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GROUPS OF MOTIONS AND MINIMAL IMMERSIONS OF SPHERES INTO SPHERES

Dedicated to Professor Kentaro Yano on his 75th birthday

BY YOSIO MUTŌ

I. Introduction.

Let f be an isometric minimal immersion: $S^m(1)\to S^{n-1}(r)$ and let g be a rotation: $S^m(1)\to S^m(1)$. Then $f\circ g: S^m(1)\to S^{n-1}(r)$ is also an isometric minimal immersion. Though the point set $f\circ g(S^m(1))$ is equal to the point set $f(S^m(1))$, $f\circ g$ is not in general the same with f. If $f\circ g$ is equivalent to f in the sense of do Carmo and Wallach [1], then we say that f is a g-invariant immersion. In general there exist isometric minimal immersions which are not g-invariant if $m\geqq 3$.

Let K be a skew $(m+1)\times(m+1)$ matrix of R^{m+1} in which $S^m(1)$ is embedded as the unit hypersphere and k be the one-parameter subgroup of SO$(m+1)$ generated by K, so that $k(t)=e^{Kt}$. If, for every t, $f\circ k(t)$ is equivalent to f, we say that f is K-invariant or k-invariant. The element C of W_2 (see [2], [3]) associated with f is then said to be K-invariant or k-invariant.

This notion can be extended as follows.

Let $K_1, \cdots, K_p$ be skew $(m+1)\times(m+1)$ matrices and k be the subgroup of SO$(m+1)$ generated by $K_1, \cdots, K_p$. If C is invariant by $K_1, \cdots, K_p$, C is said to be $(K_1, \cdots, K_p)$-invariant or k-invariant.

Then there arise, for example, the following problems.

(α) Find all continuous subgroups k of SO$(m+1)$ such that there exist non-trivial k-invariant elements of W_2.

(β) Find the set $\mathfrak{C}$ of elements C of W_2 such that, if C is an element of $\mathfrak{C}$, then there exists a non-trivial continuous subgroup of SO$(m+1)$ which leaves C invariant.

(γ) When a subgroup k of SO$(m+1)$ is given, find all k-invariant elements C of W_2.

(δ) When an element C of W_2 is given, find all subgroups k of SO$(m+1)$ such that C is k-invariant.

The purpose of the present paper is, on the one hand, to study such problems in some easier cases. On the other hand the paper contains some theorems on geodesics. Geodesics in minimal immersions were studied by some authors.

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For example, it has been reported by K. Tsukada that a standard minimal immersion, namely an isometric minimal immersion such that the corresponding C vanishes, is a helical immersion [5], [6], that is, an immersion such that every geodesic has constant curvatures which do not depend on the choice of the geodesic. In general a geodesic in a non-standard isometric minimal immersion has curvatures which are not constant and depend on the choice of the geodesic. But, in some isometric minimal immersions all geodesics have constant curvatures and there exists some set of geodesics such that each member of the set has constant curvatures which are independent of the choice of the member.

First we recollect the general theory of isometric minimal immersions such as was given in [1], [2], [3], [4]. In §2 K-invariant elements C of W_2 are defined and their property is studied. In §3 k-invariant isometric minimal immersions are studied. §4 is devoted to isometric minimal immersions of S^3 into S^{24}. First, some results in [3] are recollected and then invariant elements of $W_2(3, 4)$ are studied. In §5 some special elements of W_2 with $m=3$, $s>4$ are studied. We define there contraction of elements of W_2 and the effect of contraction is studied. §6 is devoted to the property of geodesics in the case $S^3 \to S^{24}$.

Isometric minimal immersions f of spheres into spheres in general cases were studied by M. P. do Carmo and N. R. Wallach [1]. Guided by this study the present author also studied the same subject in the form $f: S^m(1) \to S^{n-1}(r)$ using a different method and found that there exists a linear space $D^m_{s,s}$ of some bi-symmetric harmonic tensors of bi-degree (s, s) [2], [4]. As this space is essentially the space W_2 of do Carmo and Wallach, we prefer the notation $W_2(m, s)$ to $D^m_{s,s}$.

We consider isometric minimal immersions f of the form $S^m(1) \to S^{n-1}(r)$ where

$$n=(2s+m-1)(s+m-2)!/(s!(m-1)!)$$

and the radius r satisfies

$$r^2=m/(s(s+m-1)).$$

As the number s plays an important role, the immersion may be denoted by f_s. $S^m(1)$ is considered as the unit hypersphere of R^{m+1} where an orthonormal basis $\{e_1, \cdots, e_{m+1}\}$ is fixed and $S^{n-1}(r)$ is considered as a hypersphere with center at the origin of R^n where an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ is fixed. Thus f_s is given by n homogeneous harmonic polynomials f^A $(A=1, \cdots, n)$ of degree s, so that, if x is a point of $S^m(1)$ and $u(x)=u^\iota(x)e_\iota$ is the position vector in R^{m+1} of x, then we have $i \circ f_s(x)=f^A(u(x))\tilde{e}_A$, where i is the isometric embedding $i: S^{n-1}(r) \to R^n$.

From the polynomials f^A we get symmetric tensors F^A of degree s in R^{m+1} satisfying $f^A(u)=F^A(u, \cdots, u)$ and

$$\sum_\iota F^A(e_\iota, e_\iota, v, \cdots, v)=0,$$

where $\sum_i$ stands for $\sum_{i=1}^{m+1}$ and v is an arbitrary vector of R^{m+1}. F^A are called tensors of degree s associated with f_s.

From the tensors F^A we defined in [2] a tensor $f_{s,s}$ of degree $2s$ by

$$f_{s,s}=\sum_A F^A \otimes F^A \tag{1.1}$$

where $\sum_A$ stands for $\sum_{A=1}^{n}$.

As a special case we have a standard minimal immersion h_s. In this special case F^A is denoted by H^A and $f_{s,s}$ by $h_{s,s}$, hence

$$h_{s,s}=\sum_A H^A \otimes H^A. \tag{1.2}$$

The tensors F^A and H^A depend on the choice of the orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, but $f_{s,s}$ and $h_{s,s}$ do not. Moreover, if f_s and f'_s belong to the same equivalence class in the sense of do Carmo and Wallach, then $f_{s,s}=f'_{s,s}$ [2]. Hence there exists only one $h_{s,s}$.

In [2] the role of the tensor C defined by

$$C=f_{s,s}-h_{s,s} \tag{1.3}$$

is given. If we take all isometric minimal immersions f_s and all numbers $t \in R$, then $t(f_{s,s}-h_{s,s})$ fill a linear space now denoted by $W_2(m, s)$. As there exists only one $h_{s,s}$, the tensor C is called the element of $W_2(m, s)$ associated with the given immersion f_s. When an element C of $W_2(m, s)$ is taken arbitrarily, it may happen that there exist no f_s satisfying (1.3). But taking a number t suitably, we have f_s satisfying $tC=f_{s,s}-h_{s,s}$. Any such f_s is called an immersion subject to C.

A necessary and sufficient condition for a bi-symmetric tensor C of bi-degree (s, s) to be an element of $W_2(m, s)$ is that C satisfies for any vectors a, b of R^{m+1}

$$\sum_i C(e_i, e_i, a, \cdots, a; b, \cdots, b)=0,$$
$$C(a, a, b, \cdots, b; b, \cdots, b)=0.$$

DEFINITION 1.1. When C is an element of $W_2(m, s)$ we define $C(p, q)$ as the function $C(p, q): R^{m+1} \times R^{m+1} \to R$ such that

$$a \times b \mapsto C(a, \cdots, a, b, \cdots, b; a, \cdots, a, b, \cdots, b)$$

where in the right hand side a appears p times before the semicolon and q times after the semicolon.

As C is bi-symmetric C is determined when the function $C(s, 0)$ is given, namely, if C_1 and C_2 are elements of $W_2(m, s)$ such that $C_1(a, \cdots, a; b, \cdots, b)=C_2(a, \cdots, a; b, \cdots, b)$ for arbitrary vectors a, b, then $C_1=C_2$.

DEFINITION 1.2. Let C be an element of $W_2(m, s)$ and g be an element of $SO(m+1)$. Then a bi-symmetric tensor A is determined by

$$A(a, \cdots, a; b, \cdots, b)=C(g^{-1}a, \cdots, g^{-1}a; g^{-1}b, \cdots, g^{-1}b)$$

and, as it is easy to verify, A belongs to $W_2(m, s)$. A is called the transform of C by g and is denoted by gC.

Let $\langle , \rangle$ be the ordinary inner product of vectors or of tensors of the same degree. If C_1 and C_2 are elements of $W_2(m, s)$, then we have $\langle gC_1, gC_2\rangle =\langle C_1, C_2\rangle$ [3].

DEFINITION 1.3. If $gC=C$ we say that C is g-invariant.

2. K-invariant elements of $W_2(m, s)$.

Let K be a skew $(m+1)\times(m+1)$ matrix, namely, an element of the Lie algebra of SO$(m+1)$, k be the one-parameter subgroup of SO$(m+1)$ generated by K and put $k(t)=e^{Kt}$.

DEFINITION 2.1. If $C\in W_2(m, s)$ is k-invariant, that is, $k(t)$-invariant for every t, we say that C is K-invariant.

THEOREM 2.1. *An element C of $W_2(m, s)$ is K-invariant if and only if C satisfies, for every vectors a, b of R^{m+1},*

$$C(Ka, a, \cdots, a; b, \cdots, b)+C(a, \cdots, a; Kb, b, \cdots, b)=0. \tag{2.1}$$

Proof. If C is K-invariant, we have

$$C(e^{Kt}a, \cdots, e^{Kt}a; e^{Kt}b, \cdots, e^{Kt}b)=C(a, \cdots, a; b, \cdots, b). \tag{2.2}$$

Differentiating with respect to t and putting $t=0$, we get (2.1). If (2.1) is satisfied for arbitrary vectors a, b, we have, replacing a and b with $e^{Kt}a$ and $e^{Kt}b$,

$$C(Ke^{Kt}a, e^{Kt}a, \cdots, e^{Kt}a; e^{Kt}b, \cdots, e^{Kt}b)$$
$$+C(e^{Kt}a, \cdots, e^{Kt}a; Ke^{Kt}b, e^{Kt}b, \cdots, e^{Kt}b)=0,$$

hence

$$(d/dt)C(e^{Kt}a, \cdots, e^{Kt}a; e^{Kt}b, \cdots, e^{Kt}b)=0.$$

Thus we get (2.2).

As we have $gkg^{-1}gC=gkC=gC$ if C is k-invariant, we have the following theorem.

THEOREM 2.2. *Let C be a k-invariant element of $W_2(m, s)$ and g be an element of* SO$(m+1)$. *Then the transform gC is gkg^{-1}-invariant.*

The following theorem gives a way of constructing a K-invariant element of $W_2(m, s)$ in some cases.

THEOREM 2.3. *Let C be any element of $W_2(m, s)$ and K be such that the one-parameter subgroup k generated by K has τ as a period so that $e^{K\tau}=1$. Then C_K defined by*

$$\tau C_K(a, \cdots, a; b, \cdots, b)=\int_0^\tau C(e^{Kt}a, \cdots, e^{Kt}a; e^{Kt}b, \cdots, e^{Kt}b)dt \tag{2.3}$$

is a K-invariant element of $W_2(m, s)$.

Proof. The integrand is $(e^{-Kt}C)(a, \cdots, a; b, \cdots, b)$ where $e^{-Kt}C$ belongs to $W_2(m, s)$. Hence C_K belongs to $W_2(m, s)$. On the other hand we have, for any number u,

$$\begin{aligned}
&\tau e^{Ku}C_K(a, \cdots, a; b, \cdots, b)\\
&=\tau C_K(e^{-Ku}a, \cdots, e^{-Ku}a; e^{-Ku}b, \cdots, e^{-Ku}b)\\
&=\int_0^\tau C(e^{K(t-u)}a, \cdots, e^{K(t-u)}a; e^{K(t-u)}b, \cdots, e^{K(t-u)}b)dt\\
&=\int_0^\tau C(e^{Kt}a, \cdots, e^{Kt}a; e^{Kt}b, \cdots, e^{Kt}b)dt
\end{aligned}$$

as e^{Kt} has period τ.

It may happen that C_K vanishes.

3. Invariant minimal immersions.

Any element B of the space $B_{s,s}$ of bi-symmetric harmonic tensors of bi-degree (s, s) is determined when the function $B(s, 0): R^{m+1}\times R^{m+1}\rightarrow R$ such that $B(s, 0): a\times b\mapsto B(a, \cdots, a; b, \cdots, b)$ is given. The element U of $B_{s,s}$ defined by

$$\begin{aligned}
&U(a, \cdots, a; b, \cdots, b)\\
&=\langle a, b\rangle^s+a_1\langle a, b\rangle^{s-2}\langle a, a\rangle\langle b, b\rangle+\cdots+a_\sigma\langle a, b\rangle^{s-2\sigma}\langle a, a\rangle^\sigma\langle b, b\rangle^\sigma
\end{aligned} \tag{3.1}$$

where $\sigma=[s/2]$ is the largest integer satisfying $s-2\sigma\geqq 0$ and $a_1, \cdots, a_\sigma$ satisfy $a_0=1$ and

$$(s-2p+2)(s-2p+1)a_{p-1}+2p(2s+m-2p-1)a_p=0 \qquad p=1, \cdots, \sigma$$

acts as the unit element in $B_{s,s}$ [2]. U is expressed in terms of H^A by

$$U=(1/\acute{c})\textstyle\sum_A H^A\otimes H^A \tag{3.2}$$

where $c'=r^2/(1+a_1+\cdots+a_\sigma)$.

In view of (3.1) and

$$\langle e^{Kt}a, e^{Kt}b\rangle=\langle a, b\rangle$$

which is valid for any vectors a, b of R^{m+1}, we have

$$U(e^{Kt}a, \cdots, e^{Kt}a ; e^{Kt}b, \cdots, e^{Kt}b)=U(a, \cdots, a ; b, \cdots, b).$$

This shows the invariant property of U, which in fact follows naturally from the uniqueness of the unit element.

Let f_s be a full isometric minimal immersion and C be the element of $W_2(m, s)$ given by (1.3), namely, $C=\sum_A F^A \otimes F^A - \sum_A H^A \otimes H^A$ where F^A are tensors associated with f_s. When a is given let us denote $k(t)a$ by $a(t)$, hence $a=a(0)$. As we let the vector a move freely on $S^m(1)$, each of $F^A(a(t), \cdots, a(t))$ determines a tensor $F^A(t)$ such that $F^A(a(t), \cdots, a(t))=F^A(t)(a, \cdots, a)$ and $F^A= F^A(0)$. As $k(t)$ is an orthogonal matrix, $F^A(t)$ are harmonic and are the tensors associated with an isometric minimal immersion $f_s(t): S^m(1)\to S^{n-1}(r)$ such that $f_s(t): x \mapsto f_s(k(t)u(x))$, which we can write $f_s(t)=f_s(0)\circ k(t)$. As U is K-invariant, $\sum_A H^A \otimes H^A$ is also K-invariant, and we have

$$\begin{aligned}&\sum_A F^A(t)\otimes F^A(t)-\sum_A H^A \otimes H^A\\ &=\sum_A F^A(t)\otimes F^A(t)-\sum_A H^A(t)\otimes H^A(t)\\ &=k(-t)C.\end{aligned}$$

Though the image of $S^m(1)$ by $i\circ f_s(t)$ is, as a point set, the same with the image of $S^m(1)$ by $i\circ f_s(0)$, $f_s(t)$ is not in general the same with $f_s(0)$. But, if C is K-invariant, namely $k(t)C=C$, then $f_s(t)$ is equivalent to $f_s(0)$ in the sense of do Carmo and Wallach. In this case there exists one and only one isometry $T(t)$ of $S^{n-1}(r)$ such that $f_s(t)=T(t)f_s(0)$ as f_s is full. As we have $f_s(t_1+t_2) =f_s(0)k(t_1+t_2)=f_s(0)k(t_1)k(t_2)=f_s(t_1)k(t_2)$, we get

$$T(t_1+t_2)f_s(0)=T(t_1)f_s(t_2)=T(t_1)T(t_2)f_s(0).$$

Thus we can put $T(t)=e^{Tt}$ and get

$$f_s(t_1)\exp(Kt_2)=\exp(Tt_2)f_s(t_1).$$

If f is an immersion subject to C and $\hat{R}$ is an element of $\mathrm{SO}(n)$, then $\hat{R}f=f'$ is also an immersion subject to C. If C is K-invariant and f is full, f' is also full. Then there exist only one T and only one T' satisfying $f\circ e^{Kt} =e^{Tt}\circ f$, $f'\circ e^{Kt}=e^{T't}\circ f'$, hence

$$e^{T't}=\hat{R}e^{Tt}\hat{R}^{-1}.$$

Thus we have the following theorem.

THEOREM 3.1. *Let K be a skew $(m+1)\times(m+1)$ matrix and C be a K-invariant element of $W_2(m, s)$. Then for each full isometric minimal immersion f subject to C there exists a skew $n\times n$ matrix $T(f)$ satisfying*

$$f\circ e^{Kt}=e^{T(f)t}\circ f$$

and

$$\hat{R}\exp(T(f)t)\hat{R}^{-1}=\exp(T(\hat{R}f)t)$$

where $\hat{R}$ *is any element of* SO(n).

4. Isometric minimal immersions of S^3 into S^{24}.

We recollect some results from [3]. Let us fix an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ in R^4. On the other hand, we take a rectangular coordinate system in R^3 and express a point p of R^3 by $p=(x, y, z)$. If we take linear transformations $J_p=xJ_1+yJ_2+zJ_3$ defined by

$$\begin{aligned} J_pe_1&=-xe_2+ye_3-ze_4\,,\\ J_pe_2&=xe_1-ye_4-ze_3\,,\\ J_pe_3&=-xe_4-ye_1+ze_2\,,\\ J_pe_4&=xe_3+ye_2+ze_1\,,\end{aligned}$$

then J_p is an orthogonal transformation when p is a point of the unit sphere $S^2(1)$ of R^3. J_1, J_2, J_3 satisfy the well-known formula, $J_2J_3=-J_3J_2=J_1$, $J_3J_1=-J_1J_3=J_2$, $J_1J_2=-J_2J_1=J_3$.

Similarly, let $I_p=xI_1+yI_2+zI_3$ be defined by

$$\begin{aligned} I_pe_1&=-xe_2+ye_3+ze_4\,,\\ I_pe_2&=xe_1+ye_4-ze_3\,,\\ I_pe_3&=xe_4-ye_1+ze_2\,,\\ I_pe_4&=-xe_3-ye_2-ze_1\,.\end{aligned}$$

Then I_p is an orthogonal transformation when p is a point of the unit sphere and I_1, I_2, I_3 satisfy $I_2I_3=-I_3I_2=I_1$, $I_3I_1=-I_1I_3=I_2$, $I_1I_2=-I_2I_1=I_3$. Moreover we have

$$J_\kappa I_\lambda=I_\lambda J_\kappa\,, \tag{4.1}$$

$$\textstyle\sum_i\langle J_\kappa e_i, I_\lambda e_i\rangle=0 \tag{4.2}$$

where, here and in the sequel, we use indices $\kappa, \lambda, \mu, \nu=1, 2, 3$.

Denoting the identity transformation in R^4 by J_0 and also by I_0, we have

$$\begin{aligned} J_\mu J_\lambda+J_\lambda J_\mu&=-2\delta_{\mu\lambda}J_0\,,\\ I_\mu I_\lambda+I_\lambda I_\mu&=-2\delta_{\mu\lambda}I_0\,.\end{aligned} \tag{4.3}$$

Thus $\{aI_0+bI_1+cI_2+dI_3,\ a^2+b^2+c^2+d^2=1\}$ is a subgroup of SO(4) which we write O_I and $\{aJ_0+bJ_1+cJ_2+dJ_3,\ a^2+b^2+c^2+d^2=1\}$ is a subgroup of SO(4) which we write O_J. O_I and O_J commute and generate SO(4).

In what follows we denote the point p by (x^1, x^2, x^3). A homogeneous harmonic polynomial $A(x)$ in R^3 of degree 4 can be written $A(x)=A_{\kappa\lambda\mu\nu}x^\kappa x^\lambda x^\mu x^\nu$ or $A(x)=A^{\kappa\lambda\mu\nu}x^\kappa x^\lambda x^\mu x^\nu$. $A_{\kappa\lambda\mu\nu}$ are symmetric and satisfy

$$\sum_\kappa A_{\kappa\kappa\mu\nu}=0\,.$$

DEFINITION 4.1. The bi-symmetric tensor C given by

$$(C(4,0))(a,b)=A^{\kappa\lambda\mu\nu}\langle J_\kappa b, a\rangle\langle J_\lambda b, a\rangle\langle J_\mu b, a\rangle\langle J_\nu b, a\rangle \tag{4.4}$$

is an element of $W_2(3, 4)$ and is denoted by $C_J^{(A)}$. Similarly the bi-symmetric tensor C given by

$$(C(4,0))(a,b)=B^{\kappa\lambda\mu\nu}\langle I_\kappa b, a\rangle\langle I_\lambda b, a\rangle\langle I_\mu b, a\rangle\langle I_\nu b, a\rangle \tag{4.5}$$

is an element of $W_2(3, 4)$ and is denoted by $C_I^{(B)}$. The linear subspace of $W_2(3, 4)$ composed of the elements of the type $C_J^{(A)}$ is denoted by W_J. The subspace W_I is defined similarly.

Every $C_J^{(A)}$ is O_I-invariant and every $C_I^{(B)}$ is O_J-invariant because of (4.1). On the other hand it is written in page 357 of [3] that any element C of $W_2(3, 4)$ can be written in the form

$$C=C_J^{(A)}+C_I^{(B)}. \tag{4.6}$$

Thus, for example, C is J_1-invariant if and only if $C_J^{(A)}$ is J_1-invariant.

J_1 is an element of the Lie algebra of O_J and the infinitesimal action of J_1 on $C_J^{(A)}$ is given in page 356 of [3], or can be easily obtained from (4.3) and (4.4), in the form

$$\begin{aligned}4[C_J^{(A)}(J_1v, v, v, v\,;\ w, w, w, w)+C_J^{(A)}(v, v, v, v\,;\ J_1w, w, w, w)]\\ =C_J^{(A')}(v, v, v, v\,;\ w, w, w, w)\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}A'^{\kappa\lambda\mu\nu}=2(&\delta_2^\kappa A^{3\lambda\mu\nu}+\delta_2^\lambda A^{3\kappa\mu\nu}+\delta_2^\mu A^{3\kappa\lambda\nu}+\delta_2^\nu A^{3\kappa\lambda\mu}\\ &-\delta_3^\kappa A^{2\lambda\mu\nu}-\delta_3^\lambda A^{2\kappa\mu\nu}-\delta_3^\mu A^{2\kappa\lambda\nu}-\delta_3^\nu A^{2\kappa\lambda\mu})\,.\end{aligned} \tag{4.8}$$

Thus C is J_1-invariant if and only if $A'^{\kappa\lambda\mu\nu}$ vanish, hence

$$A(x, y, z)=c(8x^4-24x^2(y^2+z^2)+3(y^2+z^2)^2) \tag{4.9}$$

where c is an arbitrary number [3].

In order to get a J-invariant C where

$$J=a_1J_1+a_2J_2+a_3J_3\,,\quad (a_1)^2+(a_2)^2+(a_3)^2=1\,, \tag{4.10}$$

we take a rotation of R^3 such that

$$\begin{aligned}'x&=a_1x+a_2y+a_3z\,,\\ 'y&=m_1x+m_2y+m_3z\,,\\ 'z&=n_1x+n_2y+n_3z\,.\end{aligned} \tag{4.11}$$

Then $C_J^{(B)}$ is J-invariant if and only if B is given by

$$B(x, y, z)=c[8('x)^4-24('x)^2(('y)^2+('z)^2)+3(('y)^2+('z)^2)^2], \tag{4.12}$$

where c is an arbitrary number. It is clear that B depends only on a_1, a_2, a_3 on account of

$$('y)^2+('z)^2=x^2+y^2+z^2-(a_1x+a_2y+a_3z)^2.$$

Similarly an I-invariant C is obtained if I is given by

$$I=a_1I_1+a_2I_2+a_3I_3, \quad (a_1)^2+(a_2)^2+(a_3)^2=1 \tag{4.13}$$

in the form $C=C_I^{(B)}$ where B is given by (4.12).

Thus we get the following theorems (see [3]).

THEOREM 4.1. *Let J (resp. I) be given by* (4.10) (*resp.* (4.13)). *Then any J-invariant (resp. I-invariant) element C of $W_2(m, s)$ is given by*

$$C=C_I^{(A)}+C_J^{(B)} \quad \textit{resp. } C=C_J^{(A)}+C_I^{(B)}$$

where A is an arbitrary homogeneous harmonic polynomial in R^3 of degree 4 *and B is given by* (4.11) *and* (4.12).

THEOREM 4.2. *Let us take J and I such that*

$$J=a_1J_1+a_2J_2+a_3J_3, \quad (a_1)^2+(a_2)^2+(a_3)^2=1, \tag{4.14}$$

$$I=b_1I_1+b_2I_2+b_3I_3, \quad (b_1)^2+(b_2)^2+(b_3)^2=1 \tag{4.15}$$

and consider the subgroup G of SO(4) *generated by J and I. Then any G-invariant C is given by*

$$C=aC_J^{(A)}+bC_I^{(B)}$$

where a and b are arbitrary numbers and

$$\begin{aligned} A(x, y, z)=&8(a_1x+a_2y+a_3z)^4-24(a_1x+a_2y+a_3z)^2\\ &\times(x^2+y^2+z^2-(a_1x+a_2y+a_3z)^2)\\ &+3(x^2+y^2+z^2-(a_1x+a_2y+a_3z)^2)^2, \end{aligned} \tag{4.16}$$

$$\begin{aligned} B(x, y, z)=&8(b_1x+b_2y+b_3z)^4-24(b_1x+b_2y+b_3z)^2\\ &\times(x^2+y^2+z^2-(b_1x+b_2y+b_3z)^2)\\ &+3(x^2+y^2+z^2-(b_1x+b_2y+b_3z)^2)^2. \end{aligned} \tag{4.17}$$

DEFINITION 4.2. A one-parameter subgroup of O_J is called a J-type subgroup of SO(4). Similarly an I-type subgroup is defined. A one-parameter subgroup of SO(4) which is not J-type nor I-type is called a mixed type subgroup.

As the dimension of the space of homogeneous harmonic polynomials in R^3

of degree 4 is 9, we have the following theorem.

THEOREM 4.3. *Let k be a one-parameter subgroup of* SO(4). *Then there exist k-invariant elements of $W_2(3, 4)$. If k is of mixed type, then the set of k-invariant elements is a two-dimensional linear subspace of $W_2(3, 4)$ spanned by a one-dimensional linear subspace of W_J and a one-dimensional linear subspace of W_I. If k is of J-type or of I-type, then the set of k-invariant elements is a ten-dimensional linear subspace of $W_2(3, 4)$ containing W_I or W_J.*

Now let us take two mixed type one-parameter subgroups k_1, k_2 of SO(4) and let G be the subgroup generated by k_1, k_2. The Lie algebra of G is denoted by $\mathfrak{g}$. Taking a suitable coordinate system of R^3 we can consider that $\mathfrak{g}$ is generated by one of the following pairs of elements of $\mathfrak{g}$

(i) $J_1+a_1I_1+a_2I_2+a_3I_3$, $b_1I_1+b_2I_2+b_3I_3$,
(ii) $J_1+a_1I_1+a_2I_2+a_3I_3$, $J_2+b_1I_1+b_2I_2+b_3I_3$,
(iii) $I_1+a_1J_1+a_2J_2+a_3J_3$, $b_1J_1+b_2J_2+b_3J_3$,
(iv) $I_1+a_1J_1+a_2J_2+a_3J_3$, $I_2+b_1J_1+b_2J_2+b_3J_3$.

But, in order to find the property of G, we need to consider only (i) and (ii).

We take vectors a and b of R^3 with components (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively. We consider also that I_1, I_2, I_3 are the components of an imaginary vector I. Then we can write (i) and (ii) tersely in the form

(i)′ $K_1=J_1+(a, I)$, $K_2=(b, I)$ $(a, b\neq 0)$,
(ii)′ $K_1=J_1+(a, I)$, $K_2=J_2+(b, I)$ $(a\neq 0)$

where $(\ ,\ )$ is the inner product in R^3. Let us use notations $[a, b]$ and $[a, b, c]$ used in vector algebra of R^3.

If we have (i)′, then $\mathfrak{g}$ contains (c, I) where

$$c=[a, b].$$

The simplest case occurs when the vectors a and b are not linearly independent. Then $\mathfrak{g}$ is spanned by J_1 and (a, I). If a and b are linearly independent, we have $K_3=(c, I)\in\mathfrak{g}$ and

$$\frac{1}{2}(K_1K_3-K_3K_1)=([a, c], I)=(a, b)(a, I)-(a, a)(b, I)\in\mathfrak{g}$$

$$\frac{1}{2}(K_2K_3-K_3K_2)=([b, c], I)=(b, b)(a, I)-(a, b)(b, I)\in\mathfrak{g}.$$

As we have $(a, a)(b, b)>(a, b)^2$ by assumption, we get $(a, I)\in\mathfrak{g}$, $(b, I)\in\mathfrak{g}$, hence $\mathfrak{g}$ contains I_1, I_2, I_3.

If we have (ii)′, we define K_3 by

$$K_3=\frac{1}{2}(K_1K_2-K_2K_1)=J_3+(c, I), \qquad c=[a, b].$$

Then we have

$$K_1-\frac{1}{2}(K_2K_3-K_3K_2)=(a, I)-[b, c, I],$$

$$K_2-\frac{1}{2}(K_3K_1-K_1K_3)=(b, I)-[c, a, I],$$

hence $\mathfrak{g}$ contains

$$K_4=(1-(b, b))(a, I)+(a, b)(b, I),$$

$$K_5=(a, b)(a, I)+(1-(a, a))(b, I).$$

If a and b are not linearly independent, then $c=0$. Hence $\mathfrak{g}$ contains J_3 and consequently J_1 and J_2. Thus $\mathfrak{g}'$ is spanned by J_1, J_2, J_3 and (a, I).

Assume a and b are linearly independent. As we have

$$K_1K_4-K_4K_1=2(a, b)(c, I),$$

$$K_1K_5-K_5K_1=2(1-(a, a))(c, I),$$

we get $(c, I)\in\mathfrak{g}$ except the case $(a, b)=0$, $(a, a)=1$. Similarly, from $K_4K_2-K_2K_4=2(1-(b, b))(c, I)$ we get $(c, I)\in\mathfrak{g}$ if $(b, b)\neq 1$. If $(c, I)\in\mathfrak{g}$, then $\mathfrak{g}$ contains (a, I) and (b, I) and is spanned by $J_1, J_2, J_3, I_1, I_2, I_3$.

If

$$(a, b)=0, \quad (a, a)=1, \quad (b, b)=1 \tag{4.18}$$

is satisfied, then K_4 and K_5 vanish. In this case $\mathfrak{g}$ is spanned by $J_1+(a, I)$, $J_2+(b, I)$, $J_3+(c, I)$ where $[a, b]=c$, $[b, c]=a$, $[c, a]=b$.

To sum up we have the following cases.

(i. 1) a and b are not linearly independent. Then $\mathfrak{g}$ is spanned by J_1 and (a, I).

(i. 2) a and b are linearly independent. Then $\mathfrak{g}$ is spanned by J_1, I_1, I_2, I_3.

(ii. 1) a and b are not linearly independent. Then $\mathfrak{g}$ is spanned by J_1, J_2, J_3, (a, I).

(ii. 2) a and b are linearly independent and (4.18) is satisfied. Then $\mathfrak{g}$ is spanned by $J_1+(a, I)$, $J_2+(b, I)$, $J_3+(c, I)$.

(ii. 3) a and b are linearly independent and (4.18) is not satisfied. Then $\mathfrak{g}$ is spanned by $J_1, J_2, J_3, I_1, I_2, I_3$.

Thus we have the following theorem.

THEOREM 4.4. *Let G be the subgroup of* SO(4) *generated by two one-parameter subgroups k_1, k_2 of mixed type. Then there exist the following four cases.*

(α) *G is generated by a J-type one-parameter subgroup and an I-type one-parameter subgroup.*

(β) *G is generated by a J-type one-parameter subgroup and O_I, or by an*

I-type one-parameter subgroup and O_J.

(γ) *G is generated by* $J_1+(a, I)$, $J_2+(b, I)$, $J_3+(c, I)$ *where a and b satisfy* (4.18) *and* $c=[a, b]$.

(δ) $G=\mathrm{SO}(4)$.

If (γ) or (δ) is the case, then no G-invariant elements of $W_2(3, 4)$ exist except the trivial element 0. If (α) is the case, G-invariant elements are obtained by Theorem 4.2. If (β) is the case, the set of G-invariant elements is a one-dimensional linear subspace of W_J or of W_I and can be obtained with the use of Theorem 4.1.

5. Some invariant elements of $W_2(m, s)$.

Let K be an element of $W_2(m, 4)$. Then $K(3, 0)=0$ [2], [4]. The bi-symmetric tensor given by

$$(C(s, 0))(a, b)=\sum_{p=0}^{\sigma-2} c_p\langle a, a\rangle^p\langle b, b\rangle^p\langle a, b\rangle^{s-2p-4}(K(4, 0))(a, b), \tag{5.1}$$

where $\sigma=[s/2]$ and $c_0, c_1, \cdots, c_{\sigma-2}$ satisfy

$$2p(m+2s-2p-1)c_p+(s-2p-2)(s-2p-3)c_{p-1}=0, \tag{5.2}$$

is an element of $W_2(m, s)$. We have the following theorem.

THEOREM 5.1. *Let k be a subgroup of* $\mathrm{SO}(m+1)$ *and K be k-invariant. Then C is also k-invariant.*

Proof is easy as $\langle ga, gb\rangle=\langle a, b\rangle$ for $g\in\mathrm{SO}(m+1)$.

Remark. It is also easy to prove that C belongs to $W_2(m, s)$. C is bi-symmetric and satisfies $C(2, 0)=0$. That C is harmonic is assured by (5.2).

DEFINITION 5.1. Let C be an element of $W_2(m, s)$. The bi-symmetric tensor C^1 of bi-degree $(s-1, s-1)$ given by

$$C^1(s-1, 0)=\textstyle\sum_i C(a, \cdots, a, e_i; b, \cdots, b, e_i)$$

is called the first contraction of C.

LEMMA 5.2. C^1 *is an element of* $W_2(m, s-1)$.

Proof. As C is harmonic, C^1 is also harmonic. On the other hand, from $C(a, a, b, \cdots, b; b, \cdots, b)=0$, we get

$$\textstyle\sum_i C(a, a, b, \cdots, b, e_i; b, \cdots, b, e_i)=0,$$

hence $C^1(a, a, b, \cdots, b; b, \cdots, b)=0$.

THEOREM 5.3. *Let k be a subgroup of* $SO(m+1)$ *and C be a k-invariant element of $W_2(m, s)$. Then C^1 is also k-invariant.*

Proof is easy as $\{ge_1, \cdots, ge_{m+1}\}$, where $g\in SO(m+1)$, is also an orthonormal basis.

6. Geodesics.

We consider geodesics in isometric minimal immersions $f: S^3(1)\to S^{24}(r)$, $r^2=1/8$, namely the case $m=3$, $s=4$. A geodesic γ in $S^3(1)$ can be written

$$u(t)=a\cos t+b\sin t \tag{6.1}$$

where a and b are orthonormal vectors in R^4 and t is a parameter such that $u'=du/dt$ is a unit vector. $\Gamma=i\circ f(\gamma)$ is a geodesic in the image $i\circ f(S^3(1))$ and the unit tangent vector of Γ is $i_1^A\tilde{e}_A$ where $i_1^A=dF^A(u, u, u, u)/dt=4F^A(u, u, u, u')$.

Let us define F_p^A ($p=0, 1, 2, 3, 4$) by $F_p^A=F^A(u, \cdots, u')$ where p of u in $F^A(u, u, u, u)$ are replaced with u'. Then, as we have $u''=-u$, we get

$$d(F_p^A)/dt=-pF_{p-1}^A+(4-p)F_{p+1}^A\,, \tag{6.2}$$

hence

$$(F_0^A)''=-4F_0^A+12F_2^A,$$
$$(F_0^A)^{(3)}=-40F_1^A+24F_3^A,$$
$$(F_0^A)^{(4)}=40F_0^A-192F_2^A+24F_4^A,$$
$$(F_0^A)^{(5)}=544F_1^A-480F_3^A=-64(F_0^A)'+20(F_0^A)^{(3)}.$$

Let us define $C_{p,q}$ and $U_{p,q}$, which are obtained from $C(u, u, u, u\,;\, u, u, u, u)$ and $U(u, u, u, u\,;\, u, u, u, u)$ when some of u are replaced with u', by

$$C_{p,q}(u, u')=C(u, \cdots, u'\,;\, u, \cdots, u'),$$
$$U_{p,q}(u, u')=U(u, \cdots, u'\,;\, u, \cdots, u')$$

where in the right hand side of each formula u' appears p times before the semicolon and q times after the semicolon. As it is written in page 347 of [3], C satisfies

$$C_{4,0}=-4C_{3,1}=6C_{2,2}$$

and $C_{p,q}=0$ if $p+q\neq 4$, and this leads to $dC_{p,q}/dt=0$. On the other hand, as U is given by (3.1), $U_{p,q}$ is a constant depending only on p and q, and especially $U_{p,q}=0$ if $p+q$ is an odd number.

The relation between $C_{p,q}$, $U_{p,q}$ and F_p^A, F_q^A is, as it is obtained from (1.3) and (3.2),

$$\textstyle\sum_A F_p^A F_q^A=C_{p,q}(u, u')+c'U_{p,q}(u, u')\,. \tag{6.3}$$

This proves that $\sum_A F_p^A F_q^A$ does not depend on t, but depends only on p, q and $C(a, a, a, a; b, b, b, b)$.

The Frenet formula of Γ is written in the form

$$\begin{aligned} i_1^A &= (F_0^A)', \\ (i_1^A)' &= k_1 i_2^A, \\ (i_2^A)' &= -k_1 i_1^A + k_2 i_3^A, \\ (i_3^A)' &= -k_2 i_2^A + k_3 i_4^A, \\ (i_4^A)' &= -k_3 i_3^A. \end{aligned}$$

First, we have

$$(6.4) \qquad \begin{aligned} (k_1)^2 &= \textstyle\sum_A ((F_0^A)'')^2 \\ &= 16 \textstyle\sum_A (F_0^A)^2 - 96 \sum_A F_0^A F_2^A + 144 \sum_A (F_2^A)^2, \end{aligned}$$

and this proves that $(k_1)'=0$. Next we get

$$(F_0^A)^{(3)} = (i_1^A)'' = k_1 (i_2^A)' = -(k_1)^2 i_1^A + k_1 k_2 i_3^A,$$

hence

$$(k_1 k_2)^2 + (k_1)^4 = \textstyle\sum_A ((F_0^A)^{(3)})^2,$$

which proves $(k_2)'=0$. In this way we also get $(k_3)'=0$, hence the following theorem.

THEOREM 6.1. *Every geodesic Γ in the image $i \circ f(S^3(1))$ has constant curvatures k_1, k_2, k_3 which depend on the choice of the geodesic.*

We can compute straightforwardly the curvatures k_1, k_2, k_3 from the Frenet formula, (6.2) and (6.3) in detail and get as a result the following theorem.

THEOREM 6.2. *The curvatures k_1, k_2, k_3 of any geodesic $\Gamma = i \circ f(\gamma)$ are constants which depend only on $C(a, a, a, a; b, b, b, b)$ if γ is given by* (6.1) *and C is the element of $W_2(3, 4)$ associated with f.*

Let g be an element of SO(4) and take another geodesic $\hat{\gamma} = g\gamma$, where we have $\hat{u}(t) = ga \cos t + gb \sin t$. Then, as a result of Theorem 6.2, we find that the curvatures $\hat{k}_1$, $\hat{k}_2$, $\hat{k}_3$ satisfy $\hat{k}_1 = k_1$, $\hat{k}_2 = k_2$, $\hat{k}_3 = k_3$ if $g^{-1}C = C$.

Thus we have the following theorem.

THEOREM 6.3. *Let f be an isometric minimal immersion $S^3(1) \to S^{24}(r)$, $r^2 = 1/8$, such that the element C associated with f is g-invariant where g is an element of* SO(4). *Then, for any geodesic γ of $S^3(1)$, the geodesics $i \circ f(\gamma)$ and $i \circ f(g\gamma)$ have the same set of curvatures k_1, k_2, k_3.*

As the trivial element 0 of $W_2(3, 4)$ is SO(4)-invariant, this theorem also proves that a standard minimal immersion is a helical immersion [5], [6].

Let J be given by (4.14). When P is a point of $S^3(1)$ we say that the locus $\{e^{Jt}P,\ t\in R\}$ is a J-orbit. As any J-orbit is a geodesic and carried into a J-orbit by any element of O_I, we get the following theorem.

THEOREM 6.4. *Let J be given by* (4.14), *f be an isometric minimal immersion $S^3(1)\to S^{24}(r)$, $r^2=1/8$, subject to $C_J^{(A)}$ where A is given by* (4.16) *and let k_1, k_2, k_3 be the curvatures of the geodesic $\Gamma=i\circ f(\gamma)$ of $i\circ f(S^3(1))$ where γ is the J-orbit passing a point a of $S^3(1)$. Then each of these curvatures is independent of the choice of the point a.*

The theorem is valid when we take I instead of J.

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