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# NONCOMMUTATIVE EXTENSION OF AN INTEGRAL REPRESENTATION THEOREM OF ENTROPY

Dedicated to Professor H. Umegaki on his 60th birthday

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## Introduction

In 1964, Umegaki proved a theorem of McMillan type concerning the integral representation of entropy in the measure theoretic framework, about which we briefly review in §1. Noncommutative probability theory is important to analyse some physical systems [1, 2, 4, 5, 6, 7, 10, 11, 12, 13, 16, 17]. In this paper, using various results obtained in operator algebras, we extend this theorem to that for noncommutative systems.

## §1. Integral representation of entropy

Let $X$ be a compact metric space and $\mathfrak{B}(X)$ be the $\sigma$-field of all Borel sets in $X$. We denote a homeomorphism on $X$ by $T$ and the set of all $T$-invariant regular probability measures $p, q, \cdots$ on $X$ by $P_T$. Let $\mathcal{P}$ be a finite partition of $X$ and we put $\mathfrak{M}_n=\bigvee_{k=1}^{n} T^{-k}\mathcal{P}$ and $\mathfrak{M}_\infty=\bigvee_{k=1}^{\infty}\mathfrak{M}_k$. Then the entropy of each $p\in P_T$ is defined by

$$S(p)=-\lim\frac{1}{n}\Sigma_U p(U)\log p(U) \qquad (n\to\infty),$$

where $\Sigma_U$ means the summation over $U$ of the atomic sets in $\mathcal{P}\vee\mathfrak{M}_{n-1}$. For any $p\in P_T$, we denote the conditional probability functions of $U\in\mathcal{P}$ with respect to $\mathfrak{M}_n$ and $\mathfrak{M}_\infty$ by $P_p(U|\mathfrak{M}_n)$ and $P_p(U|\mathfrak{M}_\infty)$ respectively. Now we define the $\mathfrak{M}_\infty$-measurable function $h_p(x)$ on $X$ as follows:

$$h_p(x)=-\sum_{U\in\mathcal{P}} P_p(U|\mathfrak{M}_\infty)\log P_p(U|\mathfrak{M}_\infty)(x) \qquad p\text{-a. e.} \quad x\in X,$$

for any $p\in P_T$. Then, the next important theorem [14] of McMillan type holds.

THEOREM 1. *For any finite partition $\mathcal{P}$, there universally exists a Borel measurable function $h(x)$ on $X$ such that it is bounded, non-negative, T-invariant and satisfies*

(1) $h(x)=h_p(x)$ *$p$-a. e. $x\in X$ and for every $p\in P_T$,*

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(2) $S(p)=\int_X h(x)dp(x)$ *for every* $p\in P_T$.

*A typical example of* $X$ *is a compact message space* $A^Z$, *where* $A$ *is a set of some alphabets* [15]. *Then* $T$ *is the shift in* $A^Z$. *This case provides with a concrete description of communication processes.*

## §2. Noncommutative extension

Let $B(\mathcal{H})$ be the set of all bounded operators on a separable Hilbert space $\mathcal{H}$ and let $\mathfrak{N}$ be a von Neumann algebra (i.e. $\mathfrak{N}=(\mathfrak{N}')'$ where $\mathfrak{N}'=\{A\in B(\mathcal{H});\ AB-BA=0$ for any $B\in\mathfrak{N}\}$) acting on $\mathcal{H}$. In noncommutative systems, we use a $*$-automorphism $\alpha$ of $\mathfrak{N}$ instead of $T$. We further denote the set of all finite partitions of $I$ in $\mathfrak{N}$ by $\mathcal{P}(\mathfrak{N})$ (i.e. $\tilde{P}=\{P_j;\ j=1, 2, \cdots, n<\infty\}\in\mathcal{P}(\mathfrak{N})$ satisfies (i) $P_j\perp P_k$ $(k\neq j)$ and (ii) $\sum_{j=1}^{n} P_j=I$).

We denote the set of all normal states on $\mathfrak{N}$ by $\mathfrak{S}(\mathfrak{N})$ and the set of all $\alpha$-invariant states in $\mathfrak{S}(\mathfrak{N})$ by $\mathfrak{S}_I(\alpha)$. We assume that there exists a faithful state in $\mathfrak{S}_I(\alpha)$. Let $\mathfrak{M}$ be a von Neumann subalgebra of $\mathfrak{N}$ including $\mathfrak{N}^\alpha$, where $\mathfrak{N}^\alpha\equiv\{A\in\mathfrak{N};\ \alpha(A)=A\}$, and let $\mathfrak{M}_n$, $\mathfrak{M}_\infty$ be the von Neumann subalgebras generated by $\bigcup_{k=1}^{n}\alpha^k(\mathfrak{M})$, $\bigcup_{k=1}^{\infty}\alpha^k(\mathfrak{M})$ respectively. For each $\varphi\in\mathfrak{S}_I(\alpha)$, we further denote the conditional expectations [10, 17] of $A\in\mathfrak{N}$ with respect to $\mathfrak{M}$ and $\mathfrak{M}_n$ $(\forall n\in N)$ by $E_\varphi(A|\mathfrak{M})$ and $E_\varphi(A|\mathfrak{M}_n)$ respectively. For any faithful $\varphi\in\mathfrak{S}_I(\alpha)$, let $\{\sigma_t^\varphi;\ t\in R\}$ be the modular automorphism group [9, 17] with respect to $\varphi$ at $\beta=1$. We assume that there exists the conditional expectation $E_\varphi(\cdot|\mathfrak{M})$ for $\varphi\in\mathfrak{S}_I(\alpha)$. *We call this assumption "*$\langle A\rangle$*" for* $\varphi$ *in the sequel.*

LEMMA 2. *For any faithful* $\varphi\in\mathfrak{S}_I(\alpha)$ *with* $\langle A\rangle$, *there exists the conditional expectation* $E_\varphi(\cdot|\mathfrak{M}_n)$ *for any* $n\in N$.

*Proof.* For an $\alpha$-invariant state $\varphi$, we have

$$\sigma_t^\varphi\circ\alpha=\alpha\circ\sigma_t^\varphi \qquad \text{for any} \quad t\in R\,.$$

When $n=1$, we obtain $\sigma_t^\varphi(\mathfrak{M}_1)=\sigma_t^\varphi\circ\alpha(\mathfrak{M})=\alpha\circ\sigma_t^\varphi(\mathfrak{M})=\alpha(\mathfrak{M})=\mathfrak{M}_1$. Suppose that $\sigma_t^\varphi(\mathfrak{M}_n)=\mathfrak{M}_n$ holds for $n\in N$. Then

$$\begin{aligned}\sigma_t^\varphi(\mathfrak{M}_{n+1})&=\sigma_t^\varphi(\alpha(\mathfrak{M}_n)\vee\mathfrak{M}_1)\\&=\sigma_t^\varphi\circ\alpha(\mathfrak{M}_n)\vee\sigma_t^\varphi(\mathfrak{M}_1)\\&=\alpha(\mathfrak{M}_n)\vee\mathfrak{M}_1\\&=\mathfrak{M}_{n+1} \qquad \text{for any} \quad t\in R\,.\end{aligned}$$

Therefore there exists the conditional expectation $E_\varphi(\cdot|\mathfrak{M}_n)$ for any $n\in N$.

Q.E.D.

We here remind of two topologies in $B(\mathcal{H})$ [17]: (i) A net $\{A_\alpha\}\subset B(\mathcal{H})$ converges to $A\in B(\mathcal{H})$ in the strong operator topology (write $A_\alpha \xrightarrow{s} A$) if $\|(A_\alpha-A)x\|\to 0$ for any $x\in\mathcal{H}$, (ii) a net $\{A_\alpha\}\subset B(\mathcal{H})$ converges to $A\in B(\mathcal{H})$ in the ultrastrong operator topology (write $A_\alpha \xrightarrow{us} A$) if $\sum_n\|(A_\alpha-A)x_n\|^2\to 0$ for any sequence $\{x_n\}\subset\mathcal{H}$ such that $\sum_n\|x_n\|^2<\infty$.

From the definition of $\mathfrak{M}_n$, $\{\mathfrak{M}_n\}$ is an increasing sequence of von Neumann subalgebras. According to Lemma 2, we have (c.f. [11, 17]).

1° $E_\varphi(A|\mathfrak{M}_n) \xrightarrow{us} E_\varphi(A|\mathfrak{M}_\infty)$ for any $A\in\mathfrak{N}$ and any faithful $\varphi\in\mathfrak{S}_I(\alpha)$ with $\langle A\rangle$.

$\mathfrak{M}$ is said to be a *sufficient* [1, 2, 12, 13] for $\mathcal{S}\subset\mathfrak{S}(\mathfrak{N})$ if $E_\varphi(\cdot|\mathfrak{M})$ exists for each $\varphi\in\mathcal{S}$ and for each $A\in\mathfrak{N}$ there exists an $A_0\in\mathfrak{M}$ such that

$$A_0=E_\varphi(A|\mathfrak{M}) \quad \varphi\text{-a.e.}, \quad \varphi\in\mathcal{S},$$

where $A=B$ $\varphi$-a.e. means $\varphi(|A-B|)=0$. In [3], Nakamura and Umegaki showed that the function $\eta(A)=-A\log A$ for any positive $A\in\mathfrak{N}$ is operator concave. We assume that $\mathfrak{S}_I(\alpha)$ includes a faithful state with $\langle A\rangle$. Using this function $\eta$, we define

$$s_\varphi^{\tilde{P}}(\mathfrak{M}_n)\equiv\sum_j\eta(E_\varphi(P_j|\mathfrak{M}_n))$$

for any finite partition $\tilde{P}=\{P_j\}\in\mathcal{P}(\mathfrak{M})$ and $\varphi\in\mathfrak{S}_I(\alpha)$, which is uniquely determined in the sense of $\varphi$-a.e.. Moreover, we define $S_\varphi^{\tilde{P}}$ as follows: For any finite partition $\tilde{P}=\{P_j\}\in\mathcal{P}(\mathfrak{M})$ and $\varphi\in\mathfrak{S}_I(\alpha)$,

$$S_\varphi^{\tilde{P}}(\mathfrak{M}_n)\equiv\varphi(s_\varphi^{\tilde{P}}(\mathfrak{M}_n))=\sum_j\varphi(\eta(E_\varphi(P_j|\mathfrak{M}_n))).$$

Then the following lemma holds.

LEMMA 3. *For any faithful $\varphi\in\mathfrak{S}_I(\alpha)$ with $\langle A\rangle$, we obtain*

(1) $s_\varphi^{\tilde{P}}(\mathfrak{M}_n) \xrightarrow{s} s_\varphi^{\tilde{P}}(\mathfrak{M}_\infty)$

(2) $S_\varphi^{\tilde{P}}(\mathfrak{M}_n) \longrightarrow S_\varphi^{\tilde{P}}(\mathfrak{M}_\infty) \quad (n\to\infty)$

*for any partition $\tilde{P}\in\mathcal{P}(\mathfrak{M})$.*

*Proof.* It is known that [8] the convergence $A_n \xrightarrow{s} A$ for a bounded sequence $\{A_n\}$ implies $f(A_n) \xrightarrow{s} f(A)$ for any continuous function $f(t)$ such that $f(0)=0$ and $|f(t)|\leqq\alpha|t|+\beta$ with positive constants $\alpha$, $\beta$. Since $\eta(t)$ satisfies the above conditions, we obtain on the support of $\varphi$

$$s_\varphi^{\tilde{P}}(\mathfrak{M}_n) \xrightarrow{s} s_\varphi^{\tilde{P}}(\mathfrak{M}_\infty)$$

for any partition $\tilde{P}\in\mathcal{P}(\mathfrak{M})$ and any $\varphi\in\mathfrak{S}_I(\alpha)$. (2) is immediate for (1).

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THEOREM 4. *We assume that* $\mathfrak{S}_I(\alpha)$ *includes a faithful state with* $\langle A\rangle$. *Then there exists a positive operator* $h(\tilde{P}, \alpha)$ *satisfying*

(1) $h(\tilde{P}, \alpha)=s_{\varphi}^{\tilde{P}}(\mathfrak{M}_\infty)$ $\varphi$-*a. e.*

(2) $S_{\varphi}^{\tilde{P}}(\mathfrak{M}_\infty)=\varphi(h(\tilde{P}, \alpha))$

*for any partition* $\tilde{P}\in\mathcal{P}(\mathfrak{M})$ *and any* $\varphi\in\mathfrak{S}_I(\alpha)$.

*Proof*. By Theorem 6.49 of [17] (i. e. if $\mathfrak{S}_I(\alpha)$ includes a faithful state, then $\mathfrak{N}^\alpha$ is sufficient for $\mathfrak{S}_I(\alpha)$), $\mathfrak{N}^\alpha$ is sufficient for $\mathfrak{S}_I(\alpha)$. Moreover, the above lemma 2 and the fact 4° of [1] (i. e. if $\mathcal{S}$ ($\subset\mathfrak{S}(\mathfrak{N})$) contains a faithful state $\varphi$ and $\mathfrak{M}$ is sufficient for $\mathcal{S}$, then any subalgebra $\mathfrak{M}_0$ including $\mathfrak{M}$ is sufficient for $\mathcal{S}$ whenever $E_\varphi(\cdot\,|\,\mathfrak{M}_0)$ exists) imply that $\mathfrak{M}_n$ is sufficient for $\mathfrak{S}_I(\alpha)$ ($n\in N$). Let $\phi$ be a faithful state in $\mathfrak{S}_I(\alpha)$ with $\langle A\rangle$. Since $\mathfrak{M}_n$ is sufficient for $\mathfrak{S}_I(\alpha)$, $\varphi\circ E_\phi(\cdot\,|\,\mathfrak{M}_n)=\varphi(\cdot)$ holds for any $\varphi\in\mathfrak{S}_I(\alpha)$. By the fact 1°, the sequence $\{E_\phi(\cdot\,|\,\mathfrak{M}_n)\}$ is strongly convergent to $E_\phi(\cdot\,|\,\mathfrak{M}_\infty)$ satisfying $\varphi\circ E_\phi(\cdot\,|\,\mathfrak{M}_\infty)=\varphi(\cdot)$ for any $\varphi\in\mathfrak{S}_I(\alpha)$. Therefore $\mathfrak{M}_\infty$ is sufficient for $\mathfrak{S}_I(\alpha)$, which implies that there exists the conditional expectation $\xi$ from $\mathfrak{N}$ to $\mathfrak{M}_\infty$ such that $\varphi\circ\xi=\varphi$ for any $\varphi\in\mathfrak{S}_I(\alpha)$. From Lemma 3, the sequence $\{\eta(E_\phi(A\,|\,\mathfrak{M}_n))\}$ is strongly convergent to $\eta(E_\phi(A\,|\,\mathfrak{M}_\infty))$ for any $A\in\mathfrak{N}$. Thus we have

$$s_{\phi}^{\tilde{P}}(\mathfrak{M}_n) \xrightarrow{s} s_{\phi}^{\tilde{P}}(\mathfrak{M}_\infty)$$

for any partition $\tilde{P}\in\mathcal{P}(\mathfrak{M})$. Now we put

$$h(\tilde{P}, \alpha)\equiv\textstyle\sum_j\eta(\xi(P_j))$$

for any partition $\tilde{P}\in\mathcal{P}(\mathfrak{M})$, then $h(\tilde{P}, \alpha)$ is bounded operator. Since $\xi(\cdot)=E_\varphi(\cdot\,|\,\mathfrak{M}_\infty)$ $\varphi$-a. e. for any $\varphi\in\mathfrak{S}_I(\alpha)$, we obtain

$$h(\tilde{P}, \alpha)=s_{\varphi}^{\tilde{P}}(\mathfrak{M}_\infty) \quad \varphi\text{-a. e.}$$

for any $\tilde{P}\in\mathcal{P}(\mathfrak{M})$ and $\varphi\in\mathfrak{S}_I(\alpha)$. Finally the (2) of lemma 3 deduces the equality

$$\varphi(h(\tilde{P}, \alpha))=S_{\varphi}^{\tilde{P}}(\mathfrak{M}_\infty)$$

for any $\tilde{P}\in\mathcal{P}(\mathfrak{M})$ and $\varphi\in\mathfrak{S}_I(\alpha)$. Q. E. D.

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