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ON THE AUTOMORPHISM GROUPS OF A COMPACT BORDERED RIEMANN SURFACE OF GENUS FIVE

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§1. Introduction.

Let S be a compact bordered Riemann surface of genus g with k boundary components. If $2g+k-1\geqq 2$, the automorphism group of S is a finite group. Then, we put $N(g, k)$ the maximum order of automorphism groups of S where the maximum is taken over all S of genus g with k boundary components. It is well known that $N(g, k)$ is equal to the maximum order of automorphism groups of compact Riemann surfaces of genus g deleted k points, and every automorphism group of S is isomorphic to that of a compact Riemann surface (Oikawa [7]). For every $k\geqq 0$, $N(0, k)$, $N(1, k)$, $N(2, k)$, $N(3, k)$ and $N(4, k)$ are determined by Heins [2], Oikawa [7], Tsuji [8], Tsuji [9] and Kato [4], respectively. In the present paper, we shall determine $N(5, k)$.

Theorem. $N(5, k)$ is

(1) 192 for $k\equiv 0, 24, 64, 88 \pmod{96}$,
(2) 160 for $k\equiv 0, 32 \pmod{40}$ except the case (1),
(3) 120 for $k\equiv 0, 12, 40, 52 \pmod{60}$ except the cases (1), (2),
(4) 96 for $k\equiv 16, 32, 40, 48, 56, 72 \pmod{96}$ except the cases (2), (3),
(5) 80 for $k\equiv 16 \pmod{40}$ except the cases (1), (2), (4),
(6) 64 for $k\equiv 0 \pmod{8}$ except the cases (1)~(5),
(7) 60 for $k\equiv 20, 32 \pmod{60}$ except the cases (1), (2), (4)~(6),
(8) 48 for $k\equiv 0, 4 \pmod{12}$ except the cases (1)~(7),
(9) 40 for $k\equiv 0, 2 \pmod{10}$ except the cases (1)~(8),
(10) 32 for $k\equiv 4 \pmod{16}$ except the cases (1)~(5), (7)~(9),
(11) 30 for $k\equiv 0, 2, 5, 7 \pmod{15}$ except the cases (1)~(10),
(12) 24 for $k\equiv 2, 6, 10, 14, 20 \pmod{24}$ except the cases (1)~(5), (7), (9)~(11),
(13) 22 for $k\equiv 0, 1, 2, 3 \pmod{11}$ except the cases (1)~(12),
(14) 20 for $k\equiv 1, 5, 7, 11 \pmod{20}$ except the cases (1)~(8), (10)~(13),
(15) 16 for $k\equiv 2, 6 \pmod{16}$ except the cases (1)~(5), (7)~(9), (11)~(14),
(16) 15 for $k\equiv 1, 6 \pmod{15}$ except the cases (1)~(10), (12)~(15),
(17) 12 for $k\equiv 0, 1, 3, 4 \pmod{6}$ except the cases (1)~(5), (7), (9)~(12),
(18) 8 otherwise.

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§ 2. Notation.

Let S be a compact Riemann surface of genus $g\geqq 2$, let G be a conformal automorphism group of S and let N be the order of G. Let $S_0=S/G$ be the quotient surface with conformal structure induced from S through π, where π is the projection mapping of S onto S_0. Let g_0 be the genus of S_0. At $p\in S$ and at $p_0=\pi(p)\in S_0$, by a suitable choice of local parameters, π is represented locally by $z_0=z^{\nu}$, where ν is a positive integer, z, z_0 are the local parameters at p, p_0, respectively. If $\nu>1$, p is called a branch point of multiplicity ν. If $\pi(p_1)=\pi(p_2)$ ($p_1, p_2\in S$), then the multiplicity of p_1 is equal to that of p_2. Therefore we can define the multiplicity over $p_0\in S$ by the multiplicity of $p\in\pi^{-1}(p_0)$. Let $\{q_1, \cdots, q_t\}$ be the set of points on S_0 which are the projection of all the branch points on S. Let $\nu_1, \cdots, \nu_t$ be the multiplicities over $q_1, \cdots, q_t$, respectively. We call the set of integers $g_0, \nu_1, \cdots, \nu_t$ the signature of G and denote it by $(g_0; \nu_1, \cdots, \nu_t)$. Without loss of generality, we may assume $\nu_1\leqq\nu_2\leqq\cdots\leqq\nu_t$. For simplicity's sake, we shall denote $(0; \nu_1, \cdots, \nu_t)$ by $(\nu_1, \cdots, \nu_t)$.

§ 3. Lemmas.

LEMMA 1. (the Riemann-Hurwitz relation)

$$2g-2=N(2g_0-2)+N\sum_{j=1}^{t}(1-1/\nu_j)\,.$$

LEMMA 2. (Harvey [1]) *There exist a compact Riemann surface S and a cyclic automorphism group Z_N on S of order N with signature $(g_0; \nu_1, \cdots, \nu_t)$ if and only if this signature satisfies the following l. c. m. condition* (1)~(4), *where $M=l.c.m.(\nu_1, \cdots, \nu_t)$: the least common multiple of $\nu_1, \cdots, \nu_t$.*

(1) $M=l.c.m.(\nu_1, \cdots, \check{\nu}_j, \cdots, \nu_t)$. $(j=1, \cdots, t)$
Here, $\check{\nu}_j$ denetes the omission of ν_j.
(2) *$M|N$ and if $g_0=0$, then $M=N$.*
(3) *$t\neq 1$ and if $g_0=0$, then $t\geqq 3$.*
(4) *If $2|M$, the number of ν_j's which are divisible by the maximum power of 2 that divides M is even.*

LEMMA 3. *If S has an automorphism group of order N with signature $(g_0; \nu_1, \cdots, \nu_t)$, then for $k=mN+\sum_{j=1}^{t}\varepsilon_j N/\nu_j$, $N(g, k)\geqq N$, where m is a non-negative integer and $\varepsilon_j=0$ or 1 $(j=1, \cdots, t)$.*

By Lemma 3, $N(g, k)$ is completely determined by the signature of automorphism groups rather than by automorphism groups themselves. We do not look for the group of maximum order for a given integer k, or rather, we look for k points to be deleted from a compact Riemann surface so that these k points are invariant by the automorphism group with given signature.

LEMMA 4. *If $p \in S$ is a fixed point of some non-trivial automorphism in G, then the stabilizer subgroup of p in G is a cyclic group. Then, the order N of G must be a multiple of the order of the stabilizer subgroup of p.*

Proof. An automorphism h which fixes p is expanded locally as

$$h(z)=az+bz^2+\cdots \qquad (a\neq 0),$$

by a local parameter z at p. Here a is independent on the choice of local parameter. If the order of h is ν, a is a primitive νth root of unity. We claim that if $a=1$, then h is indeed the identity automorphism. Let $D=\{|w|<1\}$ be the universal covering surface of S and ϕ be the covering projection such that $\phi(0)=p$. By the covering surface theory, there is an automorphism H of D such that

$$h\circ\phi=\phi\circ H$$

and H fixes the origin $w=0$. Then H is an elliptic transformation and has the expansion

$$H(w)=w+\cdots.$$

Then H is the identity. This implies that h is also the identity automorphism. Since the set of all the automorphisms h_i fixing p is a group, the set of all the leading coefficients a_i of the expansions of those automorphisms also forms a group. This group of coefficients $\{a_i\}$ is a cyclic group. By the above arguement, $a_i=a_j$ implies that $h_i=h_j$. Thus, we conclude that the stabilizer subgroup is a cyclic group.

LEMMA 5. (Wiman [10], Nakagawa [6]) *If ν is the order of a stabilizer subgroup of G, then $2\leqq\nu\leqq 4g+2$.*

LEMMA 6. *There exists neither an automorphism of order 7 nor that of order 9 on any compact Riemann surface of genus 5.*

Proof. If $N=7$, by Lemma 4, $\nu_j=7$ $(j=1,\cdots,t)$. Then by the Riemann-Hurwitz relation we obtain

$$8=14(g_0-1)+6t.$$

Since $g_0\geqq 0$, $t\geqq 0$, this equation has no integer solution. Then the automorphism of order 7 does not exist. If $N=9$, by Lemma 4, $\nu_j=3$ or 9. Then, by the Riemann-Hurwitz relation we obtain

$$8=18(g_0-1)+6r+8s \qquad (r+s=t).$$

This equation has the integer solutions (1 ; 9) and (0 ; 3, 3, 3, 9). But these two solutions do not satisfy the l. c. m. condition.

LEMMA 7. *For all* $k\geqq 0$, $N(5, k)\geqq 8$.

Proof. Let S be the Riemann surface defined by

$$y^8=x^4(x-1)^2(x-\alpha)$$

where α is a complex number which is not equal to 0, 1. Let h be the automorphism of S defined by

$$h(x, y)=(x, \exp(\pi i/4)y).$$

The automorphism group $\langle h\rangle$ is of order 8 with signature (2, 4, 8, 8). Since $k=8m+4\varepsilon_1+2\varepsilon_2+\varepsilon_3+\varepsilon_4$ represents arbitrary integer by a suitable choice of m and ε_j $(j=1, \cdots, 4)$, then by Lemma 3 we obtain that $N(5, k)\geqq 8$.

From now on we are going to determine whether the automorphism group with a given signature exists or not on a compact Riemann surface of genus 5. By Lemma 7, it is not necessary to consider the groups of order $\leqq 8$. We assume $N>8$. By the Riemann-Hurwitz relation we obtain $g_0\leqq 1$, $t\leqq 5$. So by Lemma 5, it is enough to consider at most finite number of signatures. Among these signatures, say, (2, 3, 7) does not exist, since by Lemma 6, a cyclic group of order 7 does not exist. (2, 3, 15) also does not exist, for the order 80 is not a multiple of 3. In a similar way, using Lemmas 1, 4 and 6, we find that many signatures do not exist. Then, it is enough to consider the following signatures:

order signature

192 (2, 3, 8)	160 (2, 4, 5)	120 (2, 3, 10)	96 (2, 3, 12)
96 (2, 4, 6)	96 (3, 3, 4)	80 (2, 5, 5)	66 (2, 3, 22)
64 (2, 4, 8)	60 (2, 5, 6)	60 (3, 3, 5)	48 (2, 4, 12)
48 (2, 6, 6)	48 (3, 3, 6)	48 (3, 4, 4)	40 (2, 4, 20)
40 (2, 5, 10)	33 (3, 3, 11)	32 (2, 8, 8)	32 (4, 4, 4)
30 (2, 6, 15)	30 (3, 3, 15)	30 (3, 5, 5)	24 (2, 12, 12)
24 (3, 4, 12)	24 (3, 6, 6)	24 (4, 4, 6)	22 (2, 11, 22)
20 (2, 20, 20)	20 (4, 4, 10)	20 (5, 5, 5)	16 (4, 8, 8)
15 (3, 15, 15)	15 (5, 5, 15)	12 (6, 12, 12)	11 (11, 11, 11)
48 (2, 2, 2, 3)	32 (2, 2, 2, 4)	24 (2, 2, 2, 6)	24 (2, 2, 3, 3)
20 (2, 2, 2, 10)	16 (2, 2, 4, 4)	12 (2, 2, 4, 12)	12 (2, 2, 6, 6)
12 (2, 3, 3, 6)	12 (2, 3, 4, 4)	12 (3, 3, 3, 3)	10 (2, 2, 10, 10)
16 (2, 2, 2, 2, 2)	12 (2, 2, 2, 2, 3)	10 (2, 2, 2, 2, 5)	16 (1 ; 2)
12 (1 ; 3)	10 (1 ; 5)		

§ 4. The existence of hyperelliptic surfaces.

LEMMA 8. *Let* $\alpha_1, \cdots, \alpha_{2g+2}$ *be distinct complex numbers and let* f *be a linear transformation of the sphere which leaves the set* $\{\alpha_1, \cdots, \alpha_{2g+2}\}$ *invariant. Then, there are two automorphisms* h_1, h_2 *on the hyperelliptic surface defined by*

$$y^2=\prod_{n=1}^{2g+2}(x-\alpha_n)$$

such that $f\circ x=x\circ h_j$ $(j=1, 2)$.

At first, using Lemma 8, we show the existence of the group with signature (2, 3, 10) of order 120. On the Riemann sphere we choose the set of 12 points $\alpha_1, \cdots, \alpha_{12}$ which forms the verteces of the icosahedron. The icosahedral group leaves the set of these 12 points invariant and its order is 60. Then by Lemma 8 the hyperelliptic surface defined by

$$y^2=\prod_{n=1}^{12}(x-\alpha_n)$$

has the automorphism group of order 120 with signature (2, 3, 10). Secondly, we show the existence of the group with signature (2, 4, 12) of order 48. We put $\alpha_n=\exp(\pi in/6)$ $(n=0, 1, \cdots, 11)$. The dihedral group generated by the linear transformations

$$x\to\exp(\pi i/6)x\,, \qquad x\to 1/x$$

leaves $\{\alpha_n\}$ invariant and its order is 24. Thus the hyperelliptic surface defined by

$$y^2=x^{12}-1$$

has the automorphism group of order 48 with signature (2, 4, 12). By the similar way we can show the existence of the following signatures. We shall list up the order N of G, the signature, $\{\alpha_n\}$ and G_0 (the group of linear transformations of the sphere that leaves $\{\alpha_n\}$ invariant.)

N	signature	$\{\alpha_n\}$	G_0
120	(2, 3, 10)	vertices of icosahedron	icosahedral group I
48	(2, 4, 12)	$\exp(\pi in/6)$ $(n=0, 1, \cdots, 11)$	dihedral group D_{12}
40	(2, 4, 20)	$0, \infty, \exp(\pi in/5)$ $(n=0, 1, \cdots, 9)$	dihedral group D_{10}
24	(2, 12, 12)	$\exp(\pi in/6)$ $(n=0, 1, \cdots, 11)$	cyclic group Z_{12}
24	(4, 4, 6)	$\exp(\pi in/6)$ $(n=0, 1, \cdots, 11)$	dihedral group D_6
24	(2, 2, 3, 3)	12 points invariant by T	tetrahedral group T
22	(2, 11, 22)	$0, \exp(2\pi in/11)$ $(n=0, 1, \cdots, 10)$	cyclic group Z_{11}
20	(2, 20, 20)	$0, \infty, \exp(\pi in/5)$ $(n=0, 1, \cdots, 9)$	cyclic group Z_{10}
20	(4, 4, 10)	$0, \infty, \exp(\pi in/5)$ $(n=0, 1, \cdots, 9)$	dihedral group D_5
12	(2, 3, 4, 4)	$\exp(2\pi in/3)/2$, $\exp(2\pi in/3)$, $2\exp(2\pi in/3)$ $(n=0, 1, 2)$	dihedral group D_3

Finally we show the existence of the signature (6, 12, 12). On the surface defined by

$$y^{12}=x(x-1),$$

let h be the automorphism

$$h(x, y)=(x, \exp(\pi i/6)y).$$

Then $\langle h\rangle$ is a group with signature (6, 12, 12). The existence of the group of order 60 with signature (3, 3, 5) is shown later in § 5.

§ 5. The existence of non-hyperelliptic surfaces.

According to Wiman [11], there exist the automorphism groups of orders 192, 160, 96 and 64. The signature of the group of order 192 is (2, 3, 8). Then there are a Fuchsian triangle group Γ with signature (2, 3, 8) and the normal subgroup K of Γ of index 192 without elliptic elements such that G is isomorphic to Γ/K. We construct the non-Euclidean triangle ABC in the unit disk in w-plane, as follows. The angles at the vertices A, B and C are $\pi/8$, $\pi/2$ and $\pi/3$, respectively. Put A at the origin $w=0$, B on the non-Euclidean half line $\{\arg w=0\}$ and C on the half line $\{\arg w=\pi/8\}$. We define a by the rotation at A of angle $\pi/4$, b by the elliptic transformation with fixed points B and B^* (the inverse point of B with respect to the unit circle) of angle π and c by the elliptic transformation with fixed points C and C^* of angle $2\pi/3$. Then $a^8=b^2=c^3=abc=id$, and Γ is generated by a, b and c. If we put $\bar{a}$, $\bar{b}$ and $\bar{c}$ the K cosets of a, b and c, respectively, then $G=\langle\bar{a}, \bar{b}\rangle$. a^2 is the rotation at the origin of angle $\pi/2$. Then $\langle a^2, c\rangle$ is a Fuchsian group whose fundamental region has the non-Euclidean area twice that of Γ. Then $\langle\bar{a}^2, \bar{c}\rangle$ is the automorphism group of order 96 with signature (3, 3, 4). ba^2b is the elliptic transformation of angle $\pi/2$ with fixed points $b(A)$ and $b(A)^*$. Then $\langle a, ba^2b\rangle$ is a Fuchsian group whose fundamental region has the non-Euclidean area thrice that of Γ. Then $\langle\bar{a}, \bar{b}\bar{a}^2\bar{b}\rangle$ is the automorphism group of order 64 with signature (2, 4, 8). ba^4b is the elliptic transformation of angle π with fixed points $b(A)$ and $b(A)^*$. Then $\langle a, ba^4b\rangle$ is the automorphism group of order 32 with signature (2, 8, 8). $baba^4bab$ is the elliptic transformation of angle $\pi/4$ with fixed points $ba^4b(A)$ and $ba^4b(A)^*$. Then $\langle\bar{a}, \bar{b}\bar{a}\bar{b}\bar{a}^4\bar{b}\bar{a}\bar{b}\rangle$ is the automorphism group of order 16 with signature (4, 8, 8). The signature of the group of order 160 is (2, 4, 5). This group is isomorphic to Γ/K, where Γ is a Fuchsian group with signature (2, 4, 5): $\Gamma=\langle a, b, c\,|\,a^5=b^2=c^4=abc=id\rangle$. Then $\langle\bar{a}, \bar{c}^2\rangle$ is the automorphism group of order 80 with signature (2, 5, 5). Now we show the signature of the group of order 96 in Wiman's paper [11] is (2, 4, 6). We shall show later in § 6, that the group of order 96 with signature (2, 3, 12) does not exist. So the signature of the group of order 96 must be (3, 3, 4) or (2, 4, 6). On the curve in Wiman's paper:

$$\begin{cases} x_1^2+x_4^2+x_5^2=0 \\ x_2^2+jx_4^2+j^2x_5^2=0 \qquad (j^3=1) \\ x_3^2+j^2x_4^2+jx_5^2=0, \end{cases}$$

the points ${}^t(1, j, j^2, 0, \pm i)$ are the fixed points of the linear transformation of P^4:

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & -j^2 \end{bmatrix}$$

of order 6. Then the automorphism group of this curve has the signature (2, 4, 6). Finally we show the existence of (2, 6, 15) and (3, 15, 15). On the surface defined by

$$y^3=x^2(x^5-1),$$

put $h_1(x, y)=(\exp(2\pi i/5)x, \exp(4\pi i/15)y)$ and $h_2(x, y)=(1/x, -y/x^3)$. Then $\langle h_1\rangle$, $\langle h_1, h_2\rangle$ are the automorphism groups with signature (3, 15, 15), (2, 6, 15), respectively. Here, we show the existence of (3, 3, 5) described in § 4. The group of order 120 with signature (2, 3, 10) is isomorphic to Γ/K, where Γ is a Fuchsian group with signature (2, 3, 10): $\Gamma=\langle a, b, c \mid a^{10}=b^2=c^3=abc=id\rangle$. Then $\langle \bar{a}^2, \bar{c}\rangle$ is an automorphism group of order 60 with signature (3, 3, 5).

§ 6. The non-existence of signatures.

The universal covering surface of S is the unit disk $D=\{|w|<1\}$. Let $p\in S$ be a branch point of π of multiplicity ν, and ϕ the projection such that $\phi(0)=p$. A generator h of the stabilizer subgroup of p is lifted to the rotation

$$w\to\exp(2\pi i/\nu)w .$$

Then, the Dirichlet region F_K of K centered at 0 (i.e. $F_K=\{w \mid d(0, w)\leqq d(\tau 0, w), \tau\in K\}$, where $d(\,,\,)$ denotes the non-Euclidean distance in D) is symmetric with respect to the rotation $w\to\exp(2\pi i/\nu)w$. Now there is a Fuchsian group Γ such that G is isomorphic to Γ/K. So F_K is a finite union of F_Γ the Dirichlet region of Γ. The number of F_Γ's in one F_K is equal to N. Since F_K is symmetric with respect to the rotation $w\to\exp(2\pi i/\nu)w$, there are N/ν F_Γ's in the region $0\leqq\arg w<2\pi/\nu$. Using this fact, for example, (3, 3, 11) does not exist. If such a signature existed, the order of the automorphism group would be 33. Three (=33/11) fundamental regions of a Fuchsian group with signature (3, 3, 11) do not form one eleventh part of the fundamental region of any Fuchsian group since the angle at a vertex of a fundamental region must be $2\pi/m$, where m is an integer. In the same way, we find that (2, 5, 10), (3, 3, 11), (3, 3, 15), (3, 5, 5) and (5, 5, 5) do not exist. Next, we show that (5, 5, 15) and (2, 2, 4, 12) do not exist. If these signatures existed, the automorphism group would be cyclic. But these signatures do not satisfy the l. c. m. condition. Furthermore, (2, 3, 12), (2, 3, 22) and (3, 4, 12) do not exist. The surface having an automorphism of order 12 or 22 is conformally equivalent to the hyperelliptic surface defined by

$$y^2=x^{12}-1 \quad \text{or} \quad y^2=x(x^{11}-1),$$

respectively, on which 12 Weierstrass points exist. If $p\in S$ is a Weierstrass point, every point in G-orbit of p is also a Weierstrass point. Then for each signature (2, 3, 12), (2, 3, 22) and (3, 4, 12) the number of Weierstrass points should be represented as $96m+48\varepsilon_1+32\varepsilon_2+8\varepsilon_3$, $66m+33\varepsilon_1+22\varepsilon_2+3\varepsilon_3$, and $24m+8\varepsilon_1+6\varepsilon_2+2\varepsilon_3$, respectively, where m is an integer and $\varepsilon_j=0$ or 1 ($j=1, 2, 3$). But 12 cannot be represented in these ways. Then (2, 3, 12), (2, 3, 22) and (3, 4, 12) do not exist.

Finally we show the non-existence of (2, 5, 6). Suppose that the group of order 60 with signature (2, 5, 6) exists. 10 branch points of multiplicity 6 is regarded as branch points of multiplicity 3. The signature of the cyclic group generated by an automorphism of order 3 is (0 ; 3, 3, 3, 3, 3, 3, 3) or (1 ; 3, 3, 3, 3). If the signature were the former, the surface would be conformally equivalent to the surface defined by

$$y^3=(x-\alpha_1)^2(x-\alpha_2)^2(x-\alpha_3)\cdots(x-\alpha_7),$$

where $\alpha_1, \cdots, \alpha_7$ are distinct complex numbers. But the Weierstrass gap sequence at $(\alpha_1, 0)$ and at $(\alpha_3, 0)$ are different. This contradicts that 10 branch points are equivalent under the group. Therefore, the signature of the cyclic group of order 3 must be (1 ; 3, 3, 3, 3). If two automorphism groups $\langle h_1\rangle$ and $\langle h_2\rangle$ of order 3 have a common fixed point then $\langle h_1\rangle=\langle h_2\rangle$. So the branch points of multiplicity 3 are divided into equivalence classes, and each class consists of 4 points. But 10 is not divisible by 4. Thus, (2, 5, 6) does not exist.

By virture of the existence of the group of order 64 with signature (2, 4, 8), for $k\equiv 0 \pmod 8$, $N(5, k)\geqq 64$. And by virture of the existence of the group of order 48 with signature (2, 4, 12), for $k\equiv 0, 4 \pmod{12}$, $N(5, k)\geqq 48$. So it is not necessary to consider the groups of order 48 with signatures (2, 6, 6), (3, 3, 6), (3, 4, 4) and (2, 2, 2, 3). Similarly, by virture of the existence of the signatures shown in §§ 4, 5, it is not necessary to consider the following signatures.

48 (2, 6, 6)	48 (3, 3, 6)	48 (3, 4, 4)	48 (2, 2, 2, 3)
32 (4, 4, 4)	32 (2, 2, 2, 4)	24 (3, 6, 6)	24 (2, 2, 2, 6)
20 (2, 2, 2, 10)	16 (2, 2, 4, 4)	16 (2, 2, 2, 2, 2)	16 (1 ; 2)
12 (2, 2, 6, 6)	12 (2, 3, 3, 6)	12 (3, 3, 3, 3)	12 (2, 2, 2, 2, 3)
12 (1 ; 3)	11 (11, 11, 11)	10 (2, 2, 10, 10)	10 (2, 2, 2, 2, 5)
10 (1 ; 5)			

Summing up, we obtain our theorem.

REFERENCES

[1] HARVEY, W. J., Cyclic groups of automorphisms of a compact Riemann surface. Quart. J. Math. Oxford (2), **17** (1966), 86-97.

[2] HEINS, M., On the number of 1-1 directly conformal maps which a multiply-connected plane region of finite connectivity $p(>2)$ admits onto itself. Bull. Amer. Math. Soc. **52** (1946), 454-457.

[3] KATO, T., On the number of automorphisms of a compact bordered Riemann surface. Kōdai Math. Sem. Rep. **24** (1972), 224-233.

[4] KATO, T., On the order of automorphism group of a compact bordered Riemann surface of genus four. Kōdai Math. J. **7** (1984), 120-132.

[5] MACBEATH, A. M., Discontinuous groups and birational transformations. Proc. Dundee Summer School, (1961).

[6] NAKAGAWA, K., On the orders of automorphisms of a closed Riemann surface. Pasific J. Math. **115**, No. 2 (1984), 435-443.

[7] OIKAWA, K., Note on conformal mappings of a Riemann surface onto itself. Kōdai Math. Sem. Rep. **8** (1956), 23-30, 115-116.

[8] TSUJI, R., On conformal mappings of a hyperelliptic Riemann surface onto itself. Kōdai Math. Sem. Rep. **10** (1958), 127-136.

[9] TSUJI, R., Conformal automorphisms of a compact bordered Riemann surface of genus 3. Kōdai Math. Sem. Rep. **27** (1976), 271-290.

[10] WIMAN, A., Über die hyperelliptischen Curven und diejenigen vom Geschlechte $p=3$ welche eindeutigen Transformationen in sich zulassen. Bihang Till. Kongl. Svenska Veienskaps-Akademiens Hadlinger **21** (1895-96), 1-23.

[11] WIMAN, A., Über die algebraischen Curven von den Geschlechten $p=4$, 5 und 6 welche eindeutige Transformationen in sich besitzen. Bihang Till. Kongl. Svenska Veienskaps-Akademiens Hadlinger **21** (1895-96), afd 1, no. 3, 41 pp.

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