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ON ALGEBROID FUNCTIONS TAKING THE SAME VALUES AT THE SAME POINTS

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1. It is well known that if two meromorphic functions $w(z)$ and $\hat{w}(z)$ take five values at the same points, then $w(z)\equiv\hat{w}(z)$, and if $w(z)$ and $\hat{w}(z)$ share only four values, usually $w(z)\not\equiv\hat{w}(z)$, but there exists some relations between $w(z)$ and $\hat{w}(z)$. Recently G. Gundersen [1] proved that if two meromorphic functions share three values, then the proportion of their characteristic functions is finite. On the ν-valued algebroid functions G. Valiron [5] pointed out that if two ν-valued algebroid functions $w(z)$ and $\hat{w}(z)$ take $4\nu+1$ values at the same points with same multiple order, then $w(z)\equiv\hat{w}(z)$. In [3], we proved a uniqueness theorem which refined the result of Valiron. In present paper we first prove that if two ν-valued algebroid functions $w(z)$ and $\hat{w}(z)$ share 4ν values, then there exists some relations between $w(z)$ and $\hat{w}(z)$, and we construct two different ν-valued algebroid functions sharing 4ν values. Secondly we proved that if two ν-valued algebroid functions share $2\nu+\lambda$ values with $1\leqq\lambda\leqq 2\nu-1$, then the ratio of their characteristic functions is finite, and we give two ν-valued algebroid functions sharing 2ν values, but the ratio of their characteristic functions is infinite. We also obtain some results concerning the multiplicity.

2. Let $w(z)$ be a ν-valued algebroid function defined by the following irreducible equation

$$\psi(z, w)\equiv A_\nu(z)w^\nu+A_{\nu-1}(z)w^{\nu-1}+\cdots+A_0(z)=0 \tag{1}$$

where $A_j(z)$ $(j=0, 1, \cdots, \nu)$ are entire functions in $\boldsymbol{C}$. Set

$$N\left(r, \frac{1}{w-a}\right)=\frac{1}{\nu}N\left(r, \frac{1}{\psi(z, a)}\right)=\frac{1}{\nu}\int_0^r \frac{n\left(t, \frac{1}{w-a}\right)-n\left(0, \frac{1}{w-a}\right)}{t}dt + \frac{1}{\nu}n\left(0, \frac{1}{w-a}\right)\log r, \qquad a\in\boldsymbol{C}$$

and

$$N(r, w)=\frac{1}{\nu}N\left(r, \frac{1}{A_\nu}\right)=\frac{1}{\nu}\int_0^r \frac{n(t, w)-n(0, w)}{t}dt+\frac{1}{\nu}n(0, w)\log r, \qquad a=\infty$$

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where $n\left(t, \dfrac{1}{w-a}\right)$ is the number of the zeroes of $w(z)-a$ in $|z|<t$ being counted the multiply. We denote by $J(z)=[A_\nu(z)]^{2(\nu-1)}\prod_{1\leq j<k\leq\nu}[w_j(z)-w_k(z)]^2$ or

$$J(z)=\begin{vmatrix} 1,\ A_{\nu-1}(z),\ \cdots,\ A_0(z),\ 0,\ \cdots,\ 0 \\ 0,\ A_\nu(z),\ A_1(z),\ \cdots,\ A_0(z),\ 0,\ \cdots,\ 0 \\ \cdots\cdots\cdots \\ 0,\ \cdots,\ 0,\ A_\nu(z),\ A_{\nu-1}(z),\ \cdots,\ A_0(z) \\ \nu,\ (\nu-1)A_{\nu-1}(z),\ \cdots,\ A_1(z),\ 0,\ \cdots,\ 0 \\ 0,\ \nu A_\nu(z),\ (\nu-1)A_{\nu-1}(z),\ \cdots,\ A_1(z),\ 0,\ \cdots,\ 0 \\ \cdots\cdots\cdots \\ 0,\ \cdots,\ 0,\ \nu A_\nu(z),\ (\nu-1)A_{\nu-1}(z),\ \cdots,\ A_1(z) \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}\nu-1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}\nu \end{matrix}$$

the discriminant, and it is well known that each branch point of $w(z)$ is a zero of $J(z)$. Let L denote a curve joining all zeroes of $J(z)$, then the determinations $w_j(z)$ of $w(z)$ $(j=1, 2, \cdots, \nu)$ are simple-valued functions in $\boldsymbol{C}\setminus L$. We set

$$m\left(r, \frac{1}{w-a}\right)=\frac{1}{\nu}\sum_{j=1}^{\nu} m\left(r, \frac{1}{w_j-a}\right)=\frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|w_j(re^{i\varphi})-a|}d\varphi, \quad a\in\boldsymbol{C}$$

and

$$m(r, w)=\frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j)=\frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\varphi})|\,d\varphi, \qquad a=\infty$$

and we call

$$T(r, w)=m(r, w)+N(r, w)$$

the characteristic function of $w(z)$. We have (cf. [5] or [6])

THEOREM A. (The first fundamental theorem). *If a is any complex number, then*

$$m\left(r, \frac{1}{w-a}\right)+N\left(r, \frac{1}{w-a}\right)=T(r, w)+\frac{1}{\nu}\log\left|\frac{\psi(0, a)}{A_\nu(0)}\right|+\epsilon(r, a), \quad (2)$$

where

$$|\epsilon(r, a)|\leqq\log^+|a|+\log 2$$

THEOREM B. (The second fundamental theorem). *Let $w(z)$ be a ν-valued algebroid function and $a_j\in\hat{\boldsymbol{C}}$ $(j=1, 2, \cdots, p)$ be p different number, then*

$$(p-2\nu)T(r, w)<\sum_{j=1}^{p} N\left(r, \frac{1}{w-a_j}\right)-N_1(r, w)+S(r, w) \qquad (3)$$

where $N_1(r, w)$ is the counting function for all multiple value-points of $w(z)$, but a τ-fold value-point is counted only $\tau-1$ times and $S(r, w)$ is the remainder term.

We denote by $\bar{n}\left(t, \dfrac{1}{w-a}\right)$ the number of distinct roots of $w(z)=a$ in $|z|<t$

and define

$$\bar{N}\left(r, \frac{1}{w-a}\right)=\frac{1}{\nu}\int_0^r \frac{\bar{n}\left(t, \frac{1}{w-a}\right)-\bar{n}\left(0, \frac{1}{w-a}\right)}{t} dt+\frac{1}{\nu}\bar{n}\left(0, \frac{1}{w-a}\right)\log r$$

then the second fundamental theorem can be written as the form.

THEOREM B′. *Suppose $w(z)$ and $a_j \in \boldsymbol{C}$ are the same as theorem B, then*

$$(p-2\nu)T(r, w)<\sum_{j=1}^{p} \bar{N}\left(r, \frac{1}{w-a_j}\right)+O\{\log(rT(r, w))\} \tag{4}$$

outside a certain exceptional set of finite linear measure.

Let $\bar{E}(a, w)$ denote the set of distinct roots of $w(z)=a$, then we have

THEOREM 1. *Let $w(z)$ and $\hat{w}(z)$ be two algebroid functions definied by* (1) *and*

$$\Phi(z, \hat{w})\equiv B_\nu(z)\hat{w}^\nu+B_{\nu-1}(z)\hat{w}^{\nu-1}+\cdots+B_0(z)=0 \tag{1$'$}$$

respectively. If $\bar{E}(a_j, w)=\bar{E}(a_j, \hat{w})$ for $a_j \in \hat{C}$, $(j=1, 2, \cdots, 4\nu)$, then it must be

(i)
$$\lim_{\substack{r\to\infty \\ r\bar{\in} E}} \frac{T(r, w)}{T(r, \hat{w})}=1, \tag{5}$$

(ii)
$$\lim_{\substack{r\to\infty \\ r\bar{\in} E}} \sum_{j=1}^{4\nu} \frac{\bar{N}\left(r, \frac{1}{w-a_j}\right)}{T(r, w)}=\lim_{\substack{r\to\infty \\ r\in E}} \sum_{j=1}^{4\nu} \frac{\bar{N}\left(r, \frac{1}{\hat{w}-a_j}\right)}{T(r, \hat{w})}=2\nu, \tag{6}$$

(iii) *for any $a\neq a_j$, then*

$$\lim_{\substack{r\to\infty \\ r\bar{\in} E}} \frac{\bar{N}\left(r, \frac{1}{w-a}\right)}{T(r, w)}=\lim_{\substack{r\to\infty \\ r\bar{\in} E}} \frac{\bar{N}\left(r, \frac{1}{\hat{w}-a}\right)}{T(r, \hat{w})}=1, \tag{7}$$

where E is a set with finite linear measure.

Proof. (i) Suppose $w(z)\not\equiv\hat{w}(z)$ and $\bar{n}_0(t, a)$ denotes the number of the common roots of $w(z)=a$ and $\hat{w}(z)=a$ containing in $|z|<t$ and each common root is counted only once. We define

$$\bar{N}_0(r, a)=\frac{1}{\nu}\int_0^r \frac{\bar{n}_0(t, a)-\bar{n}_0(0, a)}{t} dt+\frac{1}{\nu}\bar{n}_0(0, a)\log r .$$

It is easy to know that

$$\sum \bar{n}_0(r, a_j)\leqq n\left(r, \frac{1}{R(\psi, \Phi)}\right)$$

where $R(\psi, \Phi)$ is the resultant of (1) and (1)′, i.e.

$$R(\psi, \Phi)=[A_\nu(z)B_\nu(z)]^\nu \prod_{\substack{1\leq j\leq \nu\\ 1\leq k\leq \nu}} [w_j(z)-\hat{w}_k(z)]$$

$$=\left|\begin{array}{l} A_\nu(z),\ A_{\nu-1}(z),\ \cdots,\ A_0(z),\ 0,\ \cdots,\ 0 \\ 0,\ A_\nu(z),\ A_{\nu-1}(z),\ \cdots,\ A_0(z),\ 0,\ \cdots,\ 0 \\ \cdots\cdots\cdots \\ 0,\ \cdots,\ 0,\ A_\nu(z),\ A_{\nu-1}(z),\ \cdots,\ A_0(z) \\ B_\nu(z),\ B_{\nu-1}(z),\ \cdots,\ B_0(z),\ 0,\ \cdots,\ 0 \\ 0,\ B_\nu(z),\ B_{\nu-1}(z),\ \cdots,\ B_0(z),\ 0,\ \cdots,\ 0 \\ \cdots\cdots\cdots \\ 0,\ \cdots,\ 0,\ B_\nu(z),\ B_{\nu-1}(z),\ \cdots,\ B_0(z) \end{array}\right| \begin{array}{l} \left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\}\nu \\ \left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\}\nu \end{array}$$

by using the Jensen's formula we get

$$N\left(r, \frac{1}{R(\psi, \Phi)}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log|R(\psi, \Phi)|\,d\varphi+\log\left|\frac{1}{R(\psi, \Phi)}\right|_{z=0}$$

$$=\frac{\nu}{2\pi}\int_0^{2\pi}\log|A_\nu(re^{i\varphi})|\,d\varphi+\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\nu(re^{i\varphi})|\,d\varphi$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\leq j\leq \nu\\ 1\leq k\leq \nu}}[w_j(re^{i\varphi})-\hat{w}_k(re^{i\varphi})]\right|d\varphi+\log\left|\frac{1}{R(\psi, \Phi)}\right|_{z=0}$$

$$\leqq \nu N\left(r, \frac{1}{A_\nu}\right)+\nu N\left(r, \frac{1}{B_\nu}\right)+\nu\sum_{j=1}^{\nu} m(r, w_j)+\nu\sum_{k=1}^{\nu} m(r, \hat{w}_k)+O(1)$$

$$=\nu^2[T(r, w)+T(r, \hat{w})]+O(1)$$

hence

$$\sum_{j=1}^{4\nu}\overline{N}_0(r, a_j)\leqq \nu[T(r, w)+T(r, \hat{w})]+O(1) \tag{8}$$

applying Theorem B′ to $w(z)$, $\{a_j\}$ and $\hat{w}(z)$, $\{a_j\}$, we have

$$2\nu T(r, w)<\sum_{j=1}^{4\nu}\overline{N}\left(r, \frac{1}{w-a_j}\right)+O\{T(r, w)\} \tag{4$'$}$$

and

$$2\nu T(r, \hat{w})<\sum_{j=1}^{4\nu}\overline{N}\left(r, \frac{1}{\hat{w}-a_j}\right)+O\{T(r, \hat{w})\} \tag{4$''$}$$

outside of a possible exceptional set E with a finite linear measure. Since $\overline{N}\left(r, \frac{1}{w-a_j}\right)=\overline{N}_0(r, a_j)$, noting (8) we get

$$2\nu T(r, w)<\sum_{j=1}^{4\nu}\overline{N}_0(r, a_j)+O\{T(r, w)\}$$

$$\leqq \nu[T(r, w)+T(r, \hat{w})]+O\{T(r, w)\} \tag{9}$$

thus

$$T(r, w)<T(r, \hat{w})+O\{T(r, w)\} \tag{10}$$

by a similar argument we have

$$T(r, \hat{w})<T(r, w)+O\{T(r, \hat{w})\} \tag{11}$$

From (10) and (11), it follows (5).

(ii) Combining (4)′ with (9) and (11) we get

$$(2\nu+O(1))T(r, w)\leqq \sum_{j=1}^{4\nu} \bar{N}\left(r, \frac{1}{w-a_j}\right)=\sum_{j=1}^{4\nu} \bar{N}_0(r, a_j)$$
$$\leqq(2\nu+O(1))T(r, w)$$

it shows that the first equality of (6) holds. Similarly we have the second one of (6).

(iii) Applying the theorem B′ to $w(z)$, a_j $(j=1, 2, \cdots, 4\nu)$ and a, and noting (8) and (11) we obtain

$$(2\nu+1)T(r, w)<\sum_{j=1}^{4\nu} \bar{N}\left(r, \frac{1}{w-a_j}\right)+\bar{N}\left(r, \frac{1}{w-a}\right)+S(r, w)$$
$$=\sum_{j=1}^{4\nu} \bar{N}_0(r, a_j)+\bar{N}\left(r, \frac{1}{w-a}\right)+O\{T(r, w)\}$$
$$=(2\nu+O(1))T(r, w)+\bar{N}\left(r, \frac{1}{w-a}\right)$$

therefore

$$T(r, w)\leqq\bar{N}\left(r, \frac{1}{w-a}\right)+O\{T(r, w)\}\leqq(1+O(1))T(r, w)$$

Similarly

$$T(r, \hat{w})\leqq\bar{N}\left(r, \frac{1}{\hat{w}-a}\right)+O\{T(r, w)\}\leqq(1+O(1))T(r, \hat{w})$$

it follows (7).

Now we give two different ν-valued algebroid functions which take 4ν values at the same points and satisfy the conclusions given by Theorem 1.

Let $w(z)$ and $\hat{w}(z)$ be two ν-valued algebroid functions defined by

$$\psi(z, w)\equiv(a+bc^z)w^\nu-(c+de^z)=0$$

and

$$\Phi(z, \hat{w})\equiv(a+bc^{-z})\hat{w}^\nu-(c+de^{-z})=0$$

respectively, where a, b, c and d are different non-zero complex numbers with $ad-bc\neq0$ and $\frac{c+d}{a+b}\neq0, \infty$.

It is obvious that $w(z)\not\equiv\hat{w}(z)$. Suppose $a_j=\left|\frac{c}{a}\right|^{1/\nu}e^{i(\alpha/\nu+2\pi j/\nu)}$ $(j=1, 2, \cdots, \nu)$ with $\alpha=\arg\frac{c}{a}$, $b_k=\left|\frac{d}{b}\right|^{1/\nu}e^{i(\beta/\nu+2k\pi/\nu)}$ $(k=1, 2, \cdots, \nu)$ with $\beta=\arg\frac{d}{b}$, $c_l=\left|\frac{c+d}{a+b}\right|^{1/\nu}e^{i(\gamma/\nu+2\pi l/\nu)}$ $(l=1, 2, \cdots, \nu)$ with $\gamma=\arg\frac{c+d}{a+b}$, $d_m=\left|\frac{c-d}{a-b}\right|^{1/\nu}e^{i(\delta/\nu+2\pi m/\nu)}$ $(m=1, 2, \cdots, \nu)$ with $\delta=\arg\frac{c-d}{a-b}$. We can show that $\bar{E}(a_j, w)=\bar{E}(a_j, \hat{w})$, $\bar{E}(b_k, w)=\bar{E}(b_k, \hat{w})$, $\bar{E}(c_l, w)=\bar{E}(c_l, \hat{w})$, $\bar{E}(d_m, w)=\bar{E}(d_m, \hat{w})$, $(j, k, l, m=1, 2, \cdots, \nu)$.

Noting the roots of $w(z)=a$ (or $\hat{w}(z)=a$) are the zeroes of $\psi(z, a)=0$ (or $\Phi(z, a)=0$), we have

$$\psi(z, a_j)=\frac{bc-ad}{a}e^z\neq 0, \qquad (j=1, 2, \cdots, \nu)$$

thus a_j $(j=1, 2, \cdots, \nu)$ are the Picard exceptional values of $w(z)$ i.e. $\bar{E}(a_j, w)=\emptyset$ $(j=1, 2, \cdots, \nu)$. Similarly

$$\Phi(z, a_j)=\frac{bc-ad}{a}e^{-z}\neq 0$$

a_j $(j=1, 2, \cdots, \nu)$ are also the Picard exceptional values of $\hat{w}(z)$ i.e. $\bar{E}(a_j, \hat{w})=\emptyset$ $(j=1, 2, \cdots, \nu)$. For b_k $(k=1, 2, \cdots, \nu)$, because

$$\psi(z, b_k)=a-c=\Phi(z, b_k)$$

we have b_k $(k=1, 2, \cdots, \nu)$ are also the Picard exceptional values of $w(z)$ and $\hat{w}(z)$, and therefore $\bar{E}(b_k, w)=\bar{E}(b_k, \hat{w})=\emptyset$ $(k=1, 2, \cdots, \nu)$. For c_l $(l=1, 2, \cdots, \nu)$, since

$$\psi(z, c_l)=\frac{ad-bc}{a+b}(1-e^z),$$

we have $\bar{E}(c_l, w)=\{2\pi ni, n\in \boldsymbol{Z}\}$, on the other hand, since

$$\Phi(z, c_l)=\frac{ad-bc}{a+b}(1-e^{-z})$$

we have $\bar{E}(c_l, \hat{w})=\{2\pi ni, n\in \boldsymbol{Z}\}$, it shows that $\bar{E}(c_l, w)=\bar{E}(c_l, \hat{w})$ $(l=1, 2, \cdots, \nu)$. Finally for d_m, since

$$\psi(z, d_m)=-\frac{ad-bc}{a-b}(1+e^z)$$

and

$$\Phi(z, d_m)=-\frac{ad-bc}{a-b}(1+e^{-z})$$

$(m=1, 2, \cdots, \nu)$, we get $\bar{E}(d_m, w)=\bar{E}(d_m, \hat{w})=\{(2n+1)\pi i, n\in \boldsymbol{Z}\}$. It shows that $w(z)$ and $\hat{w}(z)$ take 4ν values at the same points, but $w(z)\not\equiv \hat{w}(z)$.

We can point out that $w(z)$ and $\hat{w}(z)$ satisfy the conclusions of theorem 1. In fact, it is easy to show that

$$T(r, w)=\frac{r}{\nu\pi}+O(1) \quad \text{and} \quad T(r, \hat{w})=\frac{r}{\nu\pi}+O(1)$$

thus

$$\lim_{r\to\infty}\frac{T(r, w)}{T(r, \hat{w})}=1$$

since $\bar{N}_0(r, a_j)=\bar{N}_0(r, b_k)=0$, $(j, k=1, 2, \cdots, \nu)$, $\bar{N}_0(r, c_l)=\dfrac{r}{\nu\pi}+O(1)$ and $\bar{N}_0(r, d_m)$

$=\dfrac{r}{\nu\pi}+O(1)$, $(l, m=1, 2, \cdots, \nu)$, we get

$$\lim_{r\to\infty}\sum_{j=1}^{\nu}\frac{\bar{N}_0(r, a_j)+\bar{N}_0(r, b_j)+\bar{N}_0(r, c_j)+\bar{N}_0(r, d_j)}{T(r, w)}$$

$$=\lim_{r\to\infty}\sum_{j=1}^{\nu}\frac{\bar{N}_0(r, a_j)+\bar{N}_0(r, b_j)+\bar{N}_0(r, c_j)+\bar{N}_0(r, d_j)}{T(r, \hat{w})}=2\nu\,.$$

Finally if $s\neq a_j$, b_k, c_l and d_m, then $\dfrac{c-as^\nu}{bs^\nu-d}\neq 0, \infty$, since

$$\psi(z, s)=(a+be^z)s^\nu-(c+de^z)=0 \quad\text{and}\quad \Phi(z, s)=(a+be^{-z})s^\nu-(c+de^{-z})=0$$

we get $\bar{E}(s, w)=\left\{\log\dfrac{c-as^\nu}{bs^\nu-d}+2n\pi i,\ n\in\boldsymbol{Z}\right\}$ and $\bar{E}(s, \hat{w})=\left\{\log\dfrac{bs^\nu-d}{c-as^\nu}+2n\pi i,\ n\in\boldsymbol{Z}\right\}$ therefore $\bar{N}\left(r, \dfrac{1}{w-s}\right)=\dfrac{r}{\nu\pi}+O(1)$ and $\bar{N}\left(r, \dfrac{1}{\hat{w}-s}\right)=\dfrac{r}{\nu\pi}+O(1)$, it follows that

$$\lim_{r\to\infty}\frac{\bar{N}\left(r, \dfrac{1}{w-s}\right)}{T(r, w)}=\lim_{r\to\infty}\frac{\bar{N}\left(r, \dfrac{1}{\hat{w}-s}\right)}{T(r, \hat{w})}=1$$

it shows that $w(z)$ and $\hat{w}(z)$ satisfy all conclusions in theorem 1.

THEOREM 2. *Let $w(z)$ and $\hat{w}(z)$ be two ν-valued algebroid functions, if $\bar{E}(a_j, w)=\bar{E}(a_j, \hat{w})$ for $a_j\in\hat{\boldsymbol{C}}$ $(j=1, 2, \cdots, 2\nu+\lambda)$ with $1\leqq\lambda\leqq 2\nu-1$, then there exists a finite non-zero number $K=K(\nu, \lambda)$ such that*

$$\frac{1}{K}\leqq\lim_{\substack{r\to\infty\\ r\in E}}\frac{T(r, w)}{T(r, \hat{w})}\leqq K \tag{12}$$

where E denotes a set with a finite linear mearsure.

Proof. Applying theorem B′ to $w(z)$, $\{a_j\}$, $(j=1, 2, \cdots, 2\nu+\lambda)$, we get

$$\lambda T(r, w)<\sum_{j=1}^{2\nu+\lambda}\bar{N}\left(r, \frac{1}{w-a_j}\right)+S(r, w)$$

noting $\bar{N}\left(r, \dfrac{1}{w-a_j}\right)=\bar{N}\left(r, \dfrac{1}{\hat{w}-a_j}\right)\leqq T(r, \hat{w})+O(1)$, we have

$$\lambda T(r, w)<\sum_{j=1}^{2\nu+\lambda}\bar{N}\left(r, \frac{1}{\hat{w}-a_j}\right)+O\{T(r, w)\}$$

$$\leqq(2\nu+\lambda)T(r, \hat{w})+O\{T(r, w)\}$$

then for any $\epsilon>0$, there exists r_0 such that if $r\geqq r_0$ and $r\,\overline{\in}\,E$, we have

$$\left(\frac{\lambda}{2\nu+\lambda}-\epsilon\right)T(r, w)<T(r, \hat{w})\,.$$

Similarly we get

$$\left(\frac{\lambda}{2\gamma+\lambda}-\epsilon\right)T(r,\ \hat{w})<T(r,\ w)$$

where E is a set of finite linear measure. Thus

$$\left(\frac{\lambda}{2\nu+\lambda}-\epsilon\right)\leqq\lim_{\substack{r\to\infty\\ r\in E}}\frac{T(r,\ w)}{T(r,\ \hat{w})}\leqq\left(\frac{\lambda}{2\nu+\gamma}-\epsilon\right)^{-1}$$

because of the arbitrariness of ϵ, it follows (12).

The following example shows that if two ν-valued algebroid functions take only 2ν values at the same points, then the ratio of their characteristic functions may be infinite. Let $w(z)$ and $w(z)$ be two algebroid functions defined by

$$\psi(z,\ w)\equiv(1+e^{z})w^{\nu}-(a+be^{z})=0$$

and

$$\Phi(z,\ \hat{w})\equiv(1+e^{e^{z}})\hat{w}^{\nu}-(a+b^{e^{z}})=0$$

respectivelly, where a and b are non-zero and distinct complex numbers. Set $a=|a|e^{i\alpha}$ and $a_j=|a|^{1/\nu}e^{i(\alpha/\nu+2\pi j/\nu)}$, $(j=1,\ 2,\ \cdots,\ \nu)$, $b=|b|e^{i\beta}$ and $b_k=|b|^{1/\nu}e^{i(\beta/\nu+2\pi k/\nu)}$, $(k=1,\ 2,\ \cdots,\ \nu)$. Since $\psi(z,\ a_j)=(a-b)e^{z}\neq 0$, a_j $(j=1,\ 2,\ \cdots,\ \nu)$ are the Picard exceptional values of $w(z)$, i.e. $\bar{E}(a_j,\ w)=\emptyset$. On the other hand, $\Phi(z,\ a_j)=(a-b)e^{e^{z}}\neq 0$, it means $\bar{E}(a_j,\ \hat{w})=\emptyset$, hence $\bar{E}(a_j,\ w)=\bar{E}(a_j,\ \hat{w})$, $(j=1,\ 2,\ \cdots,\ \nu)$. Because $\psi(z,\ b_k)=\Phi(z,\ b_k)=b-a$, $k=1,\ 2,\ \cdots,\ \nu$, thus again $\bar{E}(b_k,\ w)=\bar{E}(b_k,\ \hat{w})$, $k=1,\ 2,\ \cdots,\ \nu$. In other words, $w(z)$ and $\hat{w}(z)$ share 2ν values. But it is easy to show that $T(r,\ w)=\dfrac{r}{\nu\pi}+O(1)$ and $T(r,\ \hat{w})=\dfrac{e^{r}}{\nu\sqrt{2\pi^{3}r}}(1+O(1))$ (cf. Hayman [2]), it follows that $\dfrac{T(r,\ \hat{w})}{T(r,\ w)}\to\infty$, as $r\to\infty$.

3. Let $\gamma(\geqq 1)$ be an integer and $\bar{E}^{\gamma)}(a,\ w)$ be the set of the distinct zeroes of $w(z)-a$ which multiple order $\leqq\gamma$. We denote by $\bar{N}^{\gamma)}\left(r,\ \dfrac{1}{w-a}\right)$ the counting function of the corresponding a-points of $w(z)$. We have

THEOREM 3. *Let $w(z)$ and $\hat{w}(z)$ be two ν-valued algebroid functions and γ $(\geqq 1)$ be an integer which divides exactly 2ν. If for $a_j\in\hat{C}$, $\bar{E}^{\gamma)}(a_j,\ w)=\bar{E}^{\gamma)}(a_j,\ \hat{w})$, $j=1,\ 2,\ \cdots,\ p_\gamma$, $p_\gamma=4\nu+\dfrac{2\nu}{\gamma}$, then*

(i)
$$\lim_{\substack{r\to\infty\\ r\in E}}\frac{T(r,\ w)}{T(r,\ \hat{w})}=1$$

(ii)
$$\lim_{\substack{r\to\infty\\ r\in E}}\sum_{j=1}^{p_\gamma}\frac{\bar{N}^{\gamma)}\left(r,\ \dfrac{1}{w-a_j}\right)}{T(r,\ w)}\lim_{\substack{r\to\infty\\ r\in E}}\sum_{j=1}^{p_\gamma}\frac{\bar{N}^{\gamma)}\left(r,\ \dfrac{1}{\hat{w}-a_j}\right)}{T(r,\ \hat{w})}=2\nu$$

and (iii) for $a\neq a_j$, we have

$$\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\bar{N}^{\gamma)}\left(r,\frac{1}{w-a}\right)}{T(r,w)}\lim_{\substack{r\to\infty\\ r\notin E}}\frac{\bar{N}^{\gamma)}\left(r,\frac{1}{\hat{w}-a}\right)}{T(r,\hat{w})}=1$$

where E is a set with finite linear measure.

Especially, if $\gamma=1, 2, \cdots, \nu$, then $p_\gamma=6\nu, 5\nu, 4\nu+2$, respectively.

Proof. Set

$$\bar{N}^{(\gamma}\left(r,\frac{1}{w-a}\right)=\bar{N}\left(r,\frac{1}{w-a}\right)-\bar{N}^{\gamma)}\left(r,\frac{1}{w-a}\right)$$

it is easy to know that $\bar{N}^{(\gamma}\left(r,\frac{1}{w-a}\right)\leqq\frac{1}{\gamma+1}N^{(\gamma}\left(r,\frac{1}{w-a}\right)$ where $N^{(\gamma}\left(r,\frac{1}{w-a}\right)$ is the counting function of the zeroes of $w(z)-a$ which multiple order $>\gamma$ and being counted multiply. Since

$$\begin{aligned}\bar{N}\left(r,\frac{1}{w-a}\right)&\leqq\bar{N}^{\gamma)}\left(r,\frac{1}{w-a}\right)+\frac{1}{\gamma+1}N^{(\gamma}\left(r,\frac{1}{w-a}\right)\\&\leqq\frac{\gamma}{\gamma+1}\bar{N}^{\gamma)}\left(r,\frac{1}{w-a}\right)+\frac{1}{\gamma+1}N\left(r,\frac{1}{w-a}\right)\\&\leqq\frac{\gamma}{\gamma+1}\bar{N}^{\gamma)}\left(r,\frac{1}{w-a}\right)+\frac{1}{\gamma+1}T(r,w)+O(1)\end{aligned}$$

(4) can be written as the following form

$$\begin{aligned}(p-2\nu)T(r,w)&<\sum_{j=1}^{p}\bar{N}\left(r,\frac{1}{w-a_j}\right)+S(r,w)\\&\leqq\frac{\gamma}{\gamma+1}\sum_{j=1}^{p}\bar{N}^{\gamma)}\left(r,\frac{1}{w-a_j}\right)+\frac{p}{\gamma+1}T(r,w)+S(r,w)\end{aligned}$$

thus (4) becomes

$$(p\gamma-2\nu(\gamma+1))T(r,w)<\gamma\sum_{j=1}^{p}\bar{N}^{\gamma)}\left(r,\frac{1}{w-a_j}\right)+S_\gamma(r,w), \tag{13}$$

By using (13) to $w(z)$, $\{a_j\}$ and $\hat{w}(z)$, $\{a_j\}$, $j=1, 2, \cdots, p_\gamma$ we get (13) and

$$(p_\gamma\gamma-2\nu(\gamma+1))T(r,\hat{w})<\gamma\sum_{j=1}^{p_\gamma}\bar{N}^{\gamma)}\left(r,\frac{1}{\hat{w}-a_j}\right)+S_\gamma(r,\hat{w}). \tag{13$'$}$$

By a argument similar to the proof of theorem 1, we can prove theorem 3.

Similarly we have the following

THEOREM 4. *Let $w(z)$ and $\hat{w}(z)$ be two ν-valued algebroid functions and γ ($\geqq1$) an integer which divides exactly 2ν, if $\bar{E}^{\gamma)}(a_j,w)=\bar{E}^{\gamma)}(a_j,\hat{w})$ for $a_j\in\hat{C}$, $j=1, 2, \cdots, p_{\gamma\lambda}$, $p_{\gamma\lambda}=2\nu+\lambda+\frac{2\nu}{\gamma}$ with $1\leqq\lambda\leqq2\nu-1$, then*

$$\frac{\lambda\gamma}{2\nu(\gamma+1)+\lambda\gamma} \leqq \lim_{\substack{r\to\infty \\ r\in E}} \frac{T(r, w)}{T(r, \hat{w})} \leqq \frac{2\nu(\gamma+1)+\lambda\gamma}{\lambda\gamma}.$$

Especially, if $\gamma=1, 2$ *and* ν, *then* $p_{\gamma\lambda}=4\nu+\lambda$, $3\nu+\lambda$ *and* $2(\nu+1)+\lambda$.

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