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# CONSTANT MEAN CURVATURE SUBMANIFOLDS OF HIGHER CODIMENSIONS

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## 0. Introduction

Let $\imath : S^2 \to E^3$ be a constant mean curvature immersion. Then $\imath$ must be an imbedding onto the round sphere. This is a well-known theorem of H. Hopf ([5]). In addition, the only closed hypersurface with constant mean curvature in any euclidean space is also the round sphere [1]. These phenomena, of course, are restricted to the case of codimension 1. Once we turn to the case of higher codimensions, the situation is more complicated. First of all, there does not seem to be a unanimously approved concept of constant mean curvature submanifolds. Anyway, it should be related to the mean curvature vector, because the latter appears in the first variation of the volume in the essential way [7].

The simplest concept of constant mean curvature submanifold is the one of submanifolds with $|\mathfrak{h}|=\text{const.}$, where $\mathfrak{h}$ is the mean curvature vector. This is the definition of the present paper. Hence any submanifold of $E^N$ on which a subgroup of $\text{Iso}(E^N)$, the group of euclidean motions, acts transitively is a constant mean curvature submanifold in our sense. Such constant mean curvature submanifolds will be called "of homogeneous type" in what follows.

The purpose of the present paper is to construct examples of constant mean curvature submanifolds of higher codimension which are not of homogeneous type in $E^N$ (with standard metric $g$).

Now let us state the way of constructing such manifolds in a precise manner:

Let $X$ be a $k$-dimensional algebraic manifold and $j : X \to P^n(\boldsymbol{C})$ be a projective embedding of $X$ into an $n$-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 2. Here we furnish the manifold $X$ with the metric induced from the embedding $j$. Let $z_0, \cdots, z_n$ be normal homogeneous coordinates of $z \in P^n(\boldsymbol{C})$ where the word "normal" means that $z_0\bar{z}_0 + \cdots + z_n\bar{z}_n = 1$ holds for them.

Let $S^N(c)$ denote an $N$-dimensional sphere of curvature $c$. We define an isometric imbedding $m$ of $P^n(\boldsymbol{C})$ into the round sphere $S^{n(n+2)}(1)$, in this way: $m(z)$ is the point of $E^{(n+1)^2}$ with cartesian coordinates

$$z_h\bar{z}_h,\ \frac{1}{\sqrt{2}}(z_h\bar{z}_k+\bar{z}_h z_k),\ \frac{\sqrt{-1}}{\sqrt{2}}(z_h\bar{z}_k-\bar{z}_h z_k) \quad (h,\ k=0, \cdots, n : h<k).$$

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Then the (isometric) imbedding $m\circ j$ is of constant mean curvature (not only in $E^{(n+1)^2}$, but also in $S^{n(n+2)}(1)$). Moreover, $D\mathfrak{h}$ does not vanish unless $j(X)$ is a linear subspace of $P^n(\boldsymbol{C})$, where $D$ stands for the usual covariant derivative in the normal bundle of $X$ in $E^{(n+1)^2}$. Hence most of our examples are not constant mean curvature submanifolds in the sense of [9].

An immediate consequence of this fact is that there exists a constant mean curvature imbedding: $S^2\to E^9$ whose image is not the round sphere (e.g., take the Veronese imbedding of degree 2: $P^1(\boldsymbol{C})\to P^2(\boldsymbol{C})$ for $j$).

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**1.** In this section we follow the notations of the introduction and prove the following.

THEOREM. *Let $\mathfrak{h}$ be the mean curvature vector of $m\circ j(X)$ in $E^{(n+1)^2}$. Then $|\mathfrak{h}|=\sqrt{(k+1)/k}$ and $|D\mathfrak{h}|=|B_1|/k$, where $|B_1|$ is the length of the second fundamental form $B_1$ of $X$ in $P^n(\boldsymbol{C})$.*

*Proof.* First of all we note the following: The isometric imbedding $m$ of $P^n(\boldsymbol{C})$ into $E^{(n+1)^2}$ is decomposed into two imbeddings (cf. [8]); $m: P^n(\boldsymbol{C})\xrightarrow{f_1} S^{n(n+2)-1}\left(\frac{n+1}{n}\right)\xrightarrow{f_2} E^{(n+1)^2}$, where $f_1$ is the standard minimal imbedding corresponding to the first eigenvalue of the Laplace-Bertrami operator of $P^n(\boldsymbol{C})$ and $f_2$ is a totally umbilic imbedding.

Now we denote by $B$ the second fundamental form of $P^n(\boldsymbol{C})$ in $E^{(n+1)^2}$ through $m$. By virtue of the above decomposition, $B$ is described as follows (cf. [2]):

$$g(B(X,Y),B(Z,W))=g(X,Y)g(Z,W)+\frac{1}{2}(g(X,W)g(Y,Z)+g(X,Z)g(Y,W)+g(JX,Z)g(JY,W)+g(JY,Z)g(JX,W)), \tag{1}$$

where $X$, $Y$, $Z$ and $W$ are vector fields on $P^n(\boldsymbol{C})$.

From (1) we see that $B(JX,JY)=B(X,Y)$ for all vectors $X$ and $Y$, where $J$ is the complex structure of $P^n(\boldsymbol{C})$.

Due to the above discussion, we find that $P^n(\boldsymbol{C})$ is a parallel submanifold of $E^{(n+1)^2}$ through $m$ (cf. [2]).

We choose a local field of orthonormal frame $e_1,\cdots,e_k$, $e_{k+1}=Je_1,\cdots,e_{2k}=Je_k$ on $X$. Since $X$ is a minimal submanifold of $P^n(\boldsymbol{C})$, we have $\mathfrak{h}=(1/2k)\sum_{i=1}^{2k}B(e_i,e_i)$. Consequently,

$$\mathfrak{h}=(1/k)\sum_{i=1}^{k}B(e_i,e_i) \tag{2}$$

From (1) and (2) we find $|\mathfrak{h}|^2=(k+1)/k$.

Moreover a straightforward calculation yields that

$$D_{e_j}\mathfrak{h}=(2/k)\sum_{i=1}^{k}B(B_1(e_i, e_j), e_i)$$

and

$$D_{Je_j}\mathfrak{h}=(2/k)\sum_{i=1}^{k}B(B_1(e_i, Je_j), e_i) \qquad \text{for} \quad j=1, 2, \cdots, k\,.$$

Then from (1) we obtain

$$|D\mathfrak{h}|^2=\sum_{j=1}^{k}g(D_{e_j}\mathfrak{h}, D_{e_j}\mathfrak{h})+\sum_{j=1}^{k}g(D_{Je_j}\mathfrak{h}, D_{Je_j}\mathfrak{h})=(|B_1|/k)^2,$$

where $|B_1|$ is the length of the second fundamental form $B_1$ of $X$ in $P^n(\boldsymbol{C})$.

Q. E. D.

**2.** In this section, using our theorem, we construct constant mean curvature imbeddings (with mean curvature vector not parallel) of an algebraic manifold $X$ into a euclidean space.

The theorem of the preceding section tells us that our method of constructing constant mean curvature submanifolds in a euclidean space is closely connected with the ways of embedding an algebraic manifold into complex projective spaces.

Let us recall some relevant facts on projective embeddings and construct various (compact) constant mean curvature submanifolds diffeomorphic to a given algebraic manifold $X$.

Let $L$ be a line bundle over $X$. We denote by $E$ the $\boldsymbol{C}$-vector space of holomorphic sections of $L$. Hence $E=H^0(X, L)$ and $\dim_{\boldsymbol{C}}E<+\infty$. We write $P(E)$ for the projective space of all lines in $E$ and $P(E)^*$ for its dual projective space. $E$ is called base-point free if, for every $x\in X$, there is a section $s$ of $E$ with $s(x)\neq 0$. Suppose $E$ is base-point free. Then we can define a holomorphic map $Z_L$ of $X$ into $P(E)^*$ by setting $Z_L(X)=\{s\in E\,;\, s(X)=0\}$ $(\in P(E)^*)$. $Z_L$ becomes a projective embedding if $L$ is very ample.

Write $N=\dim_{\boldsymbol{C}}E$. The problem of determing $N$ is nothing but the Riemann-Roch problem [4]. Since $P(E)^* \xrightarrow{\sim} P^{N-1}(\boldsymbol{C})$, we may consider that $Z_L$ is a map of $X$ into $P^{N-1}(\boldsymbol{C})$. Then $m\circ Z_L(X)$ is a constant mean curvature submanifold in $E^{N^2}$.

Now let us turn to the case that $X$ is a compact Riemann surface of genus $g$ $(\geqq 2)$. Let $L$ be the canonical line bundle of $X$. Then $L$ is very ample and $Z_L$ gives rise to a projective embedding, unless $X$ is hyperelliptic. Suppose that $X$ is non-hyperelliptic. It is known that $\dim_{\boldsymbol{C}}E=g$ ([3]). Hence $m\circ Z_L(X)$ is a constant mean curvature submanifold in $E^{g^2}$ and its mean curvature is $\sqrt{2}$. Now suppose $X$ is embedded in $P^n(\boldsymbol{C})$ and let Sec$(X)$ be the secant variety of $X$. We know $\dim_{\boldsymbol{C}}\text{Sec}(X)\leqq 3$. Hence we can find a point $x$ in $P^n(\boldsymbol{C})-\text{Sec}(X)$, if $n\geqq 4$. Let $p_X$ be the projection with center $X: P^n(\boldsymbol{C})\rightarrow$ a hyperplane. This yields another embedding: $X \to P^{n-1}(\boldsymbol{C})$ and we have another constant mean curvature submanifold. We can proceed consecutively and get one in $E^{16}$. (Actually any compact Riemann surface can be realized as a constant mean curvature surface in $E^4$, as will be seen from Theorem 2, [6].)

Let us take for $L$ the tensor product of the hyperplane bundle of $P^1(\boldsymbol{C})$ with itself $n$ times. Then we have $Z_L: P^1(\boldsymbol{C}) \to P^n(\boldsymbol{C})$, which is defined alternatively by $[1, t] \mapsto [1, t, \cdots, t^n]$. The image is the normal rational curve and the image of $m \circ Z_L$ is not the round sphere.

Finally let $X$ be a torus, i.e., $\boldsymbol{C}/\boldsymbol{Z}\oplus\boldsymbol{Z}$. Take $\tau$ from the upper half-plane $H$ in $\boldsymbol{C}$. Write $X_\tau$ for abelian variety $\boldsymbol{C}/\Gamma_\tau$ where $\Gamma_\tau = \boldsymbol{Z}\oplus\boldsymbol{Z}\tau$. Then we have a natural diffeomorphism $I_\tau: X \xrightarrow{\sim} X_\tau$. Let $\mathfrak{P}(z)$ be the Weierstrass $\mathfrak{P}$-function with period $\Gamma_\tau$. We define $j_\tau(\Gamma_\tau) = [0, 0, 1]$ and $j_\tau(z+\Gamma_\tau) = [1, \mathfrak{P}(z), \mathfrak{P}'(z)]$.

Then $j_\tau: X_\tau \to P^2(\boldsymbol{C})$ is a projective embedding. Let us define a map $\varphi: X \times H \to E^9$ by $\varphi(x, \tau) = m \circ j_\tau \circ I_\tau(x)$. This gives rise to a family of constant mean curvature spaces with mean curvature $\sqrt{2}$ in 9-dimensional real euclidean space.

## References


[1] A.D. Alexandrov, Uniqueness theorems for surfaces in the large, Vestnik Leningrard Univ. **13** (1958).

[2] D. Ferus, Symmetric submanifolds of Euclidean space, Math. Ann. **247** (1980), 81-93.

[3] P. Griffiths and J. Harris, Principles of algebraic geometry, John Wiley Sons, New York (1978).

[4] F. Hirzebruch, Topological methods in algebraic geometry, Grundlehren **131**, Springer-Verlag, Heiderberg (1966).

[5] H. Hopf, Über Flächen mit einer Relation zwischen den Hauptkrümmungen, Math. Nachr. **4** (1950), 232-249.

[6] H.B. Jr. Lawson, Complete minimal surfaces in $S^3$, Ann. of Math. **92** (1970), 335-374.

[7] H.B. Jr. Lawson, Lectures on minimal submanifolds, Publish or Perish (1980).

[8] S. Maeda, Imbedding of a complex projective space similar to Segre imbedding, Archiv der Math. **37** (1981) 556-560.

[9] B. Smyth, Submanifolds of constant mean curvature, Math. Ann. **205** (1973), 265-280.



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