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ON MEROMORPHIC FUNCTIONS WITH FEW POLES AND WITH REGIONS FREE OF ZEROS

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1. Introduction. In this note we improve one of the results of Edrei and Fuchs [3]. We shall adopt the terminology, notations and conventions of [3]. We shall write, for instance, [3, Lemma 4] to denote Lemma 4 of [3].

The aim of this investigation is to prove the following

THEOREM. *Suppose that $f(z)$ ($\not\equiv$ const.) is a meromorphic function of lower order $\mu(<+\infty)$, and that $\delta(\infty, f)=1$. Let the s B-regular paths*

$$L_l : z=te^{i\alpha_l(t)} \tag{1.1}$$

$$(t\geqq t_0>0\,;\ l=1, 2, \cdots, s\,;\ \alpha_1(t)<\alpha_2(t)<\cdots<\alpha_s(t)<\alpha_1(t)+2\pi=\alpha_{s+1}(t))$$

divide $|z|\geqq t_0$ into s sectors, each of which has opening $\geqq c>0$.

Let $\delta(>0)$ be fixed and let $\bar{n}_\delta(r)$ denote the number of distinct zeros of $f(z)$ which lie in $t_0\leqq|z|\leqq r$ but outside the s sectors $\mathcal{E}_l(\delta)$ $(l=1, 2, \cdots, s)$ defined by

$$\alpha_l(t)-\delta\leqq\arg z\leqq\alpha_l(t)+\delta, \qquad t_0\leqq|z|=t<+\infty.$$

Assume that for every fixed $\delta(>0)$, we have

$$\bar{n}_\delta(r)=o(T(r, f)) \qquad (r\to\infty). \tag{1.2}$$

Denote by p the number of deficient values of $f(z)$ other than 0 and ∞. Then

$$p\leqq\min\{s-1, 2\mu, \{2\mu(1-c/\pi)+1\}^+\}.$$

This is an improvement of [3, Theorem 3].

2. Proof of Theorem.

2.1. Suppose that the function $f(z)$ satisfies the hypotheses of Theorem and it has $\tau_1, \tau_2, \cdots, \tau_p(\tau_j\neq 0, \tau_j\neq\infty, j=1, 2, \cdots, p)$ among its deficient values.

The paths $L_l(l=1, 2, \cdots, s)$ divide the z-plane into s sectors S_l. Let $J_l(r)$ $(r\geqq t_0)$ be the set of arguments corresponding to the arc of $|z|=r$ in S_l. Since the τ_j are deficient, there is at least one index $l=l(j, r)$ such that for some

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fixed $\kappa>0$

(2.1) $m(r, 1/(f-\tau_j); J_l(r))>\kappa T(r, f) \qquad (r\geqq r_0\geqq t_0, l=l(j, r), j=1, 2, \cdots, p).$

With the equations (1.1) for the L_l, we shall denote by $S_l(\delta)$ $(0\leqq\delta<c/16)$ the sector

$$\alpha_l(t)+\delta<\arg z<\alpha_{l+1}(t)-\delta, \qquad t_0\leqq|z|=t<+\infty;$$

by $J_l(r, \delta)$ the set of arguments of the arc $|z|=r$ in $S_l(\delta)$ and by $I_l(r, \delta)$ the complement of $J_l(r, \delta)_{in}$ $J_l(r, 0)=J_l(r)$.

Let $\{r_m\}_1^\infty$ be a sequence of Pólya peaks of order μ of $T(r, f)$ with associated sequences $\{r'_m\}_1^\infty$, $\{r''_m\}_1^\infty$ $(r'_m<r_m<r''_m, r'_1\geqq r_0)$. (For the basic properties and existence of Pólya peaks the reader is referred to [2, p. 82].) Using [3, Lemma C], we have for $r_m\leqq r\leqq 2r_m$ $(m>m_0)$

$$\begin{aligned} m(r, 1/(f-\tau_j); I_l(r, 2\delta)) &\leqq 22T(2r, 1/(f-\tau_j))4\delta(1+\log^+(1/4\delta)) \\ &<89T(2r, f)\delta(1+\log^+(1/4\delta)) \\ &<90(2r/r_m)^\mu T(r_m, f)\delta(1+\log^+(1/4\delta)) \\ &\leqq 90\cdot 4^\mu T(r, f)\delta(1+\log^+(1/4\delta)) \\ &<(\kappa/2)T(r, f), \end{aligned}$$

provided that $0<\delta<\delta_1=\delta_1(\kappa, \mu)$. Further, we may assume that $\delta_1\leqq\pi/12\mu s$. Combining the above estimate with (2.1), we have

(2.2) $$m(r, 1/(f-\tau_j); J_l(r, 2\delta))>(\kappa/2)T(r, f)$$
$$(r_m\leqq r\leqq 2r_m, m>m_0, 0<\delta<\delta_1, l=l(j, r), j=1, 2\cdots, p).$$

If $\mu<1/2$, the condition $\delta(\infty, f)=1$ implies that $p=0$ (See [4].). In what follows we assume that $\mu\geqq 1/2$. This gives

(2.3) $$\log r=o(T(r, f)) \qquad (r\to\infty).$$

A basic fact of Nevanlinna's theory is that

$$m(r, f'/(f-a))=o(T(r, f))$$

as $r\to\infty$ outside an exceptional set E which has finite measure. It is important to note that E occurs in intervals where $T(r, f)$ grows very rapidly; in particular, E does not depend on the value a, and consideration of the growth lemma from which it arises shows that it may be taken to be disjoint from the intervals $[r_m, \sigma r_m]$ $(m>m_0)$, where $\sigma>1$ is fixed and $m_0=m_0(\sigma)$.

Combining (2.2), (2.3), [3, Lemma B] and the estimate

(2.4) $$m(r, f'/f)+m(r, f'/(f-\tau_j))=o(T(r, f))$$
$$(r_m\leqq r\leqq\sigma r_m, r\to\infty, j=1, 2, \cdots, p),$$

we obtain

$$m(r, f/f'; J_l(r, 2\delta)) > (\kappa/3)T(r, f) \tag{2.5}$$

$$(r_m \leqq r \leqq 2r_m,\ m > m_0,\ 0 < \delta < \delta_1,\ l = l(j, r),\ j = 1, 2, \cdots, p).$$

2.2. We choose now a constant $M(>0)$. For the proof of $p \leqq s-1$ we take $M=1$. For the proof of $p \leqq 2\mu$ we shall obtain a contradiction if we assume $p > 2\mu$, and if we choose M so large that

$$e^M > 16A_2^2 = U, \qquad A_2 = 5e^{4\pi}/\pi, \tag{2.6}$$

and

$$-K/2 + 6\cdot 2^\mu A_2 e^{-(p/2-\mu)M} + 4A_2 e^{-M/2} < -K/4, \tag{2.7}$$

where the positive constant $K = K(\mu, c, s, B, \kappa)$ is defined in (2.33). If $p > 2\mu(1-c/\pi)+1 > 0$, there is a number $c' \in (c/2, c)$ such that

$$p > 2\mu(1-(3c+c')/4\pi)+1. \tag{2.8}$$

In this case we choose M so large that (2.6) and

$$-K/2 + 6\cdot 2^\mu A_2 \exp\{[-(p-1)/2(1-(3c+c')/4\pi)+\mu]M\} + 4A_2 e^{-M/2} < -K/4 \tag{2.9}$$

hold. We shall seek a contradiction from (2.6), (2.8) and (2.9), and deduce $p \leqq 2\mu(1-c/\pi)+1$.

2.3. By [3, Lemma 4]

$$|f'(z)/f(z)| < A\{T(2r, f)\}^A \qquad (|z| = r > r_0 > 4\pi sB/\delta_1,\ z \notin \mathfrak{R}), \tag{2.10}$$

where $\mathfrak{R}$ is a set of discs with sum of radii less than 1. Assume that $r'_m < r < r''_m$. It follows from (2.10) and the definition of Pólya peaks that

$$|f'(z)/f(z)| < A\{(1+o(1))T(r_m, f)(2|z|)^\mu / r_m^\mu\}^A$$

$$< 2^{1+\mu A} A |z|^{\mu A} \{T(r_m, f)/r_m^\mu\}^A \qquad (r'_m \leqq |z| \leqq r''_m;\ m > m_0,\ z \notin \mathfrak{R}).$$

Hence we can find a positive integer h such that

$$|z^{-h} f'(z)/f(z)| < \{T(r_m, f)/r_m^\mu\}^A \qquad (r'_m \leqq |z| \leqq r''_m;\ m > m_0,\ z \notin \mathfrak{R}). \tag{2.11}$$

It follows now from [3, Lemma 2] that there is some $\delta \in (\delta_1/2, \delta_1)$ such that (2.11) holds on the boundaries of $S_l(\delta) \cap \left(\bigcup_{m=1}^{\infty} \{z;\ r'_m/2 < |z| < r''_m/2\} \right)$ $(l = 1, 2, \cdots, s)$. From now on we assume that δ has been chosen in this way and we shall make no further changes in the choice of δ. It is also easily seen that there are two circles $|z| = R'_m (r_m/3e^M < R'_m < r_m/2e^M)$ and $|z| = R''_m (4e^M r_m < R''_m < 6e^M r_m)$ on which (2.11) holds.

Let

$$P_m(z)=\prod_{\nu=1}^{n_m}\{(z-a_\nu)/2R''_m\}$$

be the product, taken over all the poles of $f'(z)/f(z)$ which lie in $t_0\leqq|z|\leqq R''_m$ but outside the sectors $\mathcal{E}_l(\delta)$ $(l=1, 2, \cdots, s)$.

Now, we define a sequence of functions $\{g_k(z)\}_{m_0+1}^{\infty}$. Assume first that there is a subsequence $\{r_{m_k}\}_{k=1}^{\infty}$ of $\{r_m\}_1^{\infty}$ such that $T(r_{m_k}, f)/r_{m_k}^{\mu}\leqq 1$. In this case we simply write $\{r_k\}_1^{\infty}$, $\{R'_k\}_1^{\infty}$, $\{R''_k\}_1^{\infty}$ and $\{P_k(z)\}_1^{\infty}$ instead of $\{r_{m_k}\}_{k=1}^{\infty}$, $\{R'_{m_k}\}_{k=1}^{\infty}$, $\{R''_{m_k}\}_{k=1}^{\infty}$ and $\{P_{m_k}(z)\}_{k=1}^{\infty}$, respectively, and define $g_k(z)$ $(k\geqq 1)$ by

$$g_k(z)=z^{-h}P_k(z)f'(z)/f(z).$$

Assume next that $T(r_m, f)/r_m^{\mu}>1$ for $m>m_0$. In this case we define $g_k(z)$ $(k>m_0)$ by

$$g_k(z)=\{r_k^{\mu}/T(r_k, f)\}^A z^{-h}P_k(z)f'(z)/f(z).$$

The function $g_k(z)$ $(k>m_0)$ is regular in the intersection of $|z|\leqq R''_k$ with every $S_l(\delta)$ $(l=1, 2, \cdots, s)$. In $|z|\leqq R''_k$

$$|P_k(z)|\leqq 1. \tag{2.12}$$

By (2.11), (2.12) and the maximum modulus principle

$$|g_k(z)|<1 \qquad (z\in D_{k,l}), \tag{2.13}$$

where $D_{k,l}(k>m_0, l=1, 2, \cdots, s)$ is defined by

$$R'_k\leqq r\leqq R''_k, \qquad \alpha_l(r)+\delta\leqq\theta\leqq\alpha_{l+1}(r)-\delta.$$

2.4. We first choose $c''\in(1, 2)$ such that $(c'')^{\mu}<4/3$. Next, we select a positive number b such that

$$b<\min\{(c-c')/144\pi Be^{2M+1}, (c''-1)/24e^{M+1}\}. \tag{2.14}$$

By a well-known lemma of H. Cartan

$$|P_k(z)|\geqq(b/2)^{n_k} \tag{2.15}$$

in $|z|\leqq R''_k$ but outside circles the sum of whose diameters is less than $4ebR''_k$. It follows from [3, Lemma 2] and (2.14) that (2.15) holds on the curves $C_{k,l}$ $(k>m_0, l=1, 2, \cdots, s)$ defined by

$$C_{k,l}: z=z(t)=te^{i(\alpha_l(t)+\gamma_{k,l})} \qquad (R'_k\leqq t\leqq R''_k)$$

with $c'/2<\gamma_{k,l}<c/2$. Further, we deduce from (2.14) that (2.15) holds on a circle $|z|=R_k$ with $r_k\leqq R_k\leqq c''r_k$ $(k>m_0)$. By (2.12) and (2.5)

$$m(R_k, 1/g_k; J_l(R_k, 2\delta))>m(R_k, f/f'; J_l(R_k, 2\delta))>(\kappa/3)T(R_k, f) \tag{2.16}$$

$$(k>m_0, l=l(j, R_k), j=1, 2, \cdots, p).$$

2.5. We note first, by repeating the argument following [3, (3.19)], that the image of $D_{k,l}$ ($k>m_0$, $l=1, 2, \cdots, s$) by

$$\zeta=\log z+\text{const.}$$

contains a lens $\varLambda_{k,l}$ whose center line is formed by the vertical segment which is the image of $R_k e^{i\theta}$ ($\theta\in J_l(R_k, \delta)$) and whose boundary is formed by the two circular arcs through the endpoints of this segment making a sufficiently small constant angle β with it. We choose $\beta=1/40B$ and apply [3, Lemma 5] with $H(\zeta)=g_k(z)$, $\varepsilon=\delta\in(\delta_1/2, \delta_1)$, $M^*=(\kappa/3)T(R_k, f)$ (See (2.16).), $\alpha=\{\alpha_{l+1}(R_k)-\alpha_l(R_k)\}/2-\delta>3c/8$. This yields

$$\log|g_k(z)|<-K_1T(R_k, f) \qquad (z\in \mathscr{B}_k(l)), \tag{2.17}$$

where $\mathscr{B}_k(l)$ is given by

$$z=R_k e^{i\theta}, \quad \theta\in J_l(R_k, 2\delta), \quad l=l(j, R_k), \qquad j=1, 2, \cdots, p, \tag{2.18}$$

and where the constant K_1 may be chosen as

$$K_1=(2\kappa/3\pi^3)(\delta_1/4\pi)^{2\pi/\beta}.$$

Next, we apply [3, Lemma E], first to the part of $D_{k,l}$ in $|z|\geqq R_k$ then to the part of $D_{k,l}$ in $|z|\leqq R_k$. In both cases $\mathscr{B}_k(l)$ is the arc (2.18) and $\mathscr{L}$ is a portion of the curve $C_{k,l}$. It is easily verified, with the aid of [3, Lemma 1] that for any point ζ on $C_{k,l}$, with

$$e^{-M}R_k\leqq|\zeta|\leqq e^M R_k, \tag{2.19}$$

we have

$$\rho(\zeta)>|\zeta|/K_2(c, B). \tag{2.20}$$

From now on, we denote by $C'_{k,l}$ the portion of $C_{k,l}$ which satisfies the condition (2.19). By (2.20) and the B-regularity of $C_{k,l}$

$$\int_{w_{k,l}}^{z}|d\zeta|/\rho(\zeta)\leqq BK_2\left|\int_{R_k}^{|z|}dt/t\right|=BK_2|\log(|z|/R_k)|$$
$$(z\in C'_{k,l},\ w_{k,l}=R_k e^{i\theta_{k,l}}\in C'_{k,l}).$$

Therefore, by (2.13), (2.17), [3, Lemma E] and the two-constant theorem

$$\log|g_k(z)|<-(K_1/2\pi)\exp\{-4BK_2|\log(|z|/R_k)|\}T(R_k, f) \tag{2.21}$$
$$(z\in C'_{k,l},\ k>m_0,\ l=l(j, R_k),\ j=1, 2, \cdots, p).$$

2.6. We now deduce from (2.17) and (2.21) similar inequalities with g_k replaced by f'/f. Since $\delta(\infty, f)=1$, $n(r, \infty, f)\leqq(\log 2)^{-1}N(2r, \infty, f)=o(T(2r, f))$ $(r\to\infty)$. It follows from this, (1.2) and the definition of Pólya peaks that

(2.22) $$n_k \leqq n(R_k'', \infty, f)+\bar{n}_\delta(R_k'')=o(T(2R_k'', f))$$
$$=o(T(12e^M r_k, f))=o(T(r_k, f))=o(T(R_k, f)) \qquad (k\to\infty).$$

By the definition of g_k

(2.23) $$\log|f'(z)/f(z)|=\log|g_k(z)|+h\log|z|-\log|P_k(z)|+A\log^+\{T(r_k, f)/r_k^\mu\}.$$

Combining (2.23), (2.17), (2.18), (2.15), (2.3) and (2.22), we have

(2.24) $$\log|f'(z)/f(z)|<-K_1T(R_k, f)+h\log R_k+n_k\log(2/b)+A\log T(R_k, f)$$
$$=-K_1T(R_k, f)+o(T(R_k, f))<-(K_1/2)T(R_k, f)$$
$$(z\in \mathscr{B}_k(l),\ k>m_0,\ l=l(j, R_k),\ j=1, 2, \cdots p).$$

Similarly, using (2.21) instead of (2.17), we obtain

(2.25) $$\log|f'(z)/f(z)|<-(K_1/7)\exp\{-4BK_2|\log(|z|/R_k)|\}T(R_k, f)$$
$$(z\in C'_{k,l},\ k>m_0,\ l=l(j, R_k),\ j=1, 2, \cdots, p).$$

By (2.2) with $r=R_k$, there is a point $z_{k,l}\in\mathscr{B}_k(l)$ ($l=l(j, R_k)$, $j=1, 2, \cdots, p$) such that $|f(z_{k,l})-\tau_j|<\varepsilon$ for any assigned $\varepsilon>0$, provided that $k>m_0$. If z is any other point of $\mathscr{B}_k(l)$, then by (2.24) and (2.3) $|\log f(z)-\log f(z_{k,l})|<2\pi R_k\times\exp\{-K_1/2)T(R_k, f)\}\to 0$ $(k\to\infty)$, and so for any assigned $\varepsilon\in(0, 1/2)$

(2.26) $$|f(z)-\tau_j|\leqq|f(z)-f(z_{k,l})|+|f(z_{k,l})-\tau_j|<2\varepsilon$$
$$(z\in\mathscr{B}_k(l),\ k>m_0,\ l=l(j, R_k),\ j=1, 2, \cdots, p).$$

2.7. Since $r_k\leqq R_k\leqq c''r_k$ $(k>m_0)$, we have for $r_k'\leqq r\leqq r_k''$

(2.27) $$T(r, f)/T(R_k, f)\leqq T(r, f)/T(r_k, f)\leqq(1+o(1))(r/r_k)^\mu$$
$$=(1+o(1))(R_k/r_k)^\mu(r/R_k)^\mu\leqq(1+o(1))(c'')^\mu(r/R_k)^\mu$$
$$<(3/2)(r/R_k)^\mu \qquad (k>m_0).$$

Let $\Lambda(r)=o(T(r, f))$ be a positive function tending to ∞ as $r\to\infty$. Then the following relation holds:

(2.28) $$\sigma_k\equiv\text{meas}\{\theta\,;\log|f(R_ke^{i\theta})|>\Lambda(R_k)\}>\pi/3\mu \qquad (k>m_0).$$

To prove this, we follow Baernstein's procedure in [1, pp. 430-434]. Let $\gamma=1/2\mu$ and define

$$v(z)=T^*(z^\gamma) \qquad (z=re^{i\theta},\ 0<r<\infty,\ 0\leqq\theta\leqq\pi),$$

where $T^*(z)$ is the Baernstein characteristic of f. Using (2.27) instead of [1, (4.8)], we obtain

(2.29) $$v(R_k^{1/\gamma}e^{i\theta})\leqq(3/2)T(R_k, f)[\cos(\pi-\theta)\gamma\mu+o(1)] \qquad (k\to\infty,\ 0<\theta<\pi),$$

where the $o(1)$ term is independent of θ. Clearly

$$T^*(R_k e^{i\sigma_k/2}) \leqq T(R_k, f) = m(R_k, f) + N(R_k, \infty, f) \leqq T^*(R_k e^{i\sigma_k/2}) + \Lambda(R_k),$$

and hence

$$\lim_{k\to\infty} T^*(R_k e^{i\sigma_k/2})/T(R_k, f) = 1. \tag{2.30}$$

Let $N=\{k\,;\sigma_k<\pi\gamma\}$. If N is a finite set then (2.28) holds, so we assume that N is infinite. If $k\in N$, $(R_k e^{i\sigma_k/2})^{1/\gamma}=R_k^{1/\gamma}e^{i\sigma_k/2\gamma}$ belongs to the domain of v. In this case $T^*(R_k e^{i\sigma_k/2})=v(R_k^{1/\gamma}e^{i\sigma_k/2\gamma})$ $(k\in N)$. Combining this, (2.29) and (2.30), we have

$$T(R_k, f) \sim T^*(R_k e^{i\sigma_k/2}) \leqq (3/2)T(R_k, f)[\cos(\pi-\sigma_k/2\gamma)\gamma\mu+o(1)]$$
$$(k\to\infty,\ k\in N),$$

from which we deduce that $\sigma_k>\pi/3\mu$ $(k\in N,\ k>m_0)$. If $k\notin N$, then $\sigma_k\geqq\pi\gamma=\pi/2\mu>\pi/3\mu$. Thus we have reached (2.28).

2.8. By choosing $\varepsilon(>0)$ small enough, we see from (2.26) that the index $l=l(j, R_k)$ $(k>m_0)$ cannot have the same value for different values of $j(=1, 2, \cdots, p)$. This implies $p\leqq s$. Assume that $p=s$. Then from (2.28), (2.26) and (2.18) it follows that

$$\pi/3\mu<\sigma_k\leqq 4\delta s<4\delta_1 s\leqq\pi/3\mu \qquad (k>m_0),$$

which is impossible. This proves $p\leqq s-1$.

2.9. By integrating f'/f along $C_{k,l}$ $(l=l(j, R_k))$ from the point of intersection $w_{k,l}$ of $C'_{k,l}$ with $\mathscr{B}_k(l)$ to the point z, we have, in view of (2.19) and (2.25)

$$|\log f(z)-\log f(w_{k,l})|<B(e^M-1)R_k\exp\{-(K_1/7)e^{-4BK_2M}T(R_k, f)\}$$
$$(z\in C'_{k,l},\ k>m_0,\ l=l(j, R_k),\ j=1, 2, \cdots, p).$$

Hence, by (2.26) and (2.3) we have, for any assigned $(\varepsilon>0)$

$$|f(z)-\tau_j|<\varepsilon \qquad (z\in C'_{k,l},\ k>m_0,\ l=l(j, R_k),\ j=1, 2, \cdots, p).$$

From this and (2.25) it follows that

$$\log|f'(z)|<-(K_1/8)\exp\{-4BK_2|\log(|z|/R_k)|\}T(R_k, f) \tag{2.31}$$
$$(z\in C'_{k,l},\ k>m_0,\ l=l(j, R_k),\ j=1, 2, \cdots, p).$$

We have already seen that for any fixed $k(>m_0)$ the curves $C_{k,l}$ $(l=l(j, R_k)$, $j=1, 2, \cdots, p)$ do not intersect, since they lie in different sectors S_l $(1\leqq l\leqq s)$. Therefore they divide the annulus (2.19) into p different domains.

Let S_k^* be a typical one of these domains and let $t\Theta(t)$ be the length of the arc of $|z|=t$ which lies in S_k^*, and let

$$Q_k(z)=\prod_{\nu=1}^{n'_k}\{(z-b_\nu)/2R''_k\}$$

be the product, taken over all the poles of $f'(z)$ which lie in $|z|\leqq R''_k$. By Cartan's lemma

$$|Q_k(z)|\geqq(b/2)^{n'_k} \tag{2.32}$$

in $|z|\leqq R''_k$ but outside circles the sum of whose diameters is less than $4ebR''_k$. Without loss of generality, we may assume that (2.32) as well as (2.15) holds on the circle $|z|=R_k$ $(k>m_0)$.

2.10. Let $A_1=e^{9\pi}$ and let A_2 and $U(>A_1)$ be the quantities which appear in (2.6). Denote by $\Gamma_1(k, *)$ the part of the boundary of S_k^* in $R_k/U<|z|<UR_k$, by $\Gamma_2(k, *)$ the boundary arc of S_k^* on $|z|=e^{+M}R_k$, by $\Gamma_3(k, *)$ the boundary arc of S_k^* on $|z|=e^{-M}R_k$ and by $\Gamma_4(k, *)$ the part of the boundary of S_k^* which does not belong to $\Gamma_1\cup\Gamma_2\cup\Gamma_3$.

On Γ_1 and Γ_4 (2.31) holds, so that

$$\log|f'(z)Q_k(z)|<-KT(R_k, f) \tag{2.33}$$

$$(z\in\Gamma_1(k, *),\ K=(K_1/8))\exp\{-4BK_2\log(16A_2^2)\}=K(\mu, c, s, B, \kappa)),$$

$$\log|f'(z)Q_k(z)|<0 \qquad (z\in\Gamma_4(k, *)). \tag{2.34}$$

Since $f'(z)Q_k(z)$ is regular in $|z|\leqq R''_k$, the Poisson-Jensen formula gives for $0<r<R''_k/2$

$$\log M(r, f'Q_k)\leqq 3m(2r, f'Q_k)\leqq 3m(2r, f')$$

$$\leqq 3m(2r, f'/f)+3m(2r, f).$$

Therefore by (2.4)

$$\log|f'(z)Q_k(z)|\leqq\log M(e^{-M}R_k, f'Q_k)\leqq\log M(R_k/2, f'Q_k) \tag{2.35}$$

$$\leqq o(T(R_k, f))+3T(R_k, f)<4T(R_k, f) \qquad (z\in\Gamma_3(k, *))$$

and

$$\log|f'(z)Q_k(z)|\leqq o(T(2e^MR_k, f))+3T(2e^MR_k, f) \tag{2.36}$$

$$<4T(2e^MR_k, f)<6(2e^M)^\mu T(R_k, f) \qquad (z\in\Gamma_2(k, *)).$$

Next, we denote by $\omega_q(z, k, *)$ the harmonic measure of $\Gamma_q(k, *)$ with respect to S_k^* $(q=1, 2, 3, 4)$. Then by [3, Lemma 6],

$$\sum_{q=2}^{4}\omega_q(R_ke^{i\theta}, k, *)<A_2\exp\left\{-\pi\int_{R_k/U}^{R_k}dt/t\Theta(t)\right\}+A_2\exp\left\{-\pi\int_{R_k}^{UR_k}dt/t\Theta(t)\right\}$$

$$\leqq 2A_2U^{-1/2}=1/2 \qquad (R_ke^{i\theta}\in S_k^*)$$

since $\Theta(t)\leqq 2\pi$. Hence

(2.37) $$\omega_1(R_k e^{i\theta},\ k,\ *) > 1/2.$$

Similarly,

(2.38) $$\omega_3(R_k e^{i\theta},\ k,\ *) < A_2 e^{-M/2},$$

and

(2.39) $$\omega_2(R_k e^{i\theta},\ k,\ *) < A_2 \exp\left\{-\pi\int_{R_k}^{eMR_k} dt/t\Theta(t)\right\}.$$

We show that for at least one of the sectors, say for S_k^{**},

(2.40) $$\pi\int_{R_k}^{eMR_k} dt/t\Theta(t) \geqq pM/2$$

and, say for S_k^{***},

(2.41) $$\pi\int_{R_k}^{eMR_k} dt/t\Theta(t) \geqq (p-1)M/2(1-(3c+c')/4\pi).$$

By Schwarz's inequality and the fact that $\sum_{j=1}^{p}\Theta_j(t)=2\pi$ (where the index j refers to the p different sectors S_k^*)

$$p^2=\left\{\sum_{j=1}^{p}(\Theta_j(t))^{1/2}(\Theta_j(t))^{-1/2}\right\}^2 \leqq 2\pi\sum_{j=1}^{p}1/\Theta_j(t).$$

Hence

$$p^2M/2=(p^2/2)\int_{R_k}^{eMR_k} dt/t \leqq \sum_{j=1}^{p}\pi\int_{R_k}^{eMR_k} dt/t\Theta_j(t).$$

This proves (2.40). In the same way, using the facts that $p\leqq s-1$ and $c'/2<\gamma_{k,l}<c/2$ $(k>m_0,\ l=1,2,\cdots,s)$, we have

$$(p-1)^2 \leqq (2\pi-(3c+c')/2)\sum_{j=1}^{p-1}1/\Theta_j(t).$$

Hence

$$(p-1)^2M/2=((p-1)^2/2)\int_{R_k}^{eMR_k} dt/t \leqq \sum_{j=1}^{p-1}(\pi-(3c+c')/4)\int_{R_k}^{eMR_k} dt/t\Theta_j(t).$$

This proves (2.41). Combining (2.39) with (2.40) or (2.41), we have

(2.42) $$\omega_2(R_k e^{i\theta},\ k,\ **) < A_2 e^{-pM/2} \qquad (R_k e^{i\theta}\in S_k^{**}),$$

and

(2.43) $$\omega_2(R_k e^{i\theta},\ k,\ ***) < A_2 \exp\{-(p-1)M/2(1-(3c+c')/4\pi)\} \qquad (R_k e^{i\theta}\in S_k^{***}).$$

2.11. Now, a bounded function, harmonic in S_k^*, with the following boundary values

$$-KT(R_k, f) \text{ on } \Gamma_1(k, *), \qquad 6\cdot 2^{\mu}e^{\mu M}T(R_k, f) \text{ on } \Gamma_2(k, *),$$
$$4T(R_k, f) \text{ on } \Gamma_3(k, *), \qquad 0 \text{ on } \Gamma_4(k, *)$$

dominates the subharmonic function $\log|f'(z)Q_k(z)|$ at each point of S_k^*. This follows from (2.33)-(2.36). Hence, in particular,

$$(2.44) \quad \log|f'(R_ke^{i\theta})Q_k(R_ke^{i\theta})|$$
$$<-\omega_1(R_ke^{i\theta}, k, *)KT(R_k, f)+\omega_2(R_ke^{i\theta}, k, *)6\cdot 2^{\mu}e^{\mu M}T(R_k, f)$$
$$+\omega_3(R_ke^{i\theta}, k, *)4T(R_k, f) \qquad (R_ke^{i\theta}\in S_k^*, k>m_0).$$

From (2.44), (2.37), (2.42), (2.38), (2.7) and (2.32) we deduce that

$$(2.45) \quad \log|f'(R_ke^{i\theta})|$$
$$<\{-K/2+6\cdot 2^{\mu}A_2e^{-(p/2-\mu)M}+4A_2e^{-M/2}\}T(R_k, f)-\log|Q_k(R_ke^{i\theta})|$$
$$<-(K/4)T(R_k, f)+n_k'\log(2/b)<-(K/5)T(R_k, f)$$
$$(k>m_0, R_ke^{i\theta}\in S_k^{**}),$$

where we used the estimate

$$n_k'=n(R_k'', \infty, f')\leqq 2n(R_k'', \infty, f)=o(T(2R_k'', f))=o(T(12e^Mr_k, f))$$
$$=o(T(r_k, f))=o(T(R_k, f)) \qquad (k\to\infty).$$

Similarly, from (2.44), (2.37), (2.43), (2.38), (2.9) and (2.32) it follows that

$$(2.46) \quad \log|f'(R_ke^{i\theta})|<\{-(K/2)+6\cdot 2^{\mu}A_2\times$$
$$\exp[\{-(p-1)/2(1-(3c+c')/4\pi)+\mu\}M]4A_2e^{-M/2}\}T(R_k, f)$$
$$-\log|Q_k(R_ke^{i\theta})|<-(K/4)T(R_k, f)+n_k'\log(2/b)<-(K/5)T(R_k, f)$$
$$(k>m_0, R_ke^{i\theta}\in S_k^{***}).$$

2.12. Let $\zeta_{k,1}$ and $\zeta_{k,2}$ be the endpoints of the arc of $|z|=R_k$ in S_k^{**} (S_k^{***}). Then we easily see from (2.26) that

$$|f(\zeta_{k,1})-f(\zeta_{k,2})|>\min_{\substack{i\neq j\\ i,j=1,2\cdots,p}}|\tau_i-\tau_j|/2 \qquad (k>m_0).$$

On the other hand, by integrating (2.45) ((2.46))

$$|f(\zeta_{k,1})-f(\zeta_{k,2})|\leqq 2\pi R_k\exp\{-(K/5)T(R_k, f)\}$$

and, in view of (2.3), the right— hand side of this inequality tends to 0 as $k\to\infty$. This contradicton proves $p\leqq 2\mu$ ($p\leqq 2\mu(1-c/\pi)+1$). This completes the proof of Theorem.

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