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# A GENERALIZATION OF A THEOREM OF LANDAU

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**1.** $C^n$ will denote the complex vector space with the ordinary norm $\|z\|^2=\sum_{j=1}^{n}|z_j|^2$ and $B_n$ will denote the unit ball $\{z\in C^n : \|z\|<1\}$.

Let $F=(f_1, \cdots, f_m)$ be a holomorphic mapping from $B_n$ into $B_m$. Let $A_F(z)$ denote the Jacobian matrix of $F$ at $z$:

$$A_F(z)=(a_{ij}), \qquad a_{ij}=\frac{\partial f_i}{\partial z_j}(z) \qquad (1\leqq i\leqq m,\ 1\leqq j\leqq n),$$

and let $\lambda_F(z)$ be the nonnegative square root of the smallest eigenvalue of $A_F(z)^*A_F(z)$.

Let $\lambda$ be a real number with $0<\lambda<1$. Let $\mathcal{F}(n, m, \lambda)$ denote the class of holomorphic mappings $F$ from $B_n$ into $B_m$ satisfying: $F(0)=0$, $\lambda_F(0)=\lambda$. In the case when $n=m$ we write $\mathcal{F}(n, \lambda)$ instead of $\mathcal{F}(n, n, \lambda)$. For each $F\in\mathcal{F}(n, \lambda)$, we introduce

$$r(F)=\sup\{r>0 : \text{there exists a domain } \Omega,\ 0\in\Omega\subset B_n, \text{ such that } F \text{ maps } \Omega \text{ univalently onto } rB_n\},$$

where $rB_n=\{rz : z\in B_n\}$, and let

$$L(n, \lambda)=\inf\{r(F) : F\in\mathcal{F}(n, \lambda)\}.$$

In one variable, the classical theorem of Landau [3] states that $L(1, \lambda)\geqq c\lambda^2$, where $c$ is an absolute constant. It is known that

$$L(1, \lambda)=\left(\frac{\lambda}{1+\sqrt{1-\lambda^2}}\right)^2$$

(see [2], p. 38). Hahn [1] proved that $L(n, \lambda)\geqq\sqrt{3}\lambda^2/18$ for $n\geqq 1$. In this note we prove that

$$L(n, \lambda)=\left(\frac{\lambda}{1+\sqrt{1-\lambda^2}}\right)^2 \qquad (n\geqq 1).$$

In our proof we follow the idea of Heins [2].

**2.** Firstly we prove the following:

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LEMMA. *Let* $n\leqq m$. *If* $F\in\mathcal{F}(n, m, \lambda)$, *then*

$$\|F(z)\|\geqq\frac{\|z\|(\lambda-\|z\|)}{1-\lambda\|z\|} \qquad (z\in B_n).$$

*Proof*. Let $F\in\mathcal{F}(n, m, \lambda)$. Since there are unitary matrices $U$ and $V$ such that

$$UA_F(0)V=\begin{bmatrix}\Lambda\\0\end{bmatrix}, \qquad \Lambda=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}, \qquad \lambda_j\geqq 0\ (1\leqq j\leqq n),$$

we may assume that

$$A_F(0)=\begin{bmatrix}\Lambda\\0\end{bmatrix},$$

where $\lambda_1^2, \cdots, \lambda_n^2$ are the eigenvalues of $A_F(0)^*A_F(0)$. Furthermore we may assume that $\lambda_1=\cdots=\lambda_n=\lambda$ (we may consider $F=(\lambda\lambda_1^{-1}f_1, \cdots, \lambda\lambda_n^{-1}f_n, f_{n+1}, \cdots, f_m)$ instead of $F=(f_1, \cdots, f_m)$).

Let $z\in B_n-\{0\}$ and consider the holomorphic mapping $G=(g_1, \cdots, g_m)$ from $\{\zeta\in\boldsymbol{C}: |\zeta|<1\}$ into $B_m$ defined by

$$G(\zeta)=F(\zeta t), \qquad t=\frac{z}{\|z\|}.$$

Since $G(0)=0$, there are functions $h_1, \cdots, h_m$ holomorphic in $|\zeta|<1$ such that

$$g_j(\zeta)=\zeta h_j(\zeta) \qquad (|\zeta|<1)$$

and

$$h_j(0)=c_j, \quad c_j=\lambda t_j\ (1\leqq j\leqq n), \quad c_j=0\ (n+1\leqq j\leqq m)$$

where $t=(t_1, \cdots, t_n)$. Set $H=(h_1, \cdots, h_m)$: then $G(\zeta)=\zeta H(\zeta)$. Since $\|H\|^2$ is subharmonic in $|\zeta|<1$ and since

$$\limsup_{|\zeta|\to 1}\|H(\zeta)\|^2=\limsup_{|\zeta|\to 1}\frac{\|G(\zeta)\|^2}{|\zeta|^2}\leqq 1,$$

we have

$$\|H(\zeta)\|\leqq 1 \qquad (|\zeta|<1).$$

Set $c=(c_1, \cdots, c_m)$: then $\|c\|=\lambda\|t\|=\lambda$. Let $\Phi_c$ be an automorphism of $B_m$ with $\Phi_c(0)=c$ and set $\Psi=\Phi_c^{-1}\circ H$. Since $\Psi$ is a holomorphic mapping from $\{\zeta\in\boldsymbol{C}: |\zeta|<1\}$ into $B_m$ with $\Psi(0)=0$, we have also that $\Psi(\zeta)=\zeta\Psi_1(\zeta)$ where $\|\Psi_1(\zeta)\|\leqq 1$, hence

$$\|\Psi(\zeta)\|\leqq|\zeta| \qquad (|\zeta|<1).$$

Now using the equality

$$\|\Phi_c(w)\|^2=1-\frac{(1-\|c\|^2)(1-\|w\|^2)}{|1-\langle w, c\rangle|^2} \qquad (w\in B_m)$$

(see [4], p. 26), we have, for $|\zeta|<\lambda$,

$$\|G(\zeta)\|^2=|\zeta|^2\left[1-\frac{(1-\|c\|^2)(1-\|\Psi(\zeta)\|^2)}{|1-\langle\Psi(\zeta),\, c\rangle|^2}\right]$$

$$\geqq|\zeta|^2\left[1-\frac{(1-\lambda^2)(1-|\zeta|^2)}{(1-\lambda|\zeta|)^2}\right]=\frac{|\zeta|^2(\lambda-|\zeta|)^2}{(1-\lambda|\zeta|)^2}.$$

Thus we obtain

$$\|F(z)\|=\|G(\|z\|)\|\geqq\frac{\|z\|(\lambda-\|z\|)}{1-\lambda\|z\|}.$$

**3.** Let

$$\tau(r)=\frac{r(\lambda-r)}{1-\lambda r}.$$

Then $\tau$ is strictly increasing in $0<r<\rho_\lambda$, and $\tau(\rho_\lambda)=\rho_\lambda^2$ is the maximum value of $\tau$ in $0<r<1$, where

$$\rho_\lambda=\frac{\lambda}{1+\sqrt{1-\lambda^2}}.$$

Now we prove our main theorem.

THEOREM A. *Let $F\in\mathcal{F}(n, \lambda)$ and $0<r\leqq\rho_\lambda$. Then there exists a domain $\Omega$ satisfying:*

(i) $0\in\Omega\subset rB_n$,

(ii) *$\Omega$ is mapped by $F$ univalently onto $\tau(r)B_n$.*

*Proof*. Let $\Omega$ be the component of $F^{-1}(\tau(r)B_n)$ containing the origin 0. Since $r<\lambda$, it follows from the lemma that $\Omega\subset rB_n$ and $F|\Omega$, the restriction of $F$ to $\Omega$, is a proper mapping from $\Omega$ onto $\tau(r)B_n$.

Let $\sharp(w)$ denote the number of points in the set $(F|\Omega)^{-1}(w)$. Then there is an integer $k$ such that $\sharp(w)\leqq k$ for $w\in\tau(r)B_n$ and the set $\{w\in\tau(r)B_n : \sharp(w)=k\}$ is dense in $\tau(r)B_n$ (see [4], Theorem 15.1.9). Since $J_F(0)\neq 0$, there exist neighborhoods $\Delta$ and $\Delta'$ of the origin 0 such that $F$ maps $\Delta$ univalently onto $\Delta'$. Take a small $t>0$ so that $tB_n\subset\Delta$ and $\tau(t)B_n\subset\Delta'$. Then, for each $w\in\tau(t)B_n$, $F^{-1}(w)$ has precisely one point in $\Delta$. On the other hand, if $z\in\Omega\backslash\Delta$, then $t\leqq\|z\|<r\leqq\rho_\lambda$ and hence the lemma shows that $\|F(z)\|\geqq\tau(\|z\|)\geqq\tau(t)$. Thus $\sharp(w)=1$ for $w\in\tau(t)B_n$ and so $k=1$. Consequently the theorem follows.

COROLLARY B.

$$L(n, \lambda)=\left(\frac{\lambda}{1+\sqrt{1-\lambda^2}}\right)^2.$$

*Proof*. It follows from Theorem A that $L(n, \lambda)\geqq\tau(\rho_\lambda)=\rho_\lambda^2$. Consider the mapping

$$F(z)=(f(z_1), z_2, \cdots, z_n), \qquad f(\zeta)=\frac{\zeta(\lambda-\zeta)}{1-\lambda\zeta}.$$

Then, since $|f(\zeta)|<|\zeta|$ for $|\zeta|<1$, $F\in\mathcal{F}(n,\lambda)$. Set $w^*=(\rho_\lambda^2, 0, \cdots, 0)$. The set $F^{-1}(w^*)$ consists of the point $z^*=(\rho_\lambda, 0, \cdots, 0)$ alone and $J_F(z^*)=0$. Hence $L(n,\lambda)\leqq\rho_\lambda^2$. Thus $L(n,\lambda)=\rho_\lambda^2$.

COROLLARY C. *Let $F\in\mathcal{F}(n,m,\lambda)$ $(n<m)$ and $0<r\leqq\rho_\lambda$. Then there exists a domain $\Omega$ satisfying:*

(i) $0\in\Omega\subset rB_n$,

(ii) *$F|\Omega$, the restriction of $F$ to $\Omega$, is a univalent proper mapping from $\Omega$ into $\tau(r)B_m$.*

*Proof.* Let $F=(f_1, \cdots, f_m)$. We may assume that

$$A_F(0)=\begin{bmatrix}\Lambda\\ 0\end{bmatrix},\qquad \Lambda=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}\qquad \lambda_j\geqq\lambda.$$

We consider the mapping $F^*=(f_1, \cdots, f_n)$. Then $F^*\in\mathcal{F}(n,\lambda)$. Hence Theorem A shows that there exists a domain $\Omega^*$ satisfying:

(1) $0\in\Omega^*\subset rB_n$,

(2) $\Omega^*$ is mapped by $F^*$ univalently onto $\tau(r)B_n$.

Now $F$ is univalent in $\Omega^*$. Let $\Omega$ be the component of $F^{-1}(\tau(r)B_m)$ containing the origin $O$. Since $\|F(z)\|\geqq\|F^*(z)\|=\tau(r)$ for $z\in\partial\Omega^*$, we conclude that $\Omega\subset\Omega^*$. The corollary follows.

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