Y. HIRASAWA
KODAI MATH. J.
9 (1986), 266—271

A REMARK ON HOMOGENEOUS LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

BY YOSHIKAZU HIRASAWA

1. We use the same notations as in previous papers [1], [2]. Let I be a closed interval $[\alpha, \beta]=\{t \mid \alpha \leqq t \leqq \beta,\ t \in \mathbf{R}\}$. We denote by $C^{\mu}(I, \mathbf{C})$ the totality of complex-valued functions defined and of class C^{μ} on I ($\mu=0, 1, \cdots, \infty$) and hereafter we fix some μ.

For the sake of brevity, we denote $C^{\mu}(I, \mathbf{C})$ by $K(I)$ and $K(I)^{n}$ by $M(I)$:

$$M(I)=\{\boldsymbol{f}(t)=\mathrm{col}(f_1(t), f_2(t), \cdots, f_n(t) \mid f_j(t) \in K(I), \quad j=1, 2, \cdots, n\}.$$

Now let $B(t)$ be a square matrix of degree n whose components all belong to $K(I)$:

$$B(t)=\begin{pmatrix} b_{11}(t) & b_{12}(t) \cdots b_{1n}(t) \\ b_{21}(t) & b_{22}(t) \cdots b_{2n}(t) \\ \vdots & \vdots \quad\quad \vdots \\ b_{n1}(t) & b_{n2}(t) \cdots b_{nn}(t) \end{pmatrix}, \tag{1}$$

and let us assume, throughout this paper, that for a positive integer $s: 1 \leqq s \leqq n-1$, a condition

$$\mathrm{rank}\, B(t)=n-s \tag{2}$$

is satisfied on I.

We consider a homogeneous linear equation

$$B(t)\boldsymbol{f}(t)=\boldsymbol{o} \qquad \text{on} \quad I\,; \quad \boldsymbol{f}(t) \in M(I), \tag{3}$$

and we denote the totality of solutions of equation (3) by $W(I)$:

$$W(I)=\{\boldsymbol{f}(t) \in M(I) \mid B(t)\boldsymbol{f}(t)=\boldsymbol{o} \quad \text{on } I\}.$$

Then, we know that there exist s vectors

$$\boldsymbol{p}_g(t)=\mathrm{col}(p_{1g}(t), p_{2g}(t), \cdots, p_{ng}(t)) \qquad (g=1, 2, \cdots, s)$$

belonging to $W(I)$, such that

$$\mathrm{rank}\,(\boldsymbol{p}_1(t), \boldsymbol{p}_2(t), \cdots, \boldsymbol{p}_s(t))=s \tag{4}$$

Received December 12, 1985

for any $t \in I$.

For a proof of this fact, see, for example, the proof of Theorem in the previous paper [1].

The purpose of this paper is to clarify the relation between the vectors $\boldsymbol{p}_g(t)$ $(g=1, 2, \cdots, s)$ given above and any vector

$$\boldsymbol{u}(t)=\mathrm{col}(u_1(t), u_2(t), \cdots, u_n(t))$$

belonging to $W(I)$. A result for this theme will be stated as Theorem in No. 3.

Let $Q(t)$ be an $n \times s$ matrix whose components all belong to $K(I)$:

$$Q(t)=\begin{pmatrix} q_{11}(t) & q_{12}(t) \cdots q_{1s}(t) \\ q_{21}(t) & q_{22}(t) \cdots q_{2s}(t) \\ \vdots & \vdots \quad\quad \vdots \\ q_{n1}(t) & q_{n2}(t) \cdots q_{ns}(t) \end{pmatrix},$$

where s is an integer such that $1 \leqq s \leqq n-1$. We denote, in general, a minor of degree s of the matrix $Q(t)$, by

$$Q\begin{pmatrix} i_1 & i_2 \cdots i_s \\ 1 & 2 \cdots s \end{pmatrix}=\begin{vmatrix} q_{i_1 1}(t) & q_{i_1 2}(t) \cdots q_{i_1 s}(t) \\ q_{i_2 1}(t) & q_{i_2 2}(t) \cdots q_{i_2 s}(t) \\ \vdots & \vdots \quad\quad \vdots \\ q_{i_s 1}(t) & q_{i_s 2}(t) \cdots q_{i_s s}(t) \end{vmatrix}$$

$$(1 \leqq i_1 < i_2 < \cdots < i_s \leqq n).$$

2. We give the following two lemmas:

LEMMA 1. *Let $Q(t)$ be an $n \times s$ matrix whose components all belong to $K(I)$, where s is an integer such that $1 \leqq s \leqq n-1$, and suppose that a condition*

$$\text{(5)} \qquad \operatorname{rank} Q(t)=s$$

is satisfied on I.

Let I_0 be a subinterval of I and let

$$\boldsymbol{x}(t)=\mathrm{col}(x_1(t), x_2(t), \cdots, x_s(t))$$

be an s-dimensional vector such that $x_g(t) \in K(I_0)$ $(g=1, 2, \cdots, s)$ and

$$\text{(6)} \qquad Q(t)\boldsymbol{x}(t)=\boldsymbol{o} \quad on \quad I_0 .$$

Then we have $\boldsymbol{x}(t) \equiv \boldsymbol{o}$ on I_0.

Proof. For any $t_0 \in I_0$, we can choose, by assumption, a minor $Q\begin{pmatrix} i_1 & i_2 \cdots i_s \\ 1 & 2 \cdots s \end{pmatrix}$ of degree s of $Q(t)$, not vanishing at t_0.

Putting

$$\tilde{\boldsymbol{q}}_g(t)=\mathrm{col}(q_{i_1 g}(t), q_{i_2 g}(t), \cdots, q_{i_s g}(t)) \qquad (g=1, 2, \cdots, s),$$

we have

$$(7) \qquad \det(\tilde{\boldsymbol{q}}_1(t_0), \tilde{\boldsymbol{q}}_2(t_0), \cdots, \tilde{\boldsymbol{q}}_s(t_0)) \neq 0$$

and in virtue of the relation (6), we get

$$\sum_{g=1}^{s} x_g(t_0)\tilde{\boldsymbol{q}}_g(t_0) = \boldsymbol{o}\,.$$

Hence the condition (7) implies $\boldsymbol{x}(t_0)=\boldsymbol{o}$ and since t_0 is any point in I_0, we see $\boldsymbol{x}(t)\equiv\boldsymbol{o}$ on I_0.

LEMMA 2. *Let $\boldsymbol{p}_g(t)$ $(g=1, 2, \cdots, s)$ be s vectors belonging to $W(I)$ and satisfying the condition (4) on I, and let $\boldsymbol{u}(t)$ be any vector belonging to $W(I)$.*
Then we have

$$(8) \qquad \operatorname{rank}(\boldsymbol{p}_1(t), \cdots, \boldsymbol{p}_s(t), \boldsymbol{u}(t)) = s \qquad \text{on} \quad I\,.$$

Proof. We put $P(t)=(\boldsymbol{p}_1(t), \boldsymbol{p}_2(t), \cdots, \boldsymbol{p}_s(t))$ and then we have always

$$s \leqq \operatorname{rank}(P(t), \boldsymbol{u}(t)) \leqq s+1 \qquad \text{on} \quad I\,.$$

If there were a point $t_0\in I$ such that

$$\operatorname{rank}(P(t_0), \boldsymbol{u}(t_0)) = s+1\,,$$

then, since

$$B(t_0)(P(t_0), \boldsymbol{u}(t_0)) = O\,,$$

we should obtain

$$\operatorname{rank} B(t_0) \leqq n-(s+1)\,;$$

this, however, contradicts the condition (2). Therefore we have always the equality (8) on I.

3. We shall prove the following theorem:

THEOREM. *Let*

$$\boldsymbol{p}_g(t) = \operatorname{col}(p_{1g}(t), p_{2g}(t), \cdots, p_{ng}(t)) \qquad (g=1, 2, \cdots, s)$$

be s vectors belonging to $W(I)$ and satisfying the condition (4) on I, and let

$$\boldsymbol{u}(t) = \operatorname{col}(u_1(t), u_2(t), \cdots, u_n(t))$$

be any vector belonging to $W(I)$.
Then we can represent $\boldsymbol{u}(t)$ uniquely as a linear combination of the vectors $\boldsymbol{p}_g(t)$ $(g=1, 2, \cdots, s)$ with coefficients $\zeta_g(t)$ $(g=1, 2, \cdots, s)$ belonging to $K(I)$:

$$(9) \qquad \boldsymbol{u}(t) = \sum_{g=1}^{s} \zeta_g(t)\boldsymbol{p}_g(t)$$

on I.

Proof. We put $P(t)=(\boldsymbol{p}_1(t), \boldsymbol{p}_2(t), \cdots, \boldsymbol{p}_s(t))$.

There exists, in virtue of the condition (4), a set $\{I_\kappa\}_{\kappa=1}^{\kappa_0}$ of subintervals of I, possessing the following properties:

(i) $I=\bigcup_{\kappa=1}^{\kappa_0} I_\kappa$;

(ii) $I_1=[\alpha_1, \beta_1)$, $I_{\kappa_0}=(\alpha_{\kappa_0}, \beta_{\kappa_0}]$, $\alpha_1=\alpha$, $\beta_{\kappa_0}=\beta$, $I_\kappa=(\alpha_\kappa, \beta_\kappa)$ $(\kappa=2, 3, \cdots, \kappa_0-1)$;

(iii) $I_\kappa \cap I_{\kappa+1}\neq\emptyset$ $(\kappa=1, 2, \cdots, \kappa_0-1)$, $I_\kappa\cap I_{\kappa'}=\emptyset$ $(\kappa+1<\kappa', \kappa=1, 2, \cdots, \kappa_0-2)$, that is, $\alpha_1<\alpha_2<\beta_1<\cdots<\alpha_\kappa<\beta_{\kappa-1}<\alpha_{\kappa+1}<\beta_\kappa<\cdots<\beta_{\kappa_0-2}<\alpha_{\kappa_0}<\beta_{\kappa_0-1}<\beta_{\kappa_0}$ $(\kappa=2, 3, \cdots, \kappa_0-1)$;

(iv) For each I_κ, there exists a minor of degree s of $P(t)$, which does not vanish on I_κ.

We consider first the interval I_1 and choose a minor $P\begin{pmatrix} i_1 & i_2 \cdots i_s \\ 1 & 2 \cdots s\end{pmatrix}$ of degree s of $P(t)$ such that a condition

$$P\begin{pmatrix} i_1 & i_2 \cdots i_s \\ 1 & 2 \cdots s\end{pmatrix}\neq 0 \tag{10}$$

is satisfied on I_1.

Let us define $(n-s)$-tuple $(i'_{s+1}, i'_{s+2}, \cdots, i'_n)$ for $1\leqq i_1<i_2<\cdots<i_s\leqq n$ in such a way that $1\leqq i'_{s+1}<i'_{s+2}<\cdots<i'_n\leqq n$ and $\{i_1, \cdots, i_s, i'_{s+1}, \cdots, i'_n\}=\{1, 2, \cdots, n\}$ and put

$$\tilde{\boldsymbol{p}}_g(t)=\mathrm{col}(p_{i_1 g}(t), p_{i_2 g}(t), \cdots, p_{i_s g}(t)) \qquad (g=1, 2, \cdots, s),$$

$$\tilde{\boldsymbol{p}}'_g(t)=\mathrm{col}(p_{i'_{s+1} g}(t), p_{i'_{s+2} g}(t), \cdots, p_{i'_n g}(t)) \qquad (g=1, 2, \cdots, s),$$

$$\tilde{\boldsymbol{u}}(t)=\mathrm{col}(u_{i_1}(t), u_{i_2}(t), \cdots, u_{i_s}(t)),$$

$$\tilde{\boldsymbol{u}}'(t)=\mathrm{col}(u_{i'_{s+1}}(t), u_{i'_{s+2}}(t), \cdots, u_{i'_n}(t)).$$

Since, in virtue of the fact that the condition (10) is satisfied on I_1, we know

$$\det(\tilde{\boldsymbol{p}}_1(t), \tilde{\boldsymbol{p}}_2(t), \cdots, \tilde{\boldsymbol{p}}_s(t))\neq 0 \qquad \text{on} \quad I_1,$$

we can represent the vector $\tilde{\boldsymbol{u}}(t)$ as a linear combination of the vectors $\tilde{\boldsymbol{p}}_g(t)$ $(g=1, 2, \cdots, s)$ with coefficients $\zeta_g(t)$ $(g=1, 2, \cdots, s)$ belonging to $K(I_1)$:

$$\tilde{\boldsymbol{u}}(t)=\sum_{g=1}^{s}\zeta_g(t)\tilde{\boldsymbol{p}}_g(t) \tag{11}$$

on I_1.

Furthermore, we wish to show that the vector $\boldsymbol{u}(t)$ can be represented on I_1, as a linear combination (9) of the vectors $\boldsymbol{p}_g(t)$ $(g=1, 2, \cdots, s)$ with the same coefficients $\zeta_g(t)$ $(g=1, 2, \cdots, s)$ as in the representation (11). To this end, we have only to prove that a representation

$$\tilde{\boldsymbol{u}}'(t)=\sum_{g=1}^{s}\zeta_g(t)\tilde{\boldsymbol{p}}'_g(t) \tag{12}$$

holds on I_1.

If there were a point $t_0\in I_1$ and an index ρ $(s+1\leqq\rho\leqq n)$ such that

$$u_{i'_\rho}(t_0)\neq\sum_{g=1}^{s}\zeta_g(t_0)p_{i'_\rho g}(t_0)\,,$$

then we should have

$$\mathrm{rank}\,(P(t_0),\,\boldsymbol{u}(t_0))=s+1\,.$$

This equality contradicts the result which was proved in Lemma 2. Therefore we have the representation (12) on I_1.

Next we consider the second interval I_2 and then, in the same way as for the interval I_1, we get a representation

$$\boldsymbol{u}(t)=\sum_{g=1}^{s}\theta_g(t)\boldsymbol{p}_g(t) \tag{13}$$

on I_2, where $\theta_g(t)\in K(I_2)$ $(g=1, 2, \cdots, s)$.

It follows from the representations (9) on I_1 and (13) on I_2, that

$$\sum_{g=1}^{s}(\zeta_g(t)-\theta_g(t))\boldsymbol{p}_g(t)\equiv\boldsymbol{o}$$

on $I_1\cap I_2$, and hence by Lemma 1, we obtain

$$\zeta_g(t)\equiv\theta_g(t)\qquad(g=1, 2, \cdots, s)\qquad\text{on}\quad I_1\cap I_2\,.$$

Therefore, by defining functions $\zeta_g(t)\in K(I_1\cup I_2)$ $(g=1, 2, \cdots, s)$ as follows:

$$\zeta_g(t)=\begin{cases}\zeta_g(t) & \text{on}\quad I_1\,,\\ \theta_g(t) & \text{on}\quad I_2\,,\end{cases}$$

we have the representation (9) on $I_1\cup I_2$.

By repeating the process mentioned above for the intervals I_κ $(\kappa=1, 2, \cdots, \kappa_0)$ successively, we obtain the representation (9) on the whole interval I.

Remark. The fact stated in the above theorem was used by Y. Sibuya without proof in his paper [3].

References

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY