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# THREE RESULTS IN THE VALUE-DISTRIBUTION THEORY OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS[1]

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**Abstract**

For three different classes of equations of the form $f''+A(z)f=0$, where $A(z)$ is entire, we develop new information about the distribution of zeros of all solutions $f\not\equiv 0$, and thereby obtain a complete value-distribution theory for all solutions.

**1. Introduction:** In [1], we investigated the value-distribution of solutions $f\not\equiv 0$ of an equation of the form,

$$f''+A(z)f=0, \tag{1}$$

where $A(z)$ is an arbitrary entire function. Roughly speaking, it was shown that if one possesses sufficient information on the counting functions for the zeros of all solutions of (1), then for any solution $f\not\equiv 0$ of (1), one can determine all polynomials $P(z, u_0, \cdots, u_n)$, having meromorphic coefficients of slower growth than $f$, for which the function,

$$h(z)=P(z, f(z), f'(z), \cdots, f^{(n)}(z)), \tag{2}$$

is either identically zero, or has the property that the counting function $\overline{N}(r, 1/h)$ for its distinct zeros is of slower growth than $f$. (Of course, in view of (1), the function $h$ in (2) will also be given by a first-order differential polynomial $P^*(z, f(z), f'(z))$, and so attention can be restricted to polynomials $P(z, u, v)$ in two indeterminates.) It was shown in [1] that the existence and form of these "special" polynomials for a solution of a given equation (1), depends heavily on whether (1) possesses none, one, or two linearly independent solutions $f\not\equiv 0$ having the property that $\overline{N}(r, 1/f)$ grows slower than $f$. (Examples illustrating each of the three possibilities can be found in [1].)

To take a concrete situation, if $A(z)$ is a transcendental entire function of finite order of growth, and if $f\not\equiv 0$ is any solution of (1), then the result in [1; Corollary 3, p. 510] will determine all the polynomials $P(z, u, v)$ of positive total

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degree in the indeterminates $u$ and $v$, having arbitrary meromorphic coefficients of finite order, for which the function,

$$h(z)=P(z, f(z), f'(z)), \tag{3}$$

is either identically zero, or has the property that the exponent of convergence (denoted $\bar{\lambda}(h)$) of its sequence of distinct zeros is finite. To apply this result, we must know whether (1) possesses none, one, or two linearly independent solutions $f\not\equiv 0$ for which $\bar{\lambda}(f)<\infty$. Some general results already exist to help decide this matter in certain cases where the order of $A(z)$ is small or is a positive integer. For example, in [2] it was shown that if the order of $A(z)$ is less than 1/2, then for any two linearly independent solutions $f_1, f_2$ of (1), we have,

$$\max\{\bar{\lambda}(f_1), \bar{\lambda}(f_2)\}=\infty. \tag{4}$$

(Recent preprints by L.-C. Shen and by J. Rossi indicate that the conclusion (4) also holds when the order of $A(z)$ equals 1/2.)

As a second example, it was shown in [4] that the same conclusion (4) holds whenever the coefficient function $A(z)$ has the form,

$$A(z)=\sum_{j=1}^{n} B_j(z)e^{Q_j(z)}+Q(z), \tag{5}$$

where (i) $Q_1, \cdots, Q_n$ are nonconstant polynomials whose respective degrees $d_1, \cdots, d_n$ satisfy $\deg(Q_i-Q_j)=\max\{d_i, d_j\}$ for $i\neq j$; (ii) each $B_j$ is an entire function, not identically zero, of order less that $d_j$; (iii) either $Q\equiv 0$ or $Q$ is a polynomial of degree less than $2d-2$, where $d=\max\{d_1, \cdots, d_n\}$. (Of course, the order of $A(z)$ is the positive integer $d$.)

In our first result of the present paper, we consider a class of equations (1) where the order of $A(z)$ can be any nonnegative real number, and we show that if the equation possesses a solution $f_1\not\equiv 0$ satisfying $\bar{\lambda}(f_1)<\infty$, and which is of a certain form, then for any solution $f_2$ which is linearly independent with $f_1$, we must have $\bar{\lambda}(f_2)=\infty$. (Thus for this class of equations, sufficient information now exists to apply the result in [1; Corollary 3, Parts (b), (d)], to determine for any solution $f\not\equiv 0$, the class of all $P(z, u, v)$ having meromorphic coefficients of finite order, for which $\bar{\lambda}(h)<\infty$ where $h$ is given in (3).) We will prove the following theorem in §3:

THEOREM 1. *Let $g(z)$ be an entire function of finite order for which there exist real numbers $a$ and $b$ such that $g(z)$ is positive on both $(a, +\infty)$ and $(-\infty, b)$, and assume that for every $\alpha>0$, we have*

(A) *$g(r)/r^{\alpha}\to+\infty$ as $r\to+\infty$ through real values, and*

(B) *$g(s)/|s|^{\alpha}\to+\infty$ as $s\to-\infty$ through real values.*

*Let $G(z)$ be an entire function of finite order, which is real for real $z$, and which has no zeros on an infinite strip $|\operatorname{Im} z|<\delta$, for some $\delta>0$. Then, if $f=Ge^{g}$ satisfies an equation $f''+Af=0$, where $A$ is entire, we have $\bar{\lambda}(f_1)=\infty$ for any* *solution $f_1 \not\equiv 0$ of this equation which is not a constant multiple of $f$.*

We make four brief remarks concerning this theorem. First, there is no lack of examples to which this theorem will apply, since any function of the form $e^g$, where $g$ is entire, satisfies an equation of the form (1), and, in addition, if $g$ is an even, transcendental, entire function of finite order, whose power series expansion around $z=0$ has all nonnegative coefficients, then $g$ will also satisfy the conditions (A) and (B) in the theorem. Second, it is easy to see (§4 below) that the class of equations treated in Theorem 1, contains coefficient functions of any preassigned finite order. Third, it is tempting to try to strengthen Theorem 1 by possibly deleting one of the conditions (A) or (B). However, for $g(z)=e^z-(z/2)$ (which satisfies condition (A), but not (B)), it is easy to see that $f=e^g$ satisfies the same equation (1) as $f_1=\exp(-(e^z+(z/2)))$, namely where $A(z)=-(e^{2z}+(1/4))$, and it is clear that $f$ and $f_1$ are linearly independent, and $\bar{\lambda}(f_1)=0$. Finally, we remark that there are cases of coefficient functions of the form (5), where the condition (iii) on $Q$ mentioned earlier is violated (so that the result of [4] is not applicable), but which can be treated by Theorem 1. An example is furnished by the function $f(z)=\exp(e^z+e^{-z})$ which satisfies (1) where

$$A(z)=-e^{2z}-e^z-e^{-z}-e^{-2z}+2. \tag{6}$$

(Here $Q\equiv 2$, and is not of degree less than $2d-2$ since $d=1$.). However, since $g(z)=e^z+e^{-z}$ satisfies (A) and (B) of Theorem 1, we can conclude that $\bar{\lambda}(f_1)=\infty$ for every solution $f_1 \not\equiv 0$ which is not a constant multiple of $f$.

Our second result concerns the value-distribution theory for the solutions of a class of equations (1) where $A(z)$ is a periodic entire function of the form $B(e^{\alpha z})$, where $B(\zeta)$ is a rational function, and where $\alpha$ is a nonzero constant. It was proved in [3; Theorem 3], that if the rational function $B(\zeta)$ has poles of odd order at both $\zeta=0$ and $\zeta=\infty$, then for any solution $f \not\equiv 0$ of (1), we have $\bar{\lambda}(f)=\infty$. (This class of equations contains the Mathieu equation.) This result provides sufficient information to apply [1; Corollary 3(a)] to conclude that for any solution $f \not\equiv 0$, and any polynomial $P(z, u, v)$ in $u$ and $v$, of positive total degree, having meromorphic coefficients of finite order, the function $h$ in (3) satisfies the conditions $h \not\equiv 0$, $\bar{\lambda}(h)=\infty$. However, as indicated in [1; Theorem 2], the stronger the information we possess concerning the zeros of the solutions of an equation (1), the stronger will be the value-distribution theory for the solutions that we obtain from [1; Theorem 2]. In our second result, we make two substantial improvements in [3; Theorem 3], and thereby obtain a stronger value-distribution theory for a broader class of equations. First, we prove that the conclusion $\bar{\lambda}(f)=\infty$ for every solution $f \not\equiv 0$, can be replaced by the stronger conclusion,

$$\log^+ \bar{N}(r, 1/f) \neq o(r) \quad \text{as} \quad r \to +\infty. \tag{7}$$

For our second improvement, we show that instead of requiring both of the poles of $B(\zeta)$ at $\zeta=0$ and $\zeta=\infty$ to be of odd order, our stronger conclusion (7) will hold for all solutions $f\not\equiv 0$ when at least one of these poles is of odd order. (The example, $f(z)=\exp(e^{z}+e^{-z})$ which satisfies (1) where $A(z)$ is given by (6), shows that when the poles of $B(\zeta)$ at $\zeta=0, \infty$ are both of even order, the conclusion (7) can fail for some solution $f\not\equiv 0$ of (1). However, there are examples (see §9) where the poles of $B(\zeta)$ are both of even order, and (7) does hold for all solutions $f\not\equiv 0$ of (1). In addition, in the case where $B(\zeta)$ has a pole at only one of the points $\zeta=0, \infty$, we have seen earlier that the conclusion (7) can fail to hold for some solution $f\not\equiv 0$, since $f(z)=\exp(e^{z}-(z/2))$ satisfies (1) where $A(z)=-(e^{2z}+(1/4))$. However, in §9 we show that there are examples of such equations where (7) holds for all solutions $f\not\equiv 0$.) We will prove the following theorem in §8:

THEOREM 2. *Let $B(\zeta)$ be a rational function which is analytic on $0<|\zeta|<\infty$, and which has poles at both $\zeta=0$ and $\zeta=\infty$, and assume that at least one of these poles is of odd order. Let $\alpha$ be a nonzero constant, and set $A(z)=B(e^{\alpha z})$. Then*

(A) *For any solution $f\not\equiv 0$ of $f''+A(z)f=0$, the conclusion* (7) *holds.*

(B) *Let $\Delta(r)$ be any unbounded increasing function on $(1, +\infty)$ with the property that as $r\to+\infty$ outside a possible exceptional set of finite measure, we have $r=0(\Delta(r))$ and $\log^{+}\Delta(r)=o(r)$. Let $P(z, u, v)$ be any polynomial in $u$ and $v$, of positive total degree, whose coefficients are any meromorphic functions $a(z)$ satisfying the condition that $T(r, a)=0(\Delta(r))$ as $r\to+\infty$ outside a possible exceptional set of finite measure. Then, if $f\not\equiv 0$ is any solution of $f''+A(z)f=0$, the function $h(z)=P(z, f(z), f'(z))$ cannot be identically zero, and $h(z)$ cannot have the property that $\bar{N}(r, 1/h)=0(\Delta(r))$ as $r\to+\infty$ outside a possible exceptional set of finite measure.*

We remark that convenient choices for $\Delta(r)$ in Part (B) such as $\Delta(r)=\exp(r(\log r)^{-1})$ or $\Delta(r)=\exp(r^{\alpha})$ with $0<\alpha<1$, allow us to examine polynomials $P(z, u, v)$ having coefficients of infinite order. In addition, we point out here that the conclusion (7) of Part (A) cannot be greatly improved, since it follows easily from [1; Lemma 4.1(a), p. 519] that for every solution $f\not\equiv 0$, we have $\log^{+}N(r, 1/f)=0(r)$ as $r\to+\infty$. (We remark here that various classes of linear differential equations with periodic coefficients have been treated by Frei [5], Ozawa [9], and Wittich [13].)

Our final result is also an improvement of a previous result, namely the value-distribution theory given in [1; Theorem 2] for the solutions of (1) in the case when $A(z)$ is a nonconstant polynomial of degree $n$. In this case, the solutions $f\not\equiv 0$ of (1) are of order $(n+2)/2$ (see [11; p. 106] or [12; p. 281]), and the result in [1; Theorem 2] determines the polynomials $P(z, u, v)$ in $u$ and $v$, of positive total degree, having meromorphic coefficients of order less than $(n+2)/2$, for which the function $h(z)$ given by (3) is either identically zero, or has the property that $\bar{\lambda}(h)<(n+2)/2$. However, one can permit more general coefficients for $P(z, u, v)$, namely all meromorphic functions $a(z)$ having the following property:

$$T(r,\ a)=o(r^{(n+2)/2}) \quad \text{as} \quad r\rightarrow+\infty. \tag{8}$$

In our final theorem, we determine all such $P(z,\ u,\ v)$ for which either $h(z)\equiv 0$ or

$$\bar{N}(r,\ 1/h)=o(r^{(n+2)/2}) \quad \text{as} \quad r\rightarrow+\infty. \tag{9}$$

The proof (which is given in §10) consists simply in showing that the hypothesis of [1; Theorem 4] is satisfied for the differential field consisting of all meromorphic functions satisfying (8). This follows very easily from classical results of Hille, Nevanlinna, and Wittich (see [12; pp. 282-283]), and also from a new non-asymptotic approach due to R. Kaufman [7; Theorem 2]. Before stating the theorem, we remark that there is a simple algorithm [1; §7(a)] to determine for any equation (1) where $A(z)$ is a polynomial, whether (1) possesses a solution having only finitely many zeros, and to determine the solution if it exists.

THEOREM 3. *Let $A(z)$ be a polynomial of degree $n\geqq 1$, and let $H_n$ denote the field of all meromorphic functions $a(z)$ satisfying* (8). *Let $P(z,\ u,\ v)$ be any polynomial in $u$ and $v$, of positive total degree in $u$ and $v$, having coefficients in $H_n$. Then:*

(A) *If all solutions $f\not\equiv 0$ of* (1) *have infinitely many zeros, then for any solution $f\not\equiv 0$ of* (1), *the function $h(z)$ given by* (3) *is not identically zero, and $h$ cannot satisfy* (9).

(B) *Assume that* (1) *possesses a solution $f_0\not\equiv 0$ having only finitely many zeros, and set $R_0=-f_0'/f_0$. Then:*

(a) *For any solution $f\not\equiv 0$ of* (1) *which is not a constant multiple of $f_0$, the function $h(z)$ given by* (3) *is not identically zero. In addition,* (9) *holds if and only if $P(z,\ u,\ v)$ has the special form*

$$P(z,\ u,\ v)=G(z)(v+R_0(z)u)^m, \tag{10}$$

*for some positive integer $m$, and some function $G(z)\not\equiv 0$ belonging to $H_n$.*

(b) *For any solution $f\not\equiv 0$ of* (1) *which is a constant multiple of $f_0$, the function $h(z)$ given by* (3) *is identically zero if and only if $P(z,\ u,\ v)$ has the form,*

$$P(z,\ u,\ v)=Q(z,\ u,\ v)(v+R_0(z)u), \tag{11}$$

*for some polynomial $Q(z,\ u,\ v)$ in $u$ and $v$ with coefficients belonging to $H_n$ (and some coefficient not identically zero). In addition, if $h(z)\not\equiv 0$, then* (9) *holds if and only if $P(z,\ u,\ v)$ has the form,*

$$P(z,\ u,\ v)=Q(z,\ u,\ v)(v+R_0(z)u)+E(z,\ u,\ v), \tag{12}$$

*for some polynomials $Q$ and $E$ in $u$ and $v$, having coefficients in $H_n$, and where all terms in $E(z,\ u,\ v)$ have the same total degree in $u$ and $v$, and where $E(z,\ 1,\ -R_0(z))\not\equiv 0$.*

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**2.** LEMMA A. *Let $g(z)$ be an entire function of finite order for which there is a real number $b$ such that $\mathrm{Re}(g(r))>0$ on $(b, +\infty)$, and assume that for every $\alpha>0$,*

$$r^{-\alpha}(\mathrm{Re}(g(r))) \longrightarrow +\infty \quad \text{as } r\to+\infty \text{ through real values.} \tag{13}$$

*Let $G(z)$ be an entire function of finite order which has no zeros on a set of the form, $|\mathrm{Im}(z)|<\delta$, $\mathrm{Re}(z)>a$, where $\delta>0$ and $a\geqq-\infty$, and assume that $f=Ge^{g}$ is a solution of an equation $f''+A(z)f=0$, where $A(z)$ is entire. Then, if this equation possesses a solution $f_1\not\equiv 0$ satisfying $\lambda(f_1)<\infty$, and such that $f_1$ is not a constant multiple of $f$, the following hold:*

(a) *$f_1=He^{-g}$ for some entire function $H(z)$ of finite order.*

(b) *For all $r>a$,*

$$H(r)/G(r)=-Ke^{2g(r)}\int_{r}^{+\infty}(e^{-2g(t)}/(G(t))^2)dt, \tag{14}$$

*where $K$ is the Wronskian of $f$ and $f_1$.*

*Proof.* Since $A=-f''/f$, we clearly have,

$$A=-((g')^2+g''+2g'(G'/G)+(G''/G)), \tag{15}$$

which shows that $A(z)$ is of finite order. If we set $E=ff_1$, it follows (see [3; Lemma B]) that $E$ satisfies the equation,

$$K^2-(E')^2+2EE''+4AE^2=0, \tag{16}$$

for some constant $K\neq 0$, and the relation,

$$T(r, E)=0(\bar{N}(r, 1/E)+T(r, A)+\log r), \tag{17}$$

as $r\to+\infty$ "*nearly everywhere*" (briefly, n.e., meaning outside a possible exceptional set of finite measure). From the hypothesis, we see from (17) (and [3; §4(A)]) that $E$ is of finite order, and so the representation in Part (a) follows when we set $H=E/G$.

For Part (b), we observe first that since $G$ has no zeros on $|\mathrm{Im}(z)|<\delta$, $\mathrm{Re}(z)>a$, it follows from a standard minimum modulus estimate (see [10; p. 336]) that there are positive constants $r_0$ and $\beta$ such that

$$|G(t)|\geqq\exp(-t^{\beta}) \qquad \text{for all } \ t\geqq r_0. \tag{18}$$

(We remark that when $a=-\infty$, we actually obtain

$$|G(t)|\geqq\exp(-|t|^{\beta}) \ \text{ for all real } t \text{ satisfying } |t|\geqq r_0, \tag{19}$$

by [10; p. 336].).

In view of the hypothesis (13), and the estimate (18), it easily follows that for all $r>a$, the integral,

$$F(r)=\int_r^{+\infty}(e^{-2g(t)}/(G(t))^2)dt, \tag{20}$$

converges, and represents a differentiable function on $(a, +\infty)$ with the properties,

$$F'(r)\equiv -e^{-2g(r)}/(G(r))^2, \quad \text{and} \quad F(r)\to 0 \quad \text{as } r\to +\infty. \tag{21}$$

Now, since $f$ and $f_1$ are both solutions of (1), we have by Abel's identity that the derivative of $f_1/f$ is $K/f^2$. Since $f=Ge^g$ and $f_1=He^{-g}$ (by Part (a)), we see that if we set $w=H/G$, then the derivative of $w(r)e^{-2g(r)}$ is $-KF'(r)$ by (21), and so there exists a constant $L$ such that,

$$w(r)e^{-2g(r)}\equiv -KF(r)+L \qquad \text{on} \quad (a, +\infty). \tag{22}$$

We assert that $L=0$. We know from (21) that $F(r)\to 0$ as $r\to+\infty$. Since $H(z)$ is of finite order by Part (a), it follows from (18) that there are positive constants $r_1$ and $b$ such that

$$|w(r)|\leqq \exp(r^b) \qquad \text{for all} \quad r>r_1. \tag{23}$$

Thus in view of the hypothesis (13), the left-hand side of (22) approaches zero as $r\to+\infty$ so $L=0$. Thus (14) follows from (22), and so Lemma A is proved.

**3.** *Proof of Theorem* 1. We assume the conclusion fails to hold, so there exists a solution $f_1$ which is not a constant multiple of $f$, and satisfying $\bar{\lambda}(f_1)<\infty$. Hence by Lemma A, we can write $f_1=He^{-g}$ where $H(z)$ is an entire function of finite order, and (14) holds where $a=-\infty$. Since $a=-\infty$, the estimate (19) holds, and since $H(z)$ is of finite order, we see that there are positive constants $b$ and $r_2$ such that

$$|H(s)/G(s)|\leqq \exp(|s|^b) \quad \text{for all real } s \text{ satisfying } |s|\geqq r_2. \tag{24}$$

Now, by the hypothesis, $G$ and $g$ are real on the real axis, and so the integrand in (14) is a positive continuous function of $t$ on $(-\infty, \infty)$. Hence for $s<-r_2$,

$$\left|\int_s^{+\infty}(e^{-2g(t)}/(G(t))^2)dt\right|\geqq \int_0^{+\infty}(e^{-2g(t)}/(G(t))^2)dt. \tag{25}$$

Denoting by $K_1$ the right-hand side of (25), we have $K_1>0$, and from (14) and (24) we see that

$$\exp(|s|^b)\geqq |K|K_1e^{2g(s)} \qquad \text{for all} \quad s<-r_2. \tag{26}$$

Of course, (26) is in direct contradiction to the hypothesis (B) for $\alpha=b$, proving the result.

**4.** *Remark.* In this section, we show that the class of equations (1) treated in Theorem 1 contains coefficient functions of any preassigned order. To this end, let $\sigma$ be any nonnegative real number. It is easy to construct (see [10; p. 326]) an even transcendental entire function $g$ of order $\sigma$ whose power series expansion around $z=0$ has all positive coefficients. Then $g$ satisfies (A) and (B) in Theorem 1. The function $f=e^g$ satisfies equation (1) with $A=-((g')^2+g'')$. This relation shows that the order $\sigma_1$ of $A$ is equal to $\sigma$. (The inequality $\sigma_1 \leqq \sigma$ is obvious, and the impossibility that $\sigma_1 < \sigma$ follows immediately from a variant of Clunie's lemma [3; Lemma A, p. 4].)

**5.** We remark that the proofs of the results to be presented in § 5-7 parallel very closely the proofs of the analogous results presented in [3].

LEMMA B. *Let $B(\zeta)$ be a nonconstant rational function which is analytic on $0<|\zeta|<\infty$. Let $\alpha$ be a nonzero constant, and set $w=2\pi i/\alpha$, and $A(z)=B(e^{\alpha z})$. Let $f \not\equiv 0$ be a solution of* (1) *which satisfies the condition,*

$$\log^+ N(r, 1/f)=o(r) \quad \text{as} \quad r \to +\infty. \tag{27}$$

*Then, if the functions $f(z)$ and $f(z+w)$ are linearly dependent, the function $f(z)$ can be represented in the form,*

$$f(z)=\Psi(e^{\alpha z})\exp\Big(\sum_{j=q}^{m} d_j e^{\alpha j z}+dz\Big), \tag{28}$$

*where* (i) *$\Psi(\zeta)$ is a polynomial all of whose roots are simple and nonzero;* (ii) *$m$ and $q$ are integers with $m \geqq q$;* (iii) *$d, d_q, \cdots, d_m$ are complex constants such that $d_j \neq 0$ for some $j \neq 0$.*

*Proof.* From the hypothesis that $f(z)$ and $f(z+w)$ are dependent, we can write $f(z)=e^{\beta z}U(z)$ where $U$ is entire and has period $w$. Thus $U(z)=G(e^{\alpha z})$ where $G(\zeta)$ is analytic on $0<|\zeta|<\infty$, and from (1), it is easy to check that $G(\zeta)$ satisfies the equation

$$\alpha^2\zeta^2 G''+\zeta(2\beta\alpha+\alpha^2)G'+(B(\zeta)+\beta^2)G=0. \tag{29}$$

As in [3; p. 9], we first show that $G(\zeta)$ has only finitely many zeros on $0<|\zeta| <\infty$. If we assume the contrary, the sequence of zeros of $G$ has a cluster point $\zeta^*$ at either 0 or $\infty$ (or both). If $\zeta^*=\infty$, then $G(\zeta)$ has an essential singularity at $\infty$, and by the Wiman-Valiron theory [11; pp. 15, 93-111] applied to (29), we can write $G(\zeta)$ in the form $\zeta^m \Phi(\zeta) u(\zeta)$, where $m$ is an integer, $\Phi(\zeta)$ is analytic and nonvanishing at $\infty$, and where $u(\zeta)$ is an entire transcendental function of finite order having infinitely many zeros. Letting $H(\zeta)$ denote the canonical product formed with the zeros of $u(\zeta)$, we have a representation $u(\zeta) =H(\zeta)\exp(Q(\zeta))$ where $Q(\zeta)$ is a polynomial. Now from (29), the function $G_1=Ge^{-Q}$ also satisfies a second-order linear differential equation with polynomial coefficients, and in view of the representation $G_1(\zeta)=\zeta^m\Phi(\zeta)H(\zeta)$, it again follows from the Wiman-Valiron theory that the entire transcendental function $H(\zeta)$ must have order $\delta>0$. Thus $\bar{\lambda}(u)=\delta>0$. Hence (see [6; p. 25] or [8; p. 27] or [11;

p. 51]) the counting function for the zeros of $u$ satisfies

$$\limsup_{t\to+\infty}((\log n(t, 1/u))/\log t)=\delta. \tag{30}$$

Hence there exists a sequence $(t_q)\to+\infty$ such that

$$\log n(t_q, 1/u)>(\delta/2)\log t_q \quad \text{for all } q. \tag{31}$$

Now, in view of the representation,

$$f(z)=e^{\beta z}e^{\alpha m z}\Phi(e^{\alpha z})u(e^{\alpha z}), \tag{32}$$

and the fact that $\Phi(\zeta)$ is analytic and nowhere zero on a region $|\zeta|>r_0$ where $r_0>1$, it follows that if $\zeta_0$ is a zero of $u$ satisfying $|\zeta_0|>r_0$, then for any $z_0$ satisfying $e^{\alpha z_0}=\zeta_0$, we have $f(z_0)=0$. Let $\zeta_1, \zeta_2, \cdots$ be the zero-sequence of $u(\zeta)$, ordered by increasing modulus, and let $b$ be an index such that $|\zeta_j|>r_0$ for $j\geqq b$. For an index $q$, set $p=n(t_q, 1/u)$ so that $\zeta_1, \cdots, \zeta_p$ are the zeros of $u$ in $|\zeta|\leqq t_q$. For $j\geqq b$, write $\zeta_j=|\zeta_j|e^{i\theta_j}$ where $-\pi<\theta_j\leqq\pi$, and set

$$z_j=(1/\alpha)(\log|\zeta_j|+i\theta_j). \tag{33}$$

Thus $e^{\alpha z_j}=\zeta_j$, so by our previous observation, $z_{b+1}, \cdots, z_p$ are zeros of $f$, and each of these zeros lies in the disk $|z|\leqq|\alpha|^{-1}((\log t_q)+\pi)$. Hence if $q$ is sufficiently large so that $t_q>e^{\pi}$, we have

$$n(|\alpha|^{-1}2(\log t_q), 1/f)\geqq n(t_q, 1/u)-b. \tag{34}$$

Setting $s_q=|\alpha|^{-1}2(\log t_q)$, we have $s_q\to+\infty$, and from (31), we obtain

$$n(s_q, 1/f)>e^{cs_q}-b \qquad \text{where} \quad c=|\alpha|\delta/4>0. \tag{35}$$

But this implies (see [1; (10.17), p. 530]) that for all sufficiently large $q$,

$$N(2s_q, 1/f)>(\log 2)(e^{cs_q}-b), \tag{36}$$

which obviously contradicts our assumption (27) since $c>0$ and $(s_q)\to+\infty$. This shows that $\zeta^*$ cannot be $\infty$, and a similar argument applied to $G_2(t)=G(1/t)$ in place of $G(\zeta)$ shows that $\zeta^*$ cannot be zero either. Thus $G(\zeta)$ has only finitely many zeros on $0<|\zeta|<\infty$, and each zero is simple since $f$ has only simple zeros. Letting $\Psi(\zeta)$ be the polynomial having these zeros, clearly $\varphi=G/\Psi$ is analytic and nowhere zero on $0<|\zeta|<\infty$. Hence $\varphi(e^{\alpha z})$ is an entire function with no zeros, so is of the form $e^{v(z)}$, where $v(z)$ is entire. Clearly $v'(z)$ has period $w$, and thus $v'(z)=W(e^{\alpha z})$ where $W(\zeta)$ is analytic on $0<|\zeta|<\infty$. Now since

$$G'(\zeta)/G(\zeta)=(\Psi'(\zeta)/\Psi(\zeta))+(\alpha\zeta)^{-1}W(\zeta), \tag{37}$$

it follows easily from (29) that $W(\zeta)$ satisfies a nonlinear Riccati equation with rational coefficients, and the Wiman-Valiron theory then shows that $W(\zeta)$ cannot have an essential singularity at either $\zeta=0$ or $\zeta=\infty$. Hence $W(\zeta)$ is rational, and so has the form,

$$W(\zeta)=\zeta^{-k}(c_n\zeta^n+c_{n-1}\zeta^{n-1}+\cdots+c_0). \tag{38}$$

Integrating $v'(z)=W(e^{\alpha z})$ to find $v(z)$, the representation (28) follows immediately.

**6.** LEMMA C. *Let $A(z)$ be a nonconstant entire function of period $w$, and let $f\not\equiv 0$ be a solution of* (1) *which satisfies* (27). *Then, $f(z)$ and $f(z+2w)$ are linearly dependent solutions of* (1).

*Proof*. The conclusion is immediate if $f(z)$ and $f(z+w)$ are linearly dependent, so we may assume that $f(z)$ and $f(z+w)$ are linearly independent. Hence if we assume the conclusion is false, then the solutions $f_1=f$, $f_2(z)=f(z+w)$, and $f_3(z)=f(z+2w)$ are pairwise linearly independent, so by [1; Lemma 8.1, p. 523] we have n. e. as $r\to+\infty$,

$$T(r, f_1)=0\Big(\sum_{j=1}^{3} N(r, 1/f_j)+T(r, A)\Big). \tag{39}$$

Now, it is easy to see (e. g. [1; p. 530]) that for all sufficiently large $r$,

$$N(r, 1/f_j)=0((\log r)N(2r, 1/f_1)) \tag{40}$$

for $j=1, 2, 3$, and since $A=-f_1''/f_1$, we have

$$T(r, A)=0(\log T(r, f_1)) \qquad \text{n. e. as} \quad r\to+\infty. \tag{41}$$

Hence from (39), (40), and (41), we have,

$$T(r, f_1)=0((\log r)N(2r, 1/f_1)) \qquad \text{n. e. as} \quad r\to+\infty, \tag{42}$$

and since $f_1$ satisfies (27), it follows now that $\log T(r, f_1)=o(r)$ n. e. as $r\to+\infty$. But then from (41), we have $T(r, A)=o(r)$ n. e. as $r\to+\infty$, and so it easily follows (see [3; §4(A)]) that $T(r, A)=o(r)$ as $r\to+\infty$ without an exceptional set. This shows that $A(z)$ has no zeros since if $z_0$ is a zero of $A$, then all points $z_0+nw$, (for integral $n$) are also zeros, and the counting function for these points exceeds $cr$ as $r\to+\infty$ for some $c>0$. Hence $A$ must be the exponential of a polynomial, but being periodic will then be of the form $e^{cz+d}$ for constants $c$ and $d$. Since $T(r, A)=o(r)$ as $r\to+\infty$, we must have $c=0$, and so $A$ is a constant, contrary to hypothesis.

**7.** LEMMA D. *Let $A(z)$ be a nonconstant entire function of period $w$, say $A(z)=B(e^{\alpha z})$, where $B(\zeta)$ is analytic on $0<|\zeta|<\infty$, and $\alpha=2\pi i/w$, and assume*

$$\log T(r, A)=o(r) \qquad as \quad r\to+\infty. \tag{43}$$

*Let $f\not\equiv 0$ be a solution of* (1) *which satisfies* (27), *and has the property that the functions $f(z)$ and $f(z+w)$ are linearly independent. Set $E(z)=f(z)f(z+w)$.*

*Then*.

(a) $\log T(r, E)=o(r)$ *as* $r\to+\infty$.

(b) $E(z)^2$ *is periodic of period* $w$.

(c) *Writing* $E(z)^2=\Phi(e^{\alpha z})$ *where* $\Phi(\zeta)$ *is analytic on* $0<|\zeta|<\infty$, *then* $\Phi(\zeta)$ *has at most a pole at* $\zeta=\infty$ (*resp.* $\zeta=0$) *if and only if* $B(\zeta)$ *has at most a pole at* $\zeta=\infty$ (*resp.* $\zeta=0$).

*Proof*. Since $E$ and $A$ are related by equation (16), we observe first that $E$ is transcendental since $A$ is transcendental. Now since $f(z)$ has property (27), it is easy to see (e. g. [1; p. 530]) that $f(z+w)$ also has property (27), and hence so does $E(z)$. Since $E$ satisfies (17), it now easily follows from (43) that $\log T(r, E)=o(r)$ n. e. as $r\to+\infty$, and hence (using [3; §4(a)]), we see that Part (a) holds.

For Part (b), we note that by Lemma C, we have $E(z+w)\equiv cE(z)$ for some nonzero constant $c$. Thus $E'/E$ and $E''/E$ have period $w$, and so by (16), we obtain Part (b).

For Part (c), we set $F=E^2$, so $F(z)=\Phi(e^{\alpha z})$ where $\Phi(\zeta)$ is analytic on $0<|\zeta|<\infty$. From (16) it follows that $\Phi$ satisfies the equation,

$$\alpha^2(\zeta^2\Phi\Phi''-(3/4)\zeta^2(\Phi')^2+\zeta\Phi\Phi')+4B(\zeta)\Phi^2+K^2\Phi=0. \tag{44}$$

We show first that if $B(\zeta)$ has at most a pole at $\infty$, the same is true for $\Phi(\zeta)$. (The converse is obvious from (44).) If we assume the contrary, then by the Wiman-Valiron theory [11; pp. 15, 93-111], we can write $\Phi(\zeta)$ in the form $\zeta^m\Psi(\zeta)u(\zeta)$, where $m$ is an integer, $\Psi(\zeta)$ is analytic on a region $|\zeta|>r_0$ and has a finite, nonzero limit at $\infty$, and where $u(\zeta)$ is an entire transcendental function whose maximum modulus satisfies an asymptotic relation, $\log M(r, u)\sim c_1r^\sigma$ as $r\to+\infty$, for some constants $c_1>0$ and $\sigma>0$. Hence, for sufficiently large $r$, if $\zeta_r$ denotes a point on $|\zeta|=r$ at which $|u(\zeta)|=M(r, u)$, then

$$|\Phi(\zeta_r)|\geqq\exp((c_1/2)r^\sigma). \tag{45}$$

Writing $\zeta_r=re^{i\theta_r}$ where $|\theta_r|\leqq\pi$, we set $z_r=\alpha^{-1}(\log r+i\theta_r)$, so $F(z_r)=\Phi(\zeta_r)$. Since $|z_r|\leqq 2|\alpha|^{-1}\log r$ for sufficiently large $r$, we thus have from (45),

$$\log M(|z_r|, F)\geqq(c_1/2)\exp(2^{-1}|\alpha|\sigma|z_r|), \tag{46}$$

for all sufficiently large $r$. Using a standard inequality [6; p. 18, Theorem 1.6], it now follows that,

$$\log T(2|z_r|, F)\geqq\log(c_1/6)+2^{-1}|\alpha|\sigma|z_r|, \tag{47}$$

for all sufficiently large $r$, which contradicts the result in Part (a) since $F=E^2$, and $|z_r|\to+\infty$ as $r\to+\infty$. This proves that $\Phi(\zeta)$ has at most a pole at $\zeta=\infty$ if the same is true of $B(\zeta)$. The analogous situation for $\zeta=0$ is proved by setting $\Phi_1(t)=\Phi(t^{-1})$, and using similar reasoning at $t=\infty$. This proves Part (c).

**8.** *Proof of Theorem* 2. We first prove Part (A). We assume that (1) possesses a solution $f\not\equiv 0$ which fails to satisfy (7), so that (27) holds. We divide the proof into two cases.

Case I: Suppose first that $f(z)$ and $f(z+w)$ (where $w=2\pi i/\alpha$) are linearly independent. Then Lemma D is applicable, so if we set $E(z)=f(z)f(z+w)$, then $E(z)^2=\Phi(e^{\alpha z})$, where $\Phi(\zeta)$ is analytic on $0<|\zeta|<\infty$, and $\Phi(\zeta)$ has at most poles at $\zeta=0$ and $\zeta=\infty$. Thus $\Phi(\zeta)$ is rational, and so both $\zeta\Phi'/\Phi$ and $\zeta^2\Phi''/\Phi$ tend to finite limits as $\zeta\to\infty$, and as $\zeta\to 0$. Since $\Phi$ satisfies (44) for some constant $K\neq 0$, it thus follows that $4B(\zeta)+K^2(\Phi(\zeta))^{-1}$ also tends to finite limits as $\zeta\to\infty$, and as $\zeta\to 0$. But by hypothesis $B(\zeta)\to\infty$ as $\zeta\to\infty$, and as $\zeta\to 0$, and hence we can conclude that $\Phi(\zeta)\to 0$ as $\zeta\to\infty$, and as $\zeta\to 0$. Thus $\Phi(\zeta)$ is analytic on the extended plane and so is a constant, which of course must be zero. Thus $f\equiv 0$ contradicting our hypothesis. Thus Case I is impossible.

Case II: We now assume $f(z)$ and $f(z+w)$ are linearly dependent. Then by Lemma B, the solution $f(z)$ has the form (28), where conditions (i), (ii), (iii) in Lemma B are satisfied. We can assume in (28) that $d_0=0$ by incorporating the term $\exp(d_0)$ into the polynomial $\Psi$ as a constant multiplier. Since $d_j\neq 0$ for some $j\neq 0$, we can assume in (28) that $m=\max\{j : d_j\neq 0\}$ and $q=\min\{j : d_j\neq 0\}$ so that all the following hold in (28):

$$d_m\neq 0,\quad d_q\neq 0,\quad m\neq 0,\quad q\neq 0,\quad \text{and}\quad q\leqq m. \tag{48}$$

In (28), set $G(z)=\Psi(e^{\alpha z})$, and,

$$g(z)=\sum_{j=q}^{m} d_j e^{\alpha jz}+dz, \tag{49}$$

so that $f=Ge^g$. Since $f$ satisfies (1), we see that $A$, $g$, and $G$ are related by (15). We now set $H=-2g'(G'/G)-(G''/G)$ so that by (15) we have,

$$A=-(g')^2-g''+H. \tag{50}$$

We observe first that since $A$ and $g$ are entire, it follows from (50) that $H$ is entire. On the other hand, since $\Psi(\zeta)$ is a polynomial, and since from (49), clearly $g'$ and $g''$ are rational functions of $e^{\alpha z}$, we see that $H(z)$ is a rational function of $e^{\alpha z}$, say $H(z)=R(e^{\alpha z})$, where $R(\zeta)$ is rational. We distinguish two subcases depending on whether $R\equiv 0$ or $R\not\equiv 0$.

Subcase A: If $R\equiv 0$, then $H\equiv 0$, so from (49) and (50), we have

$$A(z)=-\Big(\sum_{j=q}^{m}\alpha j d_j e^{\alpha jz}+d\Big)^2-\sum_{j=q}^{m}(\alpha j)^2 d_j e^{\alpha jz}. \tag{51}$$

Since $A(z)=B(e^{\alpha z})$, we see that as rational functions of $\zeta$, we have,

$$B(\zeta)=-\Big(\sum_{j=q}^{m}\alpha j d_j\zeta^j+d\Big)^2-\sum_{j=q}^{m}(\alpha j)^2 d_j\zeta^j. \tag{52}$$

Now from (48), we know $q$ and $m$ are nonzero. If both $m$ and $q$ are positive, it is clear from (52) that $B(\zeta)$ has no pole at $\zeta=0$, contrary to hypothesis. If both $m$ and $q$ are negative, it is clear that $B(\zeta)$ has no pole at $\zeta=\infty$, again contrary to hypothesis. If $m$ and $q$ are of opposite sign, we must have $q<0<m$, and it follows from (52) that $B(\zeta)$ has a pole at $\infty$ of order $2m$, and a pole at zero of order $2|q|$, again contradicting the hypothesis that at least one of these poles must be of odd order. Thus Subcase A is impossible.

Subcase B: Here the rational function $R(\zeta)$ is not identically zero. Hence we can write $R=R_1/R_2$ where $R_1$ and $R_2$ are polynomials in $\zeta$, having no common factor. Since $H(z)=R(e^{\alpha z})$ is an entire function, it is clear that $R_2$ can have no roots other than (possibly) zero. Hence $R_2(\zeta)$ is of the form $c\zeta^m$, and so we may write $R(\zeta)=\sum_{j=t}^{s} c_j\zeta^j$, where the $c_j$ are constants, where $t$ and $s$ are integers with $t\leqq s$, and where $c_t\neq 0$ and $c_s\neq 0$. Now, by definition of $H$, we have

$$G''+2g'G'=-GH, \tag{53}$$

where $G(z)=\Psi(e^{\alpha z})$, and $g$ is given by (49). Now if the degree $p$ of the polynomial $\Psi(\zeta)$ is zero, then $G(z)$ is a constant, and so by (53), $H\equiv 0$. But then $R\equiv 0$, contradicting this subcase. Thus $p\geqq 1$, and we may write,

$$\Psi(\zeta)=\sum_{j=0}^{p} a_j\zeta^j, \qquad \text{where} \quad a_p\neq 0, \quad a_0\neq 0, \tag{54}$$

and the $a_j$ are constants. (The fact that $a_0\neq 0$ follows from the fact that the roots of $\Psi$ are nonzero.) Thus from (49), (53), and (54), we have

$$\sum_{j=1}^{p}(\alpha j)^2 a_j\zeta^j+2\Big(\sum_{j=q}^{m}\alpha j d_j\zeta^j+d\Big)\Big(\sum_{j=1}^{p}\alpha j a_j\zeta^j\Big)=-\Big(\sum_{j=0}^{p}a_j\zeta^j\Big)\Big(\sum_{j=t}^{s}c_j\zeta^j\Big), \tag{55}$$

as rational functions in $\zeta$. In addition, from (49) and (50), we have,

$$B(\zeta)=-\Big(\sum_{j=q}^{m}\alpha j d_j\zeta^j+d\Big)^2-\Big(\sum_{j=q}^{m}(\alpha j)^2 d_j\zeta^j\Big)+\sum_{j=t}^{s}c_j\zeta^j, \tag{56}$$

as rational functions of $\zeta$. We know from (48) that $m$ and $q$ are both nonzero, and $q\leqq m$. We consider separately the three possibilities, (i) $m>0$ and $q>0$: (ii) $m<0$ and $q<0$; (iii) $q<0<m$.

In Case (i), we have $m>0$ and $q>0$. Since also $p>0$, it is clear that when the left side of (55) is expanded in powers of $\zeta$, all terms have positive powers of $\zeta$. Now the nonzero term on the right side of (55) containing the smallest power of $\zeta$ is $-a_0c_t\zeta^t$, so by (55) we have $t>0$. Thus in this case, the expansion of $B(\zeta)$ given in (56) contains only nonnegative powers of $\zeta$, so that $B(\zeta)$ has no pole at $\zeta=0$ which is contrary to the hypothesis. Thus (i) cannot occur.

In Case (ii) we have $m<0$ and $q<0$. Thus, when the left side of (55) is expanded in powers of $\zeta$, all nonzero terms which appear contain a power of $\zeta$ of at most $p$. Since the right side of (55) has the nonzero term $-a_pc_s\zeta^{p+s}$, we thus have $p+s\leqq p$, so $s\leqq 0$. Since also $m<0$, we see that the expansion of $B(\zeta)$ in (56) contains only terms with nonpositive powers of $\zeta$ which contradicts the hypothesis that $B(\zeta)$ has a pole at $\infty$. Thus (ii) cannot occur.

In Case (iii), we have $q<0<m$. Now, in (54), let $k$ denote the smallest index $j>0$ such that $a_j\neq 0$. Thus $0<k\leqq p$. Since $q<0$, the nonzero term in the expansion of the left side of (55) which has the smallest power of $\zeta$ is $2\alpha^2 qkd_q a_k\zeta^{k+q}$. Since the nonzero term on the right side of (55) which has the smallest power of $\zeta$ is $-a_0c_t\zeta^t$, we must have

$$k+q=t. \tag{57}$$

On the other hand, the nonzero terms in the expansions of the left and right sides respectively of (55), which have the largest powers of $\zeta$ are $2\alpha^2 mpd_m a_p\zeta^{p+m}$ and $-a_pc_s\zeta^{p+s}$ respectively, so we have $m=s$. Since $m>0$, we thus see that the nonzero term in the expansion of $B(\zeta)$ in (56) which has the largest power of $\zeta$ is $-(\alpha m)^2d_m^2\zeta^{2m}$. Since $q<0$ and $q<t$ (from (57), the nonzero term in $B(\zeta)$ with the smallest power of $\zeta$ is $-(\alpha q)^2d_q^2\zeta^{2q}$. Thus $B(\zeta)$ has even order poles at both $\zeta=0$ and $\zeta=\infty$ contradicting our hypothesis. Hence (iii) cannot occur, and so Subcase B is impossible. Thus, our assumption of the existence of a solution which fails to satisfy (7) has led to a contradiction, proving Part (A) of Theorem 2.

For Part (B), we observe first that if $\Delta(r)$ is as in the statement, then

$$T(2r,\ A)=0(\Delta(r)) \qquad \text{n. e. as} \quad r\to+\infty, \tag{58}$$

since $T(r,\ A)=0(r)$ as $r\to+\infty$. From Part (A), it follows that no solution $f\not\equiv 0$ of (1) can satisfy $N(r,\ 1/f)=0(\Delta(r))$ n. e. as $r\to+\infty$, since such a solution, in view of the hypothesis $\log^+\Delta(r)=o(r)$ n. e. as $r\to+\infty$, would satisfy (27) nearly everywhere as $r\to+\infty$. In view of [3; §4(A)], the solution $f$ would satisfy (27) as $r\to+\infty$ with no exceptional set, thus contradicting the conclusion (7) from Part (A). Hence, the conclusion of Part (B) now follows immediately from [1; Theorem 2A, p. 508].

**9.** *Remarks.* (a) For the case of an equation, $f''+B(e^{\alpha z})f=0$, where $B(\zeta)$ is a rational function which is analytic on $0<|\zeta|<\infty$, and has even order poles at $\zeta=0$ and $\zeta=\infty$, it is possible for conclusion (7) to hold for all solutions $f\not\equiv 0$. A general class of such equations can be constructed very simply by letting $B_1(w)$ be any rational function of $w$, analytic on $0<|w|<\infty$, and having poles at $w=0$ and $w=\infty$, at least one of which has odd order. Then by Theorem 2, all solutions $f\not\equiv 0$ of the equation $f''+B_1(e^{2\alpha z})f=0$ satisfy (7). But this equation is the same as $f''+B(e^{\alpha z})f=0$, where $B(\zeta)=B_1(\zeta^2)$, and clearly $B(\zeta)$ has poles of even order at both $\zeta=0$ and $\zeta=\infty$.

(b) For the case of an equation, $f''+B(e^{\alpha z})f=0$, where $B(\zeta)$ is a rational function, analytic on $0<|\zeta|<\infty$, and has a pole at only one of the points $\zeta=0$ or $\zeta=\infty$, it is possible for all solutions $f\not\equiv 0$ to satisfy (7). Such examples are provided by the equations $f''+(e^z-K)f=0$, where $K$ is a constant with the property that $16K$ is not the square of an odd, positive integer. (It was shown in [4] using the results in [3] that one has $\bar{\lambda}(f)=\infty$ for all solutions $f\not\equiv 0$ of such an equation. By using the stronger results given in Lemmas B and D above, in place of the corresponding results in [3], we can obtain the stronger conclusion that (7) holds for all solutions $f\not\equiv 0$. We remark that when $16K$ is the square of an odd, positive integer, it was shown in [4] that the equation possesses a fundamental set $\{f_1, f_2\}$ such that $\bar{\lambda}(f_j)\leqq 1$ for $j=1, 2$.)

**10.** *Proof of Theorem* 3. We are given that $A(z)$ is a polynomial of degree $n\geqq 1$. It is shown in [12] that if $f\not\equiv 0$ is a solution of (1) having infinitely many zeros, then there is a nonzero constant $c$ such that,

$$\bar{N}(r, 1/f)/r^{(n+2)/2} \longrightarrow c \quad \text{as} \quad r\to+\infty. \tag{59}$$

(See also Kaufman [7]. Let $H_n$ and $P$ be as in the statement of Theorem 3.

Assume now that all solutions $f\not\equiv 0$ of (1) have infinitely many zeros. Then for any solution $f\not\equiv 0$, the function $f'/f$ cannot be algebraic over $H_n$, since in the contrary case we would have (from (8)), $T(r, f'/f)=o(r^{(n+2)/2})$ as $r\to+\infty$, which would clearly contradict (59) since $c\neq 0$. Hence from [1; Theorem 4(a), p. 511], we can conclude that for the function $h(z)$ given by (3), we have that $h\not\equiv 0$ and that $h'/h$ does not belong to $H_n$. Since $m(r, h'/h)=0(\log r)$ as $r\to+\infty$, it follows that $N(r, h'/h)\neq o(r^{(n+2)/2})$ as $r\to+\infty$. Since any pole of $h$ must be a pole of a coefficient of $P$, and each such coefficient satisfies (8), we have $\bar{N}(r, h)=o(r^{(n+2)/2})$ as $r\to+\infty$, and so $h$ cannot satisfy (9) since

$$N(r, h'/h)=\bar{N}(r, h)+\bar{N}(r, 1/h). \tag{60}$$

This proves Part (A).

For Part (B), we assume that (1) possesses a solution $f_0\not\equiv 0$ having only finitely many zeros. Then, $f_0$ must be of the form $B(z)e^{V(z)}$, where $B$ and $V$ are polynomials, with $V$ of degree $(n+2)/2$. Hence $f_0$ is not algebraic over $H_n$, for in the contrary case, $f_0$ would satisfy (8), and hence $e^{V(z)}$ would satisfy (8) which is false (see [6; p. 7]). Of course, $R_0=-f_0'/f_0$ is a rational function, and so belongs to $H_n$. Finally, for any solution $f\not\equiv 0$ of (1) which is linearly independent with $f_0$, we must have $\bar{\lambda}(f)=(n+2)/2$ by [2; Theorem 1], and so $f$ satisfies (59) for some $c\neq 0$. Thus, as before, $f'/f$ cannot be algebraic over $H_n$. Hence, we may apply [1; Theorem 4(b)] to conclude that $h(z)$ given by (3) cannot be identically zero, and that $h'/h$ belongs to $H_n$ if and only if $P$ has the special form given by (10). However, using (60), clearly $h'/h$ belongs to $H_n$ if and only if (9) holds, which proves Part (a). The conclusions in Part (b) for solutions which are constant multiples of $f_0$ follow exactly as above from [1; Theorem 4(d)]. This proves Theorem 3.

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