X_1)}{n_2-X_1}+\frac{n_2X_1y(n_2,\ X_1)-(2n_2-1)X_1+n_2-y(n_2,\ X_1)}{(n_2-1)X_1(n_2-y(n_2,\ X_1))}\Big\}$$

$$+\frac{2n_2\{1-w(n_2,\ X_2)\}^3\{1-X_1w(n_2,\ X_1)\}}{n_2-w(n_2,\ X_1)}\Big]$$

$$+\frac{3(y(n_2,\ X_1))^2}{(1-y(n_2,\ X_1))^3\sqrt{n_1-y(n_2,\ X_1)}}\Big[2n_2-1-w(n_1,\ X_2)$$

$$-\Big(\frac{n_2-w(n_1,\ X_2)}{n_2-1}\Big)^{n_2-1}\{n_2-w(n_1,\ X_2)+(n_2-1)(w(n_1,\ X_2))^2\}\Big]$$

$$-\frac{X_1Q_2(n_2,\ X_1)}{(n_1-1)(X_1-1)^3\sqrt{n_1-X_1}}-\frac{X_2w(n_1,\ X_2)Q_3(n_1,\ X^*)}{(n_2-1)(X_2-1)^3\sqrt{n_2-X_2}}$$

and $X^*=X_1$ for $n\geqq 4.46$, and $X^*=X_2$ when $X>\gamma_n$ and $X^*=X_1$ when $X\leqq\gamma_n$ for $(8+\sqrt{34})/4(\fallingdotseq 3.4577)\leqq n<4.46$, and $X^*=X_2$ for $3\leqq n<(8+\sqrt{34})/4$, and $P_2(n,\ x)$, $P_3(n,\ x)$, $Q_2(n,\ x)$, $Q_3(n,\ x)$ stand for $P_2(x)$, $P_3(x)$, $Q_2(x)$, $Q_3(x)$ with n by (1.6), (1.7), (5.8), (5.9) respectively, and

$$\gamma_n=\frac{2n^3-9n^2+13n-3+\sqrt{4n^6-12n^5-23n^4+66n^3-11n^2-24n+9}}{3(n-1)(2n-3)}.$$

Remark. Regarding w and y in Proposition 7, we used the inequalities

$$w(n_1,\ X_2)\leqq w(n,\ X)\leqq w(n_2,\ X_1), \tag{5.13}$$

$$y(n_1,\ X_2)\leqq y(n,\ X)\leqq y(n_2,\ X_1) \tag{5.14}$$

by Lemma 1.2 and Lemma 1.5. Corresponding to Proposoition 6′, we consider C as a decreasing function of n satisfying

$$C=C(n)\leqq\frac{5n-1}{2(n+1)},$$

then we obtain the inequalities:

$$\Big(\frac{n_1-X_2}{n_1-1}\Big)^{n_2-k_{n_2}(C(n_2))}\leqq\Big(\frac{n-X}{n-1}\Big)^{n-k_n(C(n))}\leqq\Big(\frac{n_2-X_1}{n_2-1}\Big)^{n_1-k_{n_1}(C(n_1))} \tag{5.15}$$

by Lemma 1.3. We denote the right hand side of (5.12) replaced $y(n_2,\ X_1)$ with

$$\Big(\frac{n_2-X_1}{n_2-X_1}\Big)^{n_1-k_{n_1}(C(n_1))}$$

by the notation $\tilde{K}(n_1,\ n_2\,;\ X_1,\ X_2\,;\ C)$. Then, we obtain

$$\tilde{K}(n,\ X\,;\ C)>\tilde{K}(n_1,\ n_2\,;\ X_1,\ X_2\,;\ C) \tag{5.16}$$

$$\text{for } 3\leqq n_1\leqq n\leqq n_2 \quad \text{and} \quad a_n\leqq X_1\leqq X\leqq X_2<\min(b_n, C(n)).$$

We also call the corresponding statement Proposition 7′.

We computed approximately the values of $Y=Y(n)$ such that $K(n, Y)=0$ with $1<Y<n$ for each $n(5\leqq n\leqq 9.7)$ with step 1/100 and made Table 1 and Table 1′ of (VII) for the values of n with step 1/10. We made analogously Table 1 for $3\leqq n\leqq 5$.

Table 1.

n	Y	Z	C_1	n	Y	Z	C_1
3.0	1.4445	1.0722	1.45	4.0	1.3789	1.1458	1.38
3.1	1.4318	1.0780	1.443	4.1	1.3754	1.1510	1.378
3.2	1.4218	1.0840	1.436	4.2	1.3721	1.1562	1.376
3.3	1.4136	1.0903	1.429	4.3	1.3689	1.1615	1.374
3.4	1.4068	1.0968	1.422	4.4	1.3659	1.1668	1.372
3.5	1.4008	1.1035	1.415	4.5	1.3629	1.1722	1.37
3.6	1.3956	1.1103	1.408	4.5	1.3629	1.1757	1.37
3.7	1.3909	1.1175	1.401	4.6	1.3600	1.1802	1.368
3.8	1.3866	1.1248	1.394	4.7	1.3572	1.1847	1.366
3.9	1.3826	1.1323	1.387	4.8	1.3545	1.1891	1.364
4.0	1.3789	1.1401	1.38	4.9	1.3518	1.1935	1.362
				5.0	1.3492	1.1980	1.36

Observing this table and noticing that $(5n-1)/2(n+1)\geqq 7/4=1.75$ for $n\geqq 3$, we can prove

$$Y=Y(n)<\frac{5n-1}{2(n+1)}.$$

This inequality is implied from

$$K\left(n, \frac{5n-1}{2(n+1)}\right)>0,$$

which is proved by showing $K(n_1, n_2; X_1, X_2)>0$ for suitably selected n_1, n_2, X_1, X_2 as

$$3\leqq n_1<n_2\leqq 5 \quad \text{and} \quad X_1\leqq\frac{5n_1-1}{2(n_1+1)}<\frac{5n_2-1}{2(n_2+1)}\leqq X_2,$$

by means of Proposition 6 and Proposition 7. Therefore, we can use Proposition 6′ and Proposition 7′ with suitably selected decreasing function $C=C(n)$ of n in the following arguments.

First, setting

$$C=C_1(n)=\begin{cases}(166-7n)/100 & \text{for } 3\leqq n<4\\ (73-n)/50 & \text{for } 4\leqq n\leqq 5,\end{cases}$$

we can prove $Y(n)<C_1(n)$, by showing $K(n, C_1(n))>0$ which is assured by the same way as described above. Then, we set

$$K_1(n, X):=\tilde{K}(n, X; C_1(n))$$

and let $Y_1(n)$ be the solution of the equation

$$K_1(n, Y_1)=0 \quad \text{with } 1<Y_1<n \quad \text{for } 3\leqq n\leqq 5.$$

Setting

$$C=C_2(n)=\begin{cases}(175-11n)/100 & \text{for } 3\leqq n<4\\ (37-n)/25 & \text{for } 4\leqq n\leqq 5.\end{cases}$$

We can prove $Y_1(n)<C_2(n)$, by showing $K_1(n, C_2(n))>0$, which can be assured by the same way above, using $K_1(n, X)$ and $K_1(n_1, n_2; X_1, X_2)$ constructed suitably from $\tilde{K}(n_1, n_2; X_1, X_2; C)$ in place of $K(n, X)$ and $K(n_1, n_2; X_1, X_2)$.

By the same way, for $j\geqq 1$ we suppose inductively $Y_j(n)$ is defined by

$$K_j(n, Y_j(n))=0 \quad \text{with } 1<Y_j(n)<n \quad \text{for } 3\leqq n\leqq 5$$

and a decreasing function $C=C_{j+1}(n)$ such that

$$Y_j(n)<C_{j+1}(n)\leqq Y_j(n)+\frac{1}{10^4},$$

and then we put

$$K_{j+1}(n, X):=\tilde{K}(n, X; C_{j+1}(n)).$$

By this principle of making $K_j(n, X)$, $Y_j(n)$, $C_j(n)$ and $K_j(n_1, n_2; X_1, X_2)$, we obtained $C_j(n)$ for $3\leqq j\leqq 7$ as follows:

$$C_3(n)=\begin{cases}(1804-133n)/1000 & \text{for } 3\leqq n<4\\ (76-3n)/50 & \text{for } 4\leqq n<5,\end{cases}$$

$$C_4(n)=\begin{cases}(1848-149n)/1000 & \text{for } 3\leqq n<4\\ (189-8n)/125 & \text{for } 4\leqq n\leqq 5,\end{cases}$$

$$C_5(n)=\begin{cases}(1873-158n)/1000 & \text{for } 3\leqq n<4\\ (8n^2-152n+1723)/1000 & \text{for } 4\leqq n\leqq 5,\end{cases}$$

$$C_6(n)=\begin{cases}(18923-1647n)/10000 & \text{for } 3\leqq n<4\\ (80n^2-1428n+14065)/7800 & \text{for } 4\leqq n\leqq 5,\end{cases}$$

$$C_7(n)=(16634-1084n)/110000 \quad \text{for } 4\leqq n\leqq 4.5,$$

where we computed the values of $Y_j(n)$ for n with step 1/100.

Here, we cite the approximately computed values of $Y_j(n)$ for the values of n with step 1/10 as in Table 2.

Table 2.

n	Z	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
3.0	1.0722	1.4123	1.4044	1.4001	1.3989	1.3982	1.3980
3.1	1.0780	1.3950	1.3854	1.3799	1.3781	1.3771	1.3766
3.2	1.0840	1.3803	1.3689	1.3622	1.3596	1.3582	1.3574
3.3	1.0903	1.3675	1.3541	1.3462	1.3428	1.3409	1.3398
3.4	1.0968	1.3560	1.3406	1.3312	1.3269	1.3245	1.3230
3.5	1.1035	1.3455	1.3279	1.3171	1.3118	1.3088	1.3069
3.6	1.1103	1.3357	1.3159	1.3035	1.2971	1.2935	1.2911
3.7	1.1175	1.3265	1.3043	1.2902	1.2827	1.2784	1.2755
3.8	1.1248	1.3177	1.2931	1.2772	1.2685	1.2634	1.2600
3.9	1.1323	1.3093	1.2821	1.2644	1.2543	1.2485	1.2444
4.0	1.1401	1.3012	1.2713	1.2516	1.2402	1.2335	1.2288
4.0	1.1458	1.3012	1.2760	1.2560	1.2425	1.2347	1.2298
4.1	1.1510	1.2592	1.2689	1.2476	1.2336	1.2242	1.2183
4.2	1.1562	1.2895	1.2621	1.2394	1.2250	1.2141	1.2072
4.3	1.1615	1.2841	1.2555	1.2314	1.2166	1.2042	1.1964
4.4	1.1668	1.2788	1.2491	1.2237	1.2084	1.1946	1.1858
4.5	1.1757	1.2738	1.2429	1.2161	1.2004	1.1853	1.1756
4.6	1.1802	1.2689	1.2369	1.2087	1.1926	1.1763	1.1657
4.7	1.1847	1.2641	1.2310	1.2015	1.1849	1.1675	1.1560
4.8	1.1891	1.2595	1.2253	1.1944	1.1774	1.1590	1.1466
4.9	1.1935	1.2551	1.2198	1.1875	1.1700	1.1507	1.1376
5.0	1.1980	1.2507	1.2143	1.1806	1.1628	1.1427	1.1280

Looking over Table 2, we find that $Y_6(n)<Z(n)$ for the values of $n(4.5\leqq n\leqq 5)$, which is derived from

$$K_6(n,\ Z(n))>0, \quad \text{where} \quad Z(n)=1+\frac{(n-1)(565-14n)}{10000}.$$

This inequality can be proved by showing

$$K_6(n_1,\ n_2;\ X_1,\ X_2)>0$$

for suitably selected n_1, n_2, X_1, X_2 as

$$4.5\leqq n_1<n_2\leqq 5 \quad \text{and} \quad X_1\leqq Z(n_1)<Z(n_2)\leqq X_2,$$

by the same reason as before.

By means of Proposition 6′ and Propoaition 7′ and the above results for $4.5 \leqq n \leqq 5$, we obtain

$$V(x, X_n(x)) > 0 \quad \text{for} \quad 0 < x < 1,$$

which implies Theorem C for $4.5 \leqq n \leqq 5$. Hence, we have obtained the main theoarem of this paper as follows.

THEOREM C. *The period function T as a function of τ and n is monotone decreasing with respect to $n \geqq 4.5$ for any fixed $\tau (0 < \tau < 1)$.*

Remark. The author has expected to prove the above Theorem C foar $3 \leqq n \leqq 5$ in the early stage of this work. But, when he proceeded the same arguments as above for $3 \leqq n \leqq 4.5$, he found that it takes too long steps to obtain the same result for this interval of n, especially as n comes near 3, by means of the micro computor used up to the present. He will try it by improving Proposition 1 and Proposition 2 and using a more elaborate machine to the interval $3 \leqq n \leqq 4.5$. We know that the methods used until now does not work well for the proof of Conjecture C for $2 \leqq n \leqq 3$ by Lemma 8.1 of (III). Therefore, for this interval we have to develop new methods, which may be applicable to the interval $3 \leqq n \leqq 4.5$.

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