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ON THE NUMBER OF BRANCHES OF A PLANE CURVE GERM

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1. Introduction

The topology of the zero locus $f^{-1}(0)$ of a smooth map-germ $f:(R^n, 0)\to(R^p, 0)$ is one of the most important and interesting problems in singularity theory. Moreover it determines the homological type of f (see § 3). In this note we study the simplest case of smooth functions of two variables.

Let $f:(R^2, 0)\to(R, 0)$ be a C^∞ function with an isolated critical point 0. For a small positive number $\varepsilon>0$, set

$$B_\varepsilon=\{(x, y):\ x^2+y^2\leqq\varepsilon\}$$
$$S_\varepsilon^1=\{(x, y):\ x^2+y^2=\varepsilon\}.$$

The connected components of the set $B_\varepsilon\cap f^{-1}(0)-\{0\}$ are called branches of $f^{-1}(0)$. We adopt this definition for our convenience, though it might differ from the usual one, (if it exists.) The number of branches of $f^{-1}(0)$ coincides with the number of the connected components of $S_\varepsilon^1\cap f^{-1}(0)$, and it determines not only the topological type of $f^{-1}(0)$ but also the topological type of f. In this note we give an algebraic formula for the number of branches of a plane curve germ $f^{-1}(0)$.

Given a function $f:(R^2, 0)\to(R, 0)$, set

$$J_f=\begin{vmatrix}\partial f/\partial x & \partial f/\partial y\\ x & y\end{vmatrix}.=\frac{1}{2}\text{ Jacobian }(f,\ x^2+y^2).$$

Then our first result is

THEOREM 1. *Let $f:(R^2, 0)\to(R, 0)$ be a function germ with an isolated critical point 0 such that 0 is also an isolated critical point of J_f. Then we have the number of branches of $f^{-1}(0)=2|\deg(f, J_f)|$, where $|\deg(f, J_f)|$ denotes the absolute value of the topological degree of the mapping $(f, J_f)/\|(f, J_f)\|:S_\varepsilon^1\to S^1$.*

Thanks to Eisenbud and Levine's theorem ([1]), we can calculate $|\deg(f, J_f)|$ in terms of J. Mather's local algebra $Q(f, J_f)$ associated to the map germ $(f, J_f):(R^2, 0)\to(R^2, 0)$:

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$$|\deg(f, J_f)| = \dim_R Q(f, J_f) - 2\dim_R I(f, J_f),$$

where $I=I(f, J_f)$ is an ideal of $Q(f, J_f)$ which is maximal with respect to the property $I^2=0$. Thus we have

THEOREM 2. *If $f:(R^2, 0)\to(R, 0)$ is a C^∞ function germ with an isolated critical point such that $\dim_R Q(f, J_f)<\infty$, then the number of branches of $f^{-1}(0)=2\{\dim_R Q(f, J_f)-2\dim_R I(f, J_f)\}$.*

Our motivation of this study was a desire to discover a generalization of Eisenbud and Levine's beautiful theorem. If one wants to define degree of a map germ $f:(R^n, 0)\to(R^p, 0)$ for a general pair (n, p) of dimensions, there seems to be two ways, the homological one and the homotopical one. In § 3 we discuss the homological version and the importance of the number of connected components of $f^{-1}(0)\cap S_\varepsilon^{n-1}$ is emphasized.

It should be noted that C. T. C. Wall ([3]) proved implicitly that for a C^∞ function germ $f:(R^n, 0)\to(R, 0)$ with an isolated critical point 0, we have

$$\chi(S_\varepsilon^{n-1}\cap f^{-1}(0)) = -\deg(df)+(-1)^{n-1}\deg(df)+2-\chi(S^{n-1}),$$

where $df=(\partial f/\partial x_1, \cdots, \partial f/\partial x_n):(R^n, 0)\to(R^n, 0)$ and χ denotes the Euler characteristic.

In the case where $n=2$, the Euler characteristic of $S_\varepsilon^1\cap f^{-1}(0)$ is the number of branches of $f^{-1}(0)$ in our sense. Therefore ours is not the first formula for the number of branches of plane curve germs. Moreover it is often easier to calculate the degree of $(\partial f/\partial x, \partial f/\partial y)$ than the degree of (f, J_f). However our result possibly has the merit of generalizing in another direction, that is to a formula for the number of branches of $f^{-1}(0)$ for a map germ $f:(R^n, 0)\to(R^{n-1}, 0)$; from our experiments on examples, the following conjecture seems likely to be true.

CONJECTURE. *Let $f=(f_1, \cdots, f_{n-1}):(R^n, 0)\to(R^{n-1}, 0)$ be a map germ with a generic condition. Set*

J_f=the jacobian determinant of the map germ

$$(f_1, \cdots, f_{n-1}, x_1^2+\cdots x_n^2):(R^n, 0)\to(R^n, 0).$$

Then the number of branches of $f^{-1}(0)$ is equal to twice the absolute value of the degree of $(f, J_f):(R^n, 0)\to(R^n, 0)$.

2. Proof of theorem 1.

Theorem 1 follows from the following two lemmas.

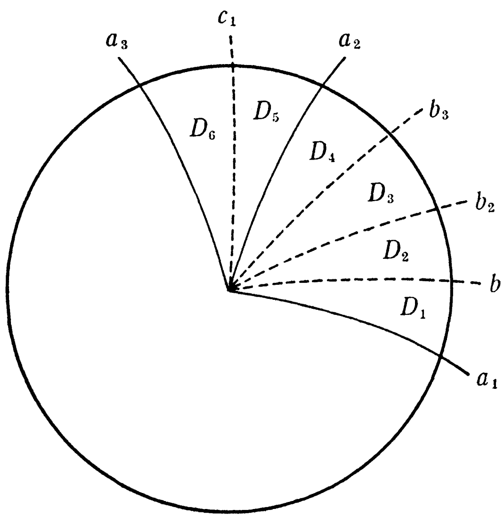
LEMMA 2.1. *Let $f, g:(R^2, 0)\to(R, 0)$ be C^∞ function germs with isolated critical points such that $f^{-1}(0)\cap g^{-1}(0)=\{0\}$. Suppose that for any two adjacent* *branches of* $f^{-1}(0)$, $g^{-1}(0)$ *has an odd number of branches between them. Then we have the number of branches of* $f^{-1}(0)=2|\deg(f, g)|$.

LEMMA 2.2. *Let* $f:(R^2, 0)\to(R, 0)$ *be a* C^∞ *function with an isolated critical point* 0. *Suppose that*

$$J=y\cdot\partial f/\partial x-x\cdot\partial f/\partial y$$

has 0 *as an isolated critical point. Then* $J^{-1}(0)$ *has an odd number of branches between any pair of adjacent branches of* $f^{-1}(0)$.

Proof of Lemma 2.1. Take any branch, say a_1, of $f^{-1}(0)$. Let a_2 be the one next to a_1 anticlockwisely and a_3 the one next to a_2. From the hypothese, there are an odd number of branches of $g^{-1}(0)$, say $b_1, \cdots, b_{2k+1}$, between a_1 and a_2, and also an odd number of branches of $g^{-1}(0)$, say $c_1, \cdots, c_{2m+1}$, between a_2 and a_3. For simplicity, we give a proof for the case where $k=1$ and $m=0$. The proof for the general case is similar. Let $D_1, D_2, \cdots, D_6$ be the regions between the branches of $f^{-1}(0)$ and $g^{-1}(0)$. (See Figure 2.1.)

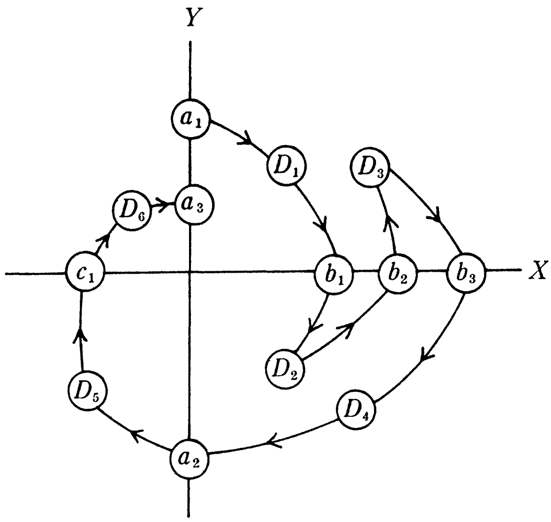


(Figure 2.1)

Let ε be a sufficiently small positive number. Since 0 is an isolated critical point of f, the function $f_\varepsilon(\theta)=f(\varepsilon\cos\theta, \varepsilon\sin\theta)$ changes the sign of its value when the path $\varepsilon e^{i\theta}=(\varepsilon\cos\theta, \varepsilon\sin\theta)$ passes through a branch of $f^{-1}(0)$. Ditto for $g(\varepsilon\cos\theta, \varepsilon\sin\theta)$. Without losing generality, we may suppose $f>0$ and $g>0$ in D_1. Then we have the following table of the signs of values of f and g.

$\varepsilon e^{i\theta}$	a_1	D_1	b_1	D_2	b_2	D_3	b_3	D_4	a_2	D_5	c_1	D_6	a_3
$X=f$	0	+	+	+	+	+	+	+	0	−	−	−	0
$Y=g$	+	+	0	−	0	+	0	−	−	−	0	+	+

From the table, we have the following figure which shows the image of the map $(f(\varepsilon\cos\theta, \varepsilon\sin\theta), g(\varepsilon\cos\theta, \varepsilon\sin\theta))$.



(Figure 2.2)

From the above figure, we see that as the path $(\varepsilon\cos\theta, \varepsilon\sin\theta)$ goes from a_1 to a_3 via a_2, the degree of (f, g) increases by -1. In the above argument it does not matter how many branches $g^{-1}(0)$ has, so long as it has an odd number of branches between each two adjacent branches of $f^{-1}(0)$.

Q. E. D. of Lemma 2.1.

Proof of Lemma 2.2. Now we use the polar coordinates (r, θ):

$$x=r\cos\theta\,,\quad y=r\sin\theta\,.$$

For a small positive number r, we set

$$f_r(\theta)=f(r\cos\theta, r\sin\theta)\,.$$

Then we have

$$df_r/d\theta = \partial f/\partial x \cdot d(r\cos\theta)/d\theta + \partial f/\partial y \cdot d(r\sin\theta)/d\theta$$
$$= -r\sin\theta \cdot \partial f/\partial x + r\cos\theta \cdot \partial f/\partial y$$
$$= -y \cdot \partial f/\partial x + x \cdot \partial f/\partial y = J_f\,.$$

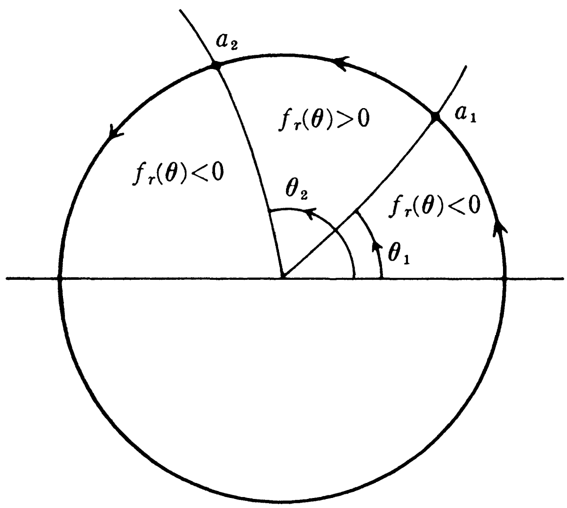
Since 0 is a common isolated critical point of f and J_f, $f_r(\theta)$ is a Morse function of the variable θ and $f_r(\theta)$ changes its sign when the path $(r\cos\theta, r\sin\theta)$ passes through a branch of $f^{-1}(0)$. Thus we have

$$J_f = -df_r/d\theta \neq 0 \quad \text{on} \quad f^{-1}(0)\,.$$

Let

$$a_1 = (r\cos\theta_1, r\sin\theta_1) \quad \text{and} \quad a_2 = (r\cos\theta_2, r\sin\theta_2)$$

be any two points of $f^{-1}(0) \cap \{x^2+y^2=r^2\}$ which are next to each other.

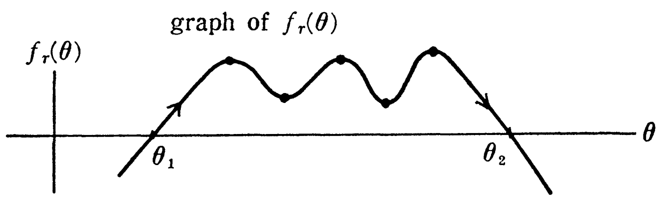


(Figure 2.3)

Since 0 is an isolated critical point of f, the sign of value of $f_r(\theta)$ changes at θ_1 and θ_2 and it does not change between them. We may suppose without losing generality that $f_r(\theta)>0$ if $\theta_1<\theta<\theta_2$. Thus we have

$$df_r/d\theta(\theta_1)>0 \quad \text{and} \quad df_r/d\theta(\theta_2)<0\,.$$

Since $f_r(\theta)$ is a Morse function, it has an odd number of critical points between θ_1 and θ_2. (See the figure below.)



(Figure 2.4)

Since

$$\{\text{critical points of } f_r(\theta)\}=\{\theta \mid df_r/d\theta=0\}$$
$$=\{J=0,\ x^2+y^2=r^2\},$$

the number of branches of $J^{-1}(0)$ between a pair of adjacent branches of $f^{-1}(0)$ is odd. Q. E. D. of Lemma 2.2.

3. Homological behaviors of a map germ

In this section, we attempt to generalize the notion of degree of map germs: $(R^n, 0)\to(R^n, 0)$ to the case of map germ: $(R^n, 0)\to(R^p, 0)$ with $n>p$ from a homological view point. We propose here to define the absolute value $|\deg f|$ of the homological degree of f to be the number of connected components of $f^{-1}(0)\cap S_\varepsilon^{n-1}$.

In the case where $n=p$, a map germ $f:(R^n, 0)\to(R^n, 0)$ such that $f^{-1}(0)=\{0\}$ induces a homomorphism

$$f_*:\ H_{n-1}(R^n-\{0\}\,;\,Z)\cong Z \to H_{n-1}(R^n-\{0\}\,;\,Z)\cong Z\,.$$

The induced homomorphism f_* may be regarded as multiplication in Z by some integer d; $f_*(a)=d.a$. The integer d thus associated to f_* is the degree of $f:(R^n, 0)\to(R^n, 0)$ (see [1]). So, if we want to generalize it to the general case, first we should observe the homology groups of $R^n-f^{-1}(0)$ and $R^p-\{0\}$ and the homomorphisms f_* between them.

Let $f:(R^n, 0)\to(R^p, 0)$ be a C^∞ map germ such that 0 is an isolated singular point of $f^{-1}(0)$. Let D_ε^n and S_ε^{n-1} be the closed n-disk and the $(n-1)$-sphere in R^n centered at 0 with radius ε respectively. Then we must observe the homomorphisms

$$f_*: H_i(D_\varepsilon^n-f^{-1}(0): Z)\to H_i(R^p-\{0\}: Z)\,, \qquad i=0,\ 1,\ \cdots,\ p-1\,.$$

Here we choose a representative mapping of f and denote it also by the same notation f.

Since

1) $$H_i(R^p-\{0\}\,;\,Z)\cong \begin{cases} Z & \text{for } i=0 \text{ and } i=p-1 \\ Z\oplus Z & \text{for } i=0 \text{ and } p=1 \\ 0 & \text{otherwise,} \end{cases}$$

the homomorphisms

$$f_*: H_i(D_\varepsilon^n - f^{-1}(0); Z) \to H_i(R^p - \{0\}; Z)$$

are 0-homomorphisms for $i \neq 0$, $p-1$. Hence it suffices to observe

$$f_*: H_0(D_\varepsilon^n - f^{-1}(0); Z) \to H_0(R^p - \{0\}; Z)$$

and

$$f_*: H_{p-1}(D_\varepsilon^n - f^{-1}(0); Z) \to H_{p-1}(R^p - \{0\}; Z) \quad \text{for} \quad p>1.$$

Now, using the Alexander-Pontrjagin duality and the Poincaré duality, we can express homology groups $H_i(D_\varepsilon^n - f^{-1}(0); Z)$ in terms of $H_j(S_\varepsilon^{n-1} \cap f^{-1}(0); Z)$, in particular we have

2) if $n-p-1>0$ and $p>1$, then

$$\begin{aligned} H_{p-1}(D_\varepsilon^n - f^{-1}(0); Z) &\cong H_{p-1}(S_\varepsilon^{n-1} - f^{-1}(0); Z) \\ &\cong H_0(S_\varepsilon^n \cap f^{-1}(0); Z) \end{aligned}$$

3) if $n-p-1=0$ and $p>1$, then

$$\begin{aligned} H_{p-1}(D_\varepsilon^n - f^{-1}(0); Z) \oplus Z &\cong H_{p-1}(S_\varepsilon^{n-1} - f^{-1}(0); Z) \oplus Z \\ &\cong H_0(S_\varepsilon^{n-1} \cap f^{-1}(0); Z) \end{aligned}$$

Thus the homology groups we want to know turn out to be

$$H_0(S_\varepsilon^{n-1} \cap f^{-1}(0); Z) \quad \text{and} \quad H_0(S_\varepsilon^{n-1} - f^{-1}(0); Z)$$

which are completely determined by the number of connected components of $S_\varepsilon^{n-1} \cap f^{-1}(0)$ and $S_\varepsilon^{n-1} - f^{-1}(0)$ respectively.

Now let us observe

$$f_*: H_0(D_\varepsilon^n - f^{-1}(0); Z) \to H_0(R^p - \{0\}; Z).$$

This homomorphism is rather trivial in the following sense:

a) If $p>1$, then $D_\varepsilon^n - f^{-1}(0)$ is connected and hence

$$f_*; H_0(D_\varepsilon^n - f^{-1}(0); Z) \cong Z \to H_0(R^p - \{0\}; Z) \cong Z$$

may be regarded as the identity map of Z.

b) If $p=1$, then

$$f_*: H_0(D_\varepsilon^n - f^{-1}(0); Z) \to H_0(R - \{0\}; Z) \cong Z \oplus Z$$

is completely determined by the number of connected components of $D_\varepsilon^n \quad f^{-1}(0)$ and the sign of values of f on each component.

Now let us observe

$$f_*: H_{p-1}(D_\varepsilon^n - f^{-1}(0); Z) \to H_{p-1}(R^p - \{0\}; Z) \cong Z.$$

Let K_i, $i=1, 2, m, \cdots, k$, be the connected components of $S_\varepsilon^{n-1} \cap f^{-1}(0)$. Since 0 is an isolated singular point of $f^{-1}(0)$, the restricted mapping $f | S_\varepsilon^{n-1}: S_\varepsilon^{n-1} \to R^p$ is submersive in a neighbourhood of K_i. Hence K_i has a tubular neighbourhood T_i in S_ε^{n-1} which is diffeomorphic to $K_i \times D_\delta^p$ for a small positive number δ. Identifying T_i with $K_i \times D_\delta^p$, the restricted mapping $f | K_i \times D_\delta^p : K_i \times D_\delta^p \to D_\delta^p \subset R^p$ becomes the canonical projection.

Thus and from the above duality isomorphisms 2) and 3), letting q_i be a point of K_i, we see that the homology class

$$[\{q_i\} \times \partial D_\delta^p] \in H_{p-1}(S_\varepsilon^{n-1} - f^{-1}(0); Z)$$
$$= H_{p-1}(D_\varepsilon^n - f^{-1}(0); Z)$$

are the generators of $H_{p-1}(D_\varepsilon^n - f^{-1}(0); Z)$ and also that

$$f_*([\{q_i\} \times \partial D_\delta^p]) = \pm[\partial D_\delta^p] = \pm 1 \in H_{p-1}(R^p - \{0\}; Z).$$

The sign in the above equality depends only on the choice of orientation of $\{q_i\} \times \partial D^p$. Thus we have

PROPOSITION. *The homological behavior of f is completely determined by the homology groups of $S_\varepsilon^{n-1} \cap f^{-1}(0)$ and the orientations. In particular the homomorphism*

$$f_*: H_{p-1}(D_\varepsilon^n - f^{-1}(0); Z) \to H_{p-1}(R^p - \{0\}; Z) \qquad p>1$$

is completely determined by the number of connected components of $S_\varepsilon^{n-1} \cap f^{-1}(0)$ and the orientations.

As a conclusion of this section, we emphasize the importance of the number of connected components of $S_\varepsilon^{n-1} \cap f^{-1}(0)$ which can be regarded as the absolute value $|\deg f|$ of the degree of $f: (R^n, 0) \to (R^p, 0)$. In particular in the case where $p=n-1$, the homological behaviour of f is completely determined by the number of connected components of $S_\varepsilon^{n-1} \cap f^{-1}(0)$ which is equal to the number of branches of $f^{-1}(0)$. Thus we are interested in the conjecture given at the end of § 1.

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