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A VARIFOLD SOLUTION TO THE NONLINEAR EQUATION OF MOTION OF A VIBRATING MEMBRANE

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§ 1. Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with the boundary $\partial\Omega$ which is a Lipschitz manifold. Then the equation of motion of a vibrating membrane is as follows:

$$D_t^2 u(t, x) - \sum_{j=1}^{n} D_j\{D_j u(t, x)(1+|Du(t, x)|^2)^{-1/2}\} = 0, \qquad x \in \Omega, \tag{1.1}$$

where D_t denotes $\partial/\partial t$ and D_j denotes $\partial/\partial x_j$, $j=1, 2, \cdots, n$. The initial and the boundary conditions we shall consider are

$$u(0, x) = u_0(x), \qquad D_t u(0, x) = u_1(x), \tag{1.2}$$

$$u(t, x) = 0 \quad \text{for} \quad x \quad \text{in} \quad \partial\Omega. \tag{1.3}$$

If $u_0(x)$ and $u_1(x)$ are sufficiently smooth, there exists a unique genuine solution of (1.1), (1.2) and (1.3) for a short time interval. (cf. Kato [9] and Shibata-Tsutsumi [10]). On the other hand, existence global in time of even a weak solution is not proved in the case $n>1$.

The purpose of the present paper is to treat the above equation by virtue of the theory of varifolds introduced by Almgren Jr. [2]. A varifold is a generalization of the notion of a function and was successfully used in the direct approach of the Plateau's problem. We shall define a generalized solution of the equation (1.1) in terms of varifolds, which we call the varifold solution. And we shall prove existence, global in time, of a varifold solution of (1.1), (1.2) and (1.3). Thus this paper is closely related with the works of Tartar [11], [12] and that of DiPerna [5].

Although a varifold solution is quite a weak notion, it satisfies a generalization of the Hamilton's principle:

$$\delta\int_0^T dt \int_\Omega \left\{\frac{1}{2}|D_t u(t, x)|^2 - (1+|Du(t, x)|^2)^{1/2}\right\} dx = 0 \tag{1.4}$$

under appropriate assumptions.

Before introducing a varifold solution, we shall formulate, in § 2, the notion

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of a weak solution of (1.1) in terms of functions of bounded variations of n-variables. (cf. De Giorgi [4] and Giusti [8]). This is interesting in itself and will help us to treat varifold solutions.

§3 is devoted to the definition of the notion of a varifold solution of (1.1).

In §4 we prove existence, global in time, of a varifold solution of (1.1), (1.2) and (1.3). This is done by the Ritz-Galerkin approximation method.

In §5 we show that the approximating sequence of Ritz-Galerkin method coverges to a function $u(t, x)$ of bounded variation in x.

In §6, we shall prove that the global varifold solution can be identified with $u(t, x)$ if $u(t, x)$ satisfies the energy conservation law. This will be done in Theorem 4.

A generalization of Hamilton's principle is proved in §7.

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Notations.

The following usual notations are used: If x and y are two vectors in $\boldsymbol{R}^k$, $x \cdot y$ is the Euclidean inner product of x and y, and $|x|$ is the length of x. If M is a Radon measure on a σ-compact metric space X and ϕ is a continuous function on X then

$$\langle M, \phi\rangle = \int_X \phi(x) dM(x)$$

and spt M is the support of M. $\mathcal{H}_n$ denotes the Hausdorff measure of dimension n. Let $m \geqq 0$ be an integer. Then

$C^m(\Omega)$ denotes the space of functions of class C^m in Ω.
$C_0^m(\Omega) = \{u \in C^m(\Omega);$ spt u is compact$\}$.

If Y is a topological vector space and U is an open subset of $\boldsymbol{R}^k$,

$C^m(U, Y)$ stands for the space of Y-valued functions of class C^m.
$C_0^m(U, Y) = \{u \in C^m(U, Y);$ spt u is compact$\}$.
$L^p(U)$, $1 \leqq p \leqq \infty$, denotes the space of p-summable functions with respect to the k-dimensional Lebesgue measure L_k.
$W^{m,p}(\Omega) = \{u \in L^p(\Omega): D^\alpha u \in L^p(\Omega)$ for $|\alpha| \leqq m\}$.
$W_0^{m,p}(\Omega) =$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

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2) Main results of the paper have already been announced briefly in [7].

§ 2. A weak solution.

We shall denote by $BV(\Omega)$ the space of all functions of bounded variation in the domain $\Omega\subset \boldsymbol{R}^n$, i. e., $u\in BV(\Omega)$ if and only if $u\in L^1(\Omega)$ and its gradient $Du=(D_1u, D_2u, \cdots, D_nu)$ in the sense of distributions is an $\boldsymbol{R}^n$-valued Radon measure. (See Giusti [8] for the detailed theory.) We denote its total variation measure by $|Du|$. Let U be an open subset of Ω. Then $|Du|(U)$ is defined by the equality

$$|Du|(U)=\sup\left|\int_\Omega u \operatorname{div} \phi(x)dx\right|, \tag{2.1}$$

where $\phi(x)=(\phi_1(x), \cdots, \phi_n(x))\in C_0^1(U; \boldsymbol{R}^n)$ satisfies $|\phi(x)|\leqq 1$ for each x. Similarly we can define the measure $(1+|Du|)^{1/2}$ by the following equality:

$$(1+|Du|)^{1/2}(U)=\sup\left|\int_\Omega \{\phi_0(x)+u \operatorname{div} \phi(x)\}\, dx\right|, \tag{2.2}$$

where $\phi(x)\in C_0^1(U; \boldsymbol{R}^n)$ and $\phi_0(x)\in C_0^1(U)$ such that

$$\phi_0(x)^2+|\phi(x)|^2\leqq 1 \qquad \text{for each} \quad x\in U.$$

If $u\in C^1(\Omega)$, then

$$\int_\Omega |Du|=\int_\Omega |Du(x)|dx\,, \quad \text{and} \quad \int_\Omega (1+|Du|)^{1/2}=\int_\Omega (1+|Du(x)|)^{1/2}dx\,.$$

The latter equals the area of the hypersurface $y=u(x)$, the graph of $u(x)$, in the space $\Omega\times\boldsymbol{R}$. If $u(x)\in BV(\Omega)$, then we can define its boundary value (the trace of u) γu to $\partial\Omega$. γu belongs to $L^1(\partial\Omega)$. Let $g\in C^1(\boldsymbol{R}^n; \boldsymbol{R}^n)$. Then we have the Green-Stokes formula

$$\int_\Omega u \operatorname{div} g\, dx=-\int_\Omega Du\cdot g+\int_{\partial\Omega}\gamma u\; g\cdot\vec{n}\; d\mathcal{H}_{n-1}\,, \tag{2.3}$$

where $\vec{n}$ is the unit outer normal to $\partial\Omega$.

If $u\in BV(\Omega)$, then $E=\{(x, y)\in\Omega\times\boldsymbol{R}: u(x)>y\}$ is the subgraph of u. The characteristic function $\chi_E(x, y)$ of E is a function of bounded variation on every bounded open subset of $\Omega\times\boldsymbol{R}$. $D\chi_E$ is an $\boldsymbol{R}^{n+1}$-valued Radon measure on $\Omega\times\boldsymbol{R}$. We know that $\operatorname{spt}|D\chi_E|\subset\partial E$.

For $\rho>0$, we set $B(x, y; \rho)=\{(z, w)\in \boldsymbol{R}^n\times\boldsymbol{R}: |z-x|^2+|w-y|^2<\rho^2\}$. Then the reduced boundary ∂^*E of E is the set of all points $(x, y)\in\Omega\times\boldsymbol{R}$ with the following properties:

(i) $\int_{B(x,y;\rho)}|D\chi_E|>0$ for each ρ.

(ii) The limit $\nu(x, y)=\lim_{\rho\to 0}\nu_\rho(x, y)$ exists, where

$$\nu_\rho(x, y)=\frac{\int_{B(x,y;\rho)}D\chi_E}{\int_{B(x,y;\rho)}|D\chi_E|} \tag{2.4}$$

and

$$|\nu(x, y)|=1.$$

It is known that $|D\chi_E|(\Omega\times\boldsymbol{R}\setminus\partial^*E)=0$ and that for each Borel subset A of $\Omega\times\boldsymbol{R}$

$$|D\chi_E|(A)=\mathcal{H}_n(A\cap\partial^*E), \tag{2.5}$$

$$D\chi_E=\nu|D\chi_E|. \tag{2.6}$$

The vector $\nu(x, y)$ is considered to be the unit inner normal at $(x, y)\in\partial^*E$ to ∂^*E in a generalized sense. In fact, if $u\in C^1(\Omega)$ then $\mathrm{spt}|D\chi_E|=$the graph of u, and

$$\begin{aligned}\nu_j(x, u(x))&=D_ju(x)(1+|Du(x)|^2)^{-1/2}, \qquad j=1, 2, \cdots, n, \\ \nu_{n+1}(x, u(x))&=-(1+|Du(x)|^2)^{-1/2}.\end{aligned} \tag{2.7}$$

If a function $u(t, x)$ is of bounded variation with respect to $x\in\Omega$ for each fixed t, then the subgraph of $u(t, *)$ will be denoted by $E(t)$. Notations $D\chi_{E(t)}$, and $\nu(t; x, y)$ etc. have obvious meanings.

DEFINITION 2.1. Let ω be an open subset of Ω and (a, b) be a time interval. Then a function $u(t, x)\in L^1_{\mathrm{loc}}((a, b)\times\omega)$ is said to be a BV-solution of the equation (1.1) in $(a, b)\times\omega$ if $u(t, x)$ is a function of bounded variation with respect to $x\in\omega$ for any fixed $t\in(a, b)$ and it satisfies the equation

$$\begin{aligned}\int_a^b dt\int_{\omega\times\boldsymbol{R}}&\Big\{D_t^2\phi(t, x)u(t, x)+\sum_{j=1}^n D_j\phi(t, x)\nu_j(t; x, y)\Big\} \\ &\times\nu_{n+1}(t; x, y)|D\chi_{E(t)}|=0\end{aligned} \tag{2.8}$$

for any function $\phi(t, x)\in C_0^\infty((a, b)\times\omega)$.

As to the initial-boundary value problem (1.1), (1.2) and (1.3) we use the following definition.

DEFINITION 2.2. Assume that $u_0\in BV(\Omega)$ and $u_1\in L^2(\Omega)$. Let $T>0$ be any number. Then a function $u(t, x)\in L^1_{\rm loc}(\boldsymbol{R}\times\Omega)$ is called a BV-solution of the equations (1.1), (1.2) and (1.3) for $0\leqq t<T$ if the following conditions hold:

(i) For each $t\in\boldsymbol{R}$, $u(t, x)$ is a function of bounded variation with respect to x such that $\gamma u=0$.
(ii) For each $\phi(t, x)\in C^2([0, T); C_0(\Omega))\cap C([0, T); C^2(\Omega))$ vanishing near $t=T$, we have

$$\int_0^T dt\int_{\Omega\times\boldsymbol{R}}\Big\{D_t^2\phi(t, x)u(t, x)+\sum_{j=1}^n D_j\phi(t, x)\nu_j(t; x, y)\Big\}\nu_{n+1}(t; x, y)|D\chi_{E(t)}| \tag{2.9}$$

$$=-\int_\Omega\phi(0, x)u_1(x)dx+\int_\Omega D_t\phi(0, x)u_0(x)dx\,.$$

If $u(t, x)\in C^1([0, T)\times\Omega)$, then the above definition coincides with the usual definition of a weak solution.

§ 3. Definition of a varifold solution.

Let $G=G(n+1, n)$ be the Grassmann manifold of all n-dimensional vector subspaces of $\boldsymbol{R}^{n+1}$. Let $S\in G$ be an n-dimensional vector subspace in $\boldsymbol{R}^{n+1}$. Then we denote the unit normal to S by $\nu(S)=(\nu_1(S), \cdots, \nu_{n+1}(S))$. We choose $\nu(S)$ so that $\nu_{n+1}(S)\leqq 0$. If $\nu_{n+1}(S)=0$, then $\nu(S)$ is not unique. We call the set $\mathrm{irr}(G)=\{S\in G:\nu_{n+1}(S)=0\}$ the set of irregularity. Functions $\nu_{n+1}(S)$ and $\nu_{n+1}(S)\nu_j(S)$, $j=1, 2, \cdots, n$, are single-valued continuous functions on G. A point of $\Omega\times\boldsymbol{R}\times G$ is denoted by (x, y, S).

A varifold (an n-varifold, more precisely), $V(x, y, S)$ is a positive Radon measure on $\Omega\times\boldsymbol{R}\times G$. (See Allard [1] for detailed discussions).

Example 3.1. If $u\in BV(\Omega)$, then u (or the graph of u, more precisely) is identified with a varifold $V(x, y, S)$ in the following manner: For any $\phi(x, y, S)\in C_0(\Omega\times\boldsymbol{R}\times G)$,

$$\int_{\Omega\times\boldsymbol{R}\times G}\phi(x, y, S)dV(x, y, S)=\int_{\partial^*E}\phi(x, y, \mathrm{Tan}_{(x,y)}(\partial^*E))|D\chi_E|, \tag{3.1}$$

where $\mathrm{Tan}_{(x,y)}\partial^*E$ is the tangent hyperplane at (x, y) to the reduced boundary ∂^*E. We call this identification canonical.

Keeping this example in mind, we can introduce the following

DEFINITION 3.1. Let ω be an open subset of Ω. A varifold $V(t; x, y, S)$ depending on a parameter $t\in(a, b)$ is called a varifold solution of the equation (1.1) for $(a, b)\times\omega\subset\boldsymbol{R}\times\Omega$ if and only if the following two conditions hold:

$$\int_a^b dt\int_{\omega\times G}dV(t; x, y, S)<\infty\,. \tag{3.2}$$

And the equality

(3.3)
$$0=\int_a^b dt\int_{\omega\times\boldsymbol{R}\times G} D_t^2\phi(t,\ x)y\nu_{n+1}(S)dV(t\,;\ x,\ y,\ S)$$
$$+\int_a^b dt\int_{\omega\times\boldsymbol{R}\times G}\left\{\sum_{j=1}^n D_j\phi(t,\ x)\nu_j(S)\nu_{n+1}(S)\right\}dV(t\,;\ x,\ y,\ S)$$

holds for any function $\phi(t,\ x)$ in $C_0^\infty((a,\ b)\times\omega)$.

Corresponding to Definition 2.2 we introduce the following

DEFINITION 3.3. Let T be a positive number. A varifold $V(t\,;x,\ y,\ S)$ depending on a parameter $t\in\boldsymbol{R}$ is called a varifold solution of the equation (1.1) and (1.2) for $[0,\ T)$ if and only if the following two conditions hold:

(3.4)
$$\int_0^T dt\int_{\omega\times\boldsymbol{R}\times G} dV(t\,;\ x,\ y,\ S)<\infty\,.$$

And the equality

(3.5)
$$\int_0^T dt\int_{\Omega\times\boldsymbol{R}\times G} D_t^2\phi(t,\ x)y\nu_{n+1}(S)dV(t\,;\ x,\ y,\ S)$$
$$+\int_0^T dt\int_{\Omega\times\boldsymbol{R}\times G}\left\{\sum_{j=1}^n D_j\phi(t,\ x)\nu_j(S)\nu_{n+1}(S)\right\}dV(t\,;\ x,\ y,\ S)$$
$$=-\int_\Omega\phi(0,\ x)u_1(x)dx+\int_\Omega D_t\phi(0,\ x)u_0(x)dx$$

holds for any function $\phi(t,\ x)$ in $C^2([0,\ T)\,;\ C_0(\Omega))\cap C([0,\ T)\,;\ C^2(\Omega))$ vanishing near $t=T$.

If a varifold solution $V(t\,;x,\ y,\ S)$ can be canonically identified with a function $u(t,\ x)$ of bounded variation as in Example 3.1, then $u(t,\ x)$ is a BV-solution of (1.1) and (1.3). This is because

(3.6)
$$\int_{\Omega\times\boldsymbol{R}\times G} D_t^2\phi(t,\ x)y\nu_{n+1}(S)dV(t\,;\ x,\ y,\ S)$$
$$=\int_{\Omega\times\boldsymbol{R}} D_t^2\phi(t,\ x)u(t,\ x)\nu_{n+1}(t\,;\ x,\ y)|D\chi_{E(t)}|,$$

and

(3.7)
$$\int_{\Omega\times\boldsymbol{R}\times G} D_j\phi(t,\ x)\nu_j(S)\nu_{n+1}(S)dV(t\,;\ x,\ y,\ S)$$
$$=\int_{\Omega\times\boldsymbol{R}} D_j\phi(t,\ x)\nu_j(t\,;\ x,\ y)\nu_{n+1}(t\,;\ x,\ y)|D\chi_{E(t)}|.$$

§ 4. Existence of a global varifold solution.

Now we state the main theorem.

THEOREM 1. *Assume that* $u_0 \in W_0^{1,2}(\Omega)$ *and* $u_1 \in L^2(\Omega)$. *Then there exists a varifold solution* $V(t; x, y, S)$ *of* (1.1) *and* (1.2), *that is*, $V(t; x, y, S)$ *satisfies* (3.2) *and* (3.3) *for any* $T>0$.

Proof is done by the Ritz-Galerkin method, which occupies the rest of this section.

Let $\phi_k(x)$, $k=1, 2, \cdots$, be the normalized eigen-functions of the Dirichlet problem in Ω:

$$-\Delta\phi_k(x)=\lambda_k\phi_k(x), \qquad x\in\Omega, \tag{4.1}$$
$$\phi_k(x)=0 \qquad \text{if} \quad x\in\partial\Omega .$$

The system $\{\phi_k\}_{k=1}^{\infty}$ forms a complete ortho-normal system in $L^2(\Omega)$. For $m=1, 2, \cdots$, we put

$$P_m f(x)=\sum_{k=1}^{m}(f, \phi_k)\phi_k(x) .$$

The m-th approximate solution of (1.1) is of the form

$$u^m(t, x)=\sum_{k=1}^{m}a_k^m(t)\phi_k(x) \tag{4.2}$$

and satisfies the equation

$$P_m\Big\{D_t^2u^m(t, x)-\sum_{j=1}^{n}D_j(D_ju^m(t, x)(1+|Du^m(t, x)|^2)^{-1/2})\Big\}=0, \tag{4.3}$$

$$u^m(0, x)=P_mu_0, \qquad D_tu^m(0, x)=P_mu_1. \tag{4.4}$$

This is equivalent to the system of equations

$$D_t^2a_k^m(t)+\sum_{j=1}^{n}\int_\Omega D_j\phi_k(x)\{D_ju^m(t, x)(1+|Du^m(t, x)|^2)^{-1/2}\}dx=0, \tag{4.5}$$

$$a_k^m(0, x)=(u_0, \phi_k), \qquad D_ta_k^m(0, x)=(u_1, \phi_k), \tag{4.6}$$

for $k=1, 2, \cdots, m$.

PROPOSITION 4.1. *The m-th approximate solution* $u^m(t, x)$ *exists for all* $t\in\boldsymbol{R}$.

Proof. Let $A_m(t)=(a_1^m(t), a_2^m(t), \cdots, a_m^m(t))$. Then the correspondence

$$A_m(t)\to F_{jk}(A^m)=\int_\Omega D_j\phi_k(x)D_ju^m(t, x)(1+|Du^m(t, x)|^2)^{-1/2}dx$$

is uniformly Lipschitz continuous for $k=1, 2, \cdots, m$, and $j=1, 2, \cdots, n$. This proves Proposition.

PROPOSITION 4.2. (Energy estimate). *For* $m=1, 2, \cdots$,

(4.7)
$$\frac{1}{2}\int_{\Omega}|D_t u^m(t,\ x)|^2 dx+\int_{\Omega}(1+|Du^m(t,\ x)|^2)^{1/2}dx$$
$$=\frac{1}{2}\int_{\Omega}|P_m u_1(x)|^2 dx+\int_{\Omega}(1+|DP_m u_0(x)|^2)^{1/2}dx\ .$$

In particular,

(4.8)
$$\frac{1}{2}\int_{\Omega}|D_t u^m(t,\ x)|^2 dx+\int_{\Omega}(1+|Du^m(t,\ x)|^2)^{1/2}dx\leqq M\ ,$$

where

(4.9)
$$M=\frac{1}{2}\int_{\Omega}|u_1(x)|^2 dx+(|\Omega|+\|u_0\|_{W^{1,2}(\Omega)})^{1/2}|\Omega|^{1/2}.$$

Proof. Multiply both sides of (4.3) by $D_t u^m(t,\ x)$ and integrate with respect to x. Then

$$D_t\left\{\frac{1}{2}\int_{\Omega}|D_t u^m(t,\ x)|^2 dx+\int_{\Omega}(1+|Du^m(t,\ x)|^2)^{1/2}dx\right\}=0\ .$$

This and the initial condition (4.4) give (4.7).

To prove (4.8) we note that $\int|P_m u_1(x)|^2 dx\leqq\int|u_1(x)|^2 dx$ and that

$$\int_{\Omega}(1+|DP_m u_0(x)|^2)^{1/2}dx\leqq\left\{\int_{\Omega}(1+|DP_m u_0(x)|^2)dx\right\}^{1/2}|\Omega|^{1/2}.$$

Since ϕ_k satisfies (4.1), we have

$$\int_{\Omega}|DP_m u_0(x)|^2 dx=(-\Delta P_m u_0,\ u_0)=\sum_{j=1}^{m}\lambda_k(u_0,\ \phi_k)^2$$
$$\leqq\sum_{j=1}^{\infty}\lambda_j(u_0,\ \phi_k)^2=-(\Delta u_0,\ u_0)=\int_{\Omega}|Du_0|^2 dx\ .$$

This proves (4.8) and (4.9).

For each $m=1,\ 2,\ 3,\ \cdots$, the function $u^m(t,\ x)$ is of class C^∞. We identify this with a varifold $V^m(t;\ x,\ y,\ S)$ as in the Example 3.1 of § 3. We rewrite (4.3) and (4.4) in terms of $V^m(t;\ x,\ y,\ S)$. Let $\phi(t)\in C^2(\boldsymbol{R})$ vanishing near $t=T$. Then we multiply both sides of (4.3) by $\phi(t)\phi_k(x)$, $k\leqq m$. After integration by parts we have

(4.10)
$$\phi(0)\int_{\Omega}u_1(x)\phi_k(x)dx-D_t\phi(0)\int_{\Omega}u_0(x)\phi_k(x)dx$$
$$=\int_0^T dt\ D_t^2\phi(t)\int_{\Omega}\phi_k(x)u^m(t,\ x)dx$$
$$+\int_0^T dt\ \phi(t)\int_{\Omega}\sum_{j=1}^{n}D_j\phi_k(x)D_j u^m(t,\ x)(1+|Du^m(t,\ x)|^2)^{-1/2}dx\ .$$

On the other hand, we have, by definition,

$$\int_{\Omega}\psi_k(x)u^m(t,\ x)dx=-\int_{\Omega\times\boldsymbol{R}\times G}\psi_k(x)y\nu_{n+1}(S)dV^m(t;\ x,\ y,\ S)$$

and

$$\int_{\Omega}D_j\psi_k(x)\{D_ju^m(t,\ x)(1+|Du^m(t,\ x)|^2)^{-1/2}\}dx$$
$$=-\int_{\Omega\times\boldsymbol{R}\times G}D_j\psi_k(x)\nu_j(S)\nu_{n+1}(S)dV^m(t;\ x,\ y,\ S)\,.$$

Therefore $V^m(t;x,y,S)$ satisfies the following equation:

$$\begin{aligned}(4.11)\qquad &\int_0^T D_t^2\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\psi_k(x)y\nu_{n+1}(S)dV^m(t;\ x,\ y,\ S)\\ &+\int_0^T\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\left\{\sum_{j=1}^n D_j\psi_k(x)\nu_j(S)\nu_{n+1}(S)\right\}dV^m(t;\ x,\ y,\ S)\\ &=-\int_{\Omega}\phi(0)u_1(x)\psi_k(x)dx+\int_{\Omega}D_t\phi(0)u_0(x)\psi_k(x)dx\,,\end{aligned}$$

where $k=1, 2, \cdots, m$ and $\phi(t)$ is an arbitrary function in $C_0^2(\boldsymbol{R})$ vanishing near $t=T$.

We wish to choose a subsequence $\{m'\}\subset\{m\}$ so that $\lim_{m'\to\infty}V^{m'}(t;x,y,S)$ exists. In fact we have

PROPOSITION 4.3. *There exist a subset $\boldsymbol{R}_1$ of $\boldsymbol{R}$, subsequence $\{m'\}$ of $\{m\}$ and a varifold $V(t;x,y,S)$ depending on a parameter $t\in\boldsymbol{R}_1$ with the following properties: $L_1(\boldsymbol{R}\setminus\boldsymbol{R}_1)=0$ and*

$$\begin{aligned}(4.12)\qquad &\int_{-\infty}^{\infty}\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\xi(x,\ y,\ S)dV(t;\ x,\ y,\ S)\\ &=\lim_{m'\to\infty}\int_{-\infty}^{\infty}\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\xi(x,\ y,\ S)dV^{m'}(t;\ x,\ y,\ S)\,,\end{aligned}$$

for any $\phi(t)\in L^1(\boldsymbol{R})$ and $\xi(x,y,S)\in C_0(\Omega\times\boldsymbol{R}\times G)$. We have

$$(4.13)\qquad \int_{\Omega\times\boldsymbol{R}\times G}dV(t;\ x,\ y,\ S)\leqq M\,.$$

Proof. Let M be the constant in (4.9). Then we note that

$$\begin{aligned}\int_{\Omega\times\boldsymbol{R}\times G}dV^m(t;\ x,\ y,\ S)&=\int_{\{y=u^m(t,x)\}}d\mathcal{H}_n\\ &=\int_{\Omega}(1+|Du^m(t,\ x)|^2)^{1/2}dx\\ &\leqq M\,.\end{aligned}$$

If $\xi(x, y, S) \in \mathcal{C}_0(\Omega \times \boldsymbol{R} \times G)$ then

$$\langle \xi, V^m(t) \rangle = \int_{\Omega \times \boldsymbol{R} \times G} \xi(x, y, S) dV^m(t; x, y, S)$$

is a bounded function of t, because we have the estimate

$$|\langle \xi, V^m(t) \rangle| \leqq M \max |\xi(x, y, S)|. \tag{4.14}$$

We consider the family of mappings $\mathcal{C}_0(\Omega \times \boldsymbol{R} \times G) \ni \xi \to \langle \xi, V^m(t) \rangle \in L^\infty(\boldsymbol{R})$. The estimate (4.14) implies that this family of mappings is equicontinuous and that for each ξ the image of mappings is relatively compact in the weak* topology of $L^\infty(\boldsymbol{R})$. We can apply the Ascoli-Arzela theorem because $\mathcal{C}_0(\Omega \times \boldsymbol{R} \times G)$ is separable. And there exists a subsequence $\{V^{m'}(t; x, y, S)\}_{m'}$ such that

$$w^*\text{-}\lim_{m' \to \infty} \langle \xi, V^{m'}(t) \rangle = f(t; \xi) \tag{4.15}$$

exists in $L^\infty(\boldsymbol{R})$ for each ξ. It is clear that $f(t; \xi) \geqq 0$ if $\xi \geqq 0$. And we have

$$\|f(t; \xi)\|_{L^\infty} \leqq M \max |\xi(x, y, S)|. \tag{4.16}$$

The function $f(t; \xi)$ may not be defined for t in an exceptional set of L_1-measure 0 and this exceptional set may depend on ξ. To avoid this inconvenience we choose a good representative $V(t; \xi)$ of $f(t; \xi)$ as a function of t: We define

$$V(t; \xi) = \lim_{h \to +0} \frac{1}{2h} \int_{t-h}^{t+h} f(t; \xi) dt. \tag{4.17}$$

This exists and is equal to $f(t; \xi)$ at L_1-almost every $t \in \boldsymbol{R}$ if ξ is fixed. Let $\{\xi_k\}_{k=1}^\infty$ be a countable dense subset of $\mathcal{C}_0(\Omega \times \boldsymbol{R} \times G)$. Then the set

$$\boldsymbol{R}_1 = \{t \in \boldsymbol{R} : V(t; \xi_k) \text{ exists and is finite for all } k\}$$

is measurable and $L_1(\boldsymbol{R} \setminus \boldsymbol{R}_1) = 0$.

We claim that $V(t; \xi)$ exists for all $\xi \in \mathcal{C}_0(\Omega \times \boldsymbol{R} \times G)$ and for $t \in \boldsymbol{R}_1$. In fact, for any $\varepsilon > 0$, there exists a function ξ_k such that

$$|\xi(x, y, S) - \xi_k(x, y, S)| < \varepsilon \quad \text{for any} \quad (x, y, S) \in \Omega \times \boldsymbol{R} \times G. \tag{4.18}$$

Then we have for each $t \in \boldsymbol{R}_1$

$$\begin{aligned} \frac{1}{2h} \left| \int_{t-h}^{t+h} f(t; \xi) dt - \int_{t-h}^{t+h} f(t; \xi_k) dt \right| \\ \leqq \frac{1}{2h} \int_{t-h}^{t+h} |f(t; \xi - \xi_k)| dt \\ \leqq M\varepsilon. \end{aligned} \tag{4.19}$$

The last estimate follows from (4.16) and (4.18). Hence

$$V(t;\xi_k)-\varepsilon\leqq\liminf_{h\to+0}\frac{1}{2h}\int_{t-h}^{t+h}f(t;\xi)dt$$

$$\leqq\limsup_{h\to+0}\frac{1}{2h}\int_{t-h}^{t+h}f(t;\xi)dt\leqq V(t;\xi_k)+\varepsilon$$

Since ε is arbitrary,

$$\lim_{h\to+0}\frac{1}{2h}\int_{t-h}^{t+h}f(t;\xi)dt=V(t;\xi)$$

exists at every $t\in\boldsymbol{R}_1$.

If $t\in\boldsymbol{R}_1$, then it follows from (4.16) and (4.17) that

$$|V(t;\xi)|\leqq M\max|\xi(x,y,S)|.$$

This implies that the correspondence $\xi\to V(t;\xi)$, $t\in\boldsymbol{R}_1$, defines a Radon measure $V(t;x,y,S)$ such that

$$V(t;\xi)=\int_{\Omega\times\boldsymbol{R}\times G}\xi(x,y,S)dV(t;x,y,S).$$

We know that $V(t;\xi)\geqq 0$ if $\xi\geqq 0$. Therefore $V(t;x,y,S)$ is a varifold. Clearly we have

$$\int_{\Omega\times\boldsymbol{R}\times G}dV(t;x,y,S)\leqq M. \tag{4.20}$$

Equality (4.15) leads us to the equality

$$w^*\text{-}\lim_{m'\to\infty}\langle\xi,V^{m'}(t)\rangle=\langle\xi,V(t)\rangle$$

as an element of $L^\infty(\boldsymbol{R})$. This proves Proposition.

End of the Proof of Theorem 1. We complete the proof of Theorem 1 by showing that the varifold $V(t;x,y,S)$ satisfies the equality (3.3). We choose the subsequence $\{m'\}$ as in Proposition 4.3 and denote it as $\{m\}$ in the following for the sake of brevity. Take $\phi(t)\in C^2(\boldsymbol{R})$ which vanishes near $t=T$. Then $D_t^2\phi(t)\in L^1(\boldsymbol{R})$. On the other hand, we know that $\psi_j(x)y\nu_{n+1}(S)\in C_0(\Omega\times\boldsymbol{R}\times G)$. Therefore the above Proposition 4.3 asserts that

$$\lim_{m\to\infty}\int_0^T D_t^2\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\psi_k(x)y\nu_{n+1}(S)dV^m(t;x,y,S) \tag{4.21}$$

$$=\int_0^T D_t^2\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\psi_k(x)y\nu_{n+1}(S)dV(t;x,y,S).$$

Similarly we have

$$\lim_{m\to\infty}\int_0^T\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\sum_{j=1}^n D_j\psi_k(x)\nu_j(S)\nu_{n+1}(S)dV^m(t;x,y,S) \tag{4.22}$$

$$=\int_0^T\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\sum_{j=1}^n D_j\psi_k(x)\nu_j(S)\nu_{n+1}(S)dV(t;x,y,S),$$

because $\phi\in L^1(\boldsymbol{R})$ and $D_j\psi_k(x)\nu_j(S)\nu_{n+1}(S)\in C_0(\Omega\times\boldsymbol{R}\times G)$. Letting m go to ∞ in (4.11), and using (4.21) and (4.22), we have

$$
\begin{aligned}
(4.23)\qquad & \int_0^T D_t^2\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\psi_k(x)y\nu_{n+1}(S)dV(t;x,y,S) \\
&+\int_0^T\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\sum_{j=1}^n D_j\psi_k(x)\nu_j(S)\nu_{n+1}(S)dV(t;x,y,S) \\
&=-\int_\Omega\phi(0)u_1(x)\psi_k(x)dx+\int_\Omega D_t\phi(0)u_0(x)\psi_k(x)dx\,,
\end{aligned}
$$

for $k=1, 2, \cdots$. Since functions of the form $\phi(t)\psi_k(x)$ are total in the space $C^2([0, T); C_0(\Omega))\cap C([0, T); C^2(\Omega))$, the equality (3.5) follows from (4.23). Inequality (3.4) is a consequence of (4.13). This proves our theorem.

§ 5. Convergence of $u^m(t, x)$ in the BV-space.

As we have proved the global existence of a varifold solution, we wish to identify $V(t; x, y, S)$ with a graph of a function. A graph of a function is, measure theoretically, a special case of an n-rectifiable subset of $\Omega\times\boldsymbol{R}$. Thus we can state our problem in the following form:

(Q) *Can one identify the varifold solution $V(t; x, y, S)$ of the preceding section with an H_n rectifiable subset of $\Omega\times\boldsymbol{R}$ for all t?*

Unfortunately we did not succeed in giving answer to this fundamental question. Of course the most probable candidate of the H_n-rectifiable subset of $\Omega\times\boldsymbol{R}$ is the graph of the function $u(t, x)=\lim_{m\to\infty} u^m(t, x)$ if the limit exists. In the present section, we prove that $u(t, x)=\lim_{m\to\infty} u^m(t, x)$ actually exists in the space $BV(\Omega)$. We shall discuss the relationship of $V(t; x, y, S)$ and $u(t, x)$ in the next section.

In the following we choose the subsequence $\{m'\}$ as in Proposition 4.3 and denote it by $\{m\}$ for the sake of brevity. For any fixed $t\in\boldsymbol{R}$ the sequence $\{u^m(t, x)\}$ of BV-functions are bounded because of Proposition 4.2.

PROPOSITION 5.1. *There exists a subsequence $\{m''\}\subset\{m\}$ such that $\{u^{m''}(t, x)\}$ converges strongly, for any fixed t, to a function $u(t, x)$ in $L^p(\Omega)$, $1\leq p<\dfrac{n}{n-1}$, and that $\{Du^{m''}(t, x)\}$ converges to $Du(t, x)$ with respect to the w^*-topology of measures. $u(t, *)\in BV(\Omega)$ for fixed $t\in\boldsymbol{R}$. The function $u(t, x)$ is a Lipschitz continuous function of t with values in $L^2(\Omega)$.*

Proof. Since

$$u^m(t, x)=\int_0^t D_s u^m(s, x)ds+u_0(x),$$

we have

$$\|u^m(t, x)\|_{L^2(\Omega)}\leqq t\sup\|D_s u^m(s, *)\|_{L^2(\Omega)}+\|u_0\|_{L^2(\Omega)}$$
$$\leqq(2M)^{1/2}t+\|u_0\|_{L^2(\Omega)}.$$

For any $t,\ t'\in \boldsymbol{R}$,

$$\|u^m(t', *)-u^m(t, *)\|_{L^2(\Omega)}\leqq\left\|\int_t^{t'} D_s u^m(s, x)ds\right\|\leqq(2M)^{1/2}|t'-t|. \tag{5.1}$$

Hence $t\rightarrow\{u^m(t, *)\}\in L^2(\Omega)$ is an equicontinuous family. The Ascoli-Arzela theorem enables us to choose a subsequence $\{u^{m''}(t, x)\}$ such that

$$w\text{-}\lim_{m''\rightarrow\infty} u^{m''}(t, *)=u(t, *) \quad \text{in} \quad L^2(\Omega)$$

exists for each $t\in\boldsymbol{R}$. As a consequence of this and (5.1), we have

$$\|u(t', *)-u(t, *)\|_{L^2(\Omega)}\leqq(2M)^{1/2}|t'-t|. \tag{5.2}$$

Therefore $u(t, *)$ is an $L^2(\Omega)$-valued Lipschitz continuous function.

We know from Proposition 4.2 that $\{u^{m''}(t, x)\}$ is a bounded set in $BV(\Omega)$. Since the inclusion $BV(\Omega)\subset L^p(\Omega)$, $1\leqq p<\dfrac{n}{n-1}$, is a compact map, every subsequence of $\{u^{m''}(t, x)\}$ contains a subsequence which converges strongly to $u(t, *)$ in $L^p(\Omega)$ because $\{u^{m''}(t, *)\}$ converges weakly to $u(t, *)$ in $L^2(\Omega)$. This implies that $\{u^{m''}(t, x)\}$ converges to $u(t, *)$ strongly in $L^p(\Omega)$. It is clear that $u(t, *)\in BV(\Omega)$ for each t. For $j=1, 2, \cdots, n$, $\{D_j u^{m''}(t, *)\}$ converges to $D_j u(t, *)$ in the sense of distribution. Therefore $\{D_j u^{m''}(t, *)\}$ converges to $D_j u(t, *)$ in the sense of w^*-topology of measures.

Remark 5.2. We expect that $u(t, x)$ above is a BV-solution of the equation (1.1). However we failed in proving it. We shall prove later in Theorem 4 that $u(t, x)$ is a BV-solution if it satisfies the energy conservation law.

We let $E_m(t)$ and $\chi_t^m(x, y)$ denote the subgraph of $u^m(t, x)$ and its characteristic function, respectively. Similarly $E(t)$ and $\chi_t(x, y)$ stand for the subgraph of $u(t, x)$ and its characteristic function, respectively.

COROLLARY 5.3. *We may choose the subsequence m'' so that $\{D\chi_t^{m''}\}$ converges to $D\chi_t$ in the w^*-topology of measures.*

Proof. Let $\phi(x, y)\in C_0^\infty(\Omega\times\boldsymbol{R})$. Then

$$\left|\int_{\Omega\times\boldsymbol{R}}(\chi_t^{m''}(x, y)-\chi_t(x, y))\phi(x, y)dxdy\right|$$

$$=\left|\int_{\Omega}dx\int_{u(t,x)}^{u^{m''}(t,x)}\phi(x, y)dy\right|$$

$$\leqq\max|\phi(x, y)|\int_{\Omega}|u^{m''}(t, x)-u(t, x)|dx .$$

As a consequence of this and Proposition 5.1, $\{\chi_t^{m''}\}$ converges to χ_t in the sense of distribution. Hence $\{D\chi_t^{m''}\}$ converges to $D\chi_t$ in the sense of distribution. This implies that $\{D\chi_t^{m''}\}$ converges to $D\chi_t$ in the w^*-topology of measures, because $|D\chi_t^{m''}|$ are bounded.

For the sake of brevity we denote $\{m\}$ instead of $\{m''\}$.

PROPOSITION 5.4. *There exists a set* $\boldsymbol{R}_2\subset\boldsymbol{R}$ *and a function* $\boldsymbol{R}_2\ni t\rightarrow D_t u(t, *)\in L^2(\Omega)$ *such that* $L_1(\boldsymbol{R}\setminus\boldsymbol{R}_2)=0$ *and*

$$\int_{\Omega}D_t u(t, x)\phi(x)dx=\lim_{h\to 0}h^{-1}\left\{\int_{\Omega}u(t+h, x)\phi(x)dx-\int_{\Omega}u(t, x)\phi(x)dx\right\} \tag{5.3}$$

exists for all $\phi\in L^2(\Omega)$ *and* $t\in\boldsymbol{R}_2$. *At* L_1*-almost all* t *we have*

$$\|D_t u(t, *)\|_{L^2(\Omega)}\leqq\limsup_{m\to\infty}\|D_t u^m(t, *)\|_{L^2(\Omega)} . \tag{5.4}$$

For any $T>0$, $D_t u(t, x)$ *is the weak limit of* $\{D_t u^m(t, x)\}_m$ *in the space* $L^2((0, T)\times\Omega)$.

Proof. For any $\phi\in L^2(\Omega)$, we put

$$F(t, \phi)=\lim_{h\to 0}h^{-1}\left\{\int_{\Omega}u(t+h, x)\phi(x)dx-\int_{\Omega}u(t, x)\phi(x)dx\right\}$$

if the right hand side exists. As a result of (5.1), we have

$$\left|h^{-1}\left\{\int_{\Omega}u(t+h, x)\phi(x)dx-\int_{\Omega}u(t, x)\phi(x)dx\right\}\right|\leqq(2M)^{1/2}\|\phi\|_{L^2(\Omega)} . \tag{5.5}$$

Let $\{\xi_k(x)\}_{k=1}^{\infty}$ be a countable dense subset of $L^2(\Omega)$. Then by virtue of (5.5), we see that there exists a set $\boldsymbol{R}_2\subset\boldsymbol{R}$ such that $L_1(\boldsymbol{R}\setminus\boldsymbol{R}_2)=0$ and $F(t, \xi_k)$ exists at $t\in\boldsymbol{R}_2$ and $k=1, 2, \cdots, .$

We claim that for any $\phi\in L^2(\Omega)$, $F(t, \phi)$ exists at all $t\in\boldsymbol{R}_2$. In fact for given $\phi\in L^2(\Omega)$ and $\varepsilon>0$, there exists ξ_k such that

$$\|\xi_k-\phi\|_{L^2(\Omega)}<\varepsilon/(4M)^{1/2}.$$

Applying (5.5) to $\phi-\xi_k$, we have

$$\begin{aligned} &h^{-1}\left\{\int_{\Omega}u(t+h, x)\xi_k(x)dx-\int_{\Omega}u(t, x)\xi_k(x)dx\right\}-\varepsilon \\ &\leqq h^{-1}\left\{\int_{\Omega}u(t+h, x)\phi(x)dx-\int_{\Omega}u(t, x)\phi(x)dx\right\} \end{aligned} \tag{5.6}$$

$$\leqq h^{-1}\left\{\int_\Omega u(t+h,\ x)\xi_k(x)dx-\int_\Omega u(t,\ x)\xi_k(x)dx\right\}+\varepsilon .$$

If $t\in \boldsymbol{R}_2$, then

$$F(t,\ \xi_k)-\varepsilon\leqq\liminf_{m\to\infty} h^{-1}\left\{\int_\Omega u(t+h,\ x)\phi(x)dx-\int_\Omega u(t,\ x)\phi(x)dx\right\}$$

$$\leqq\limsup_{m\to\infty} h^{-1}\left\{\int_\Omega u(t+h,\ x)\phi(x)dx-\int_\Omega u(t,\ x)\phi(x)dx\right\}$$

$$\leqq F(t,\ \xi_k)+\varepsilon .$$

Since ε is arbitrary, $F(t,\ \phi)$ exists.

From the estimate (5.5), we have

$$|F(t,\ \phi)|\leqq(2M)^{1/2}\|\phi\|_{L^2(\Omega)} .$$

$F(t,\ \phi)$ is a continuous linear functional of $\phi\in L^2(\Omega)$. Therefore there exists $D_t u(t,\ *)\in L^2(\Omega)$ such that

$$\int_\Omega D_t u(t,\ x)\phi(x)dx=F(t,\ \phi).$$

By definition we have

$$\int_\Omega u(t,\ x)\phi(x)dx-\int_\Omega u_0(x)\phi(x)dx=\int_0^t ds\int_\Omega D_s u(s,\ x)\phi(x)dx . \tag{5.7}$$

Let $v(x)=D_t u(t,\ x)$. Then

$$\|v\|^2_{L^2(\Omega)}=\lim_{h\to 0}\frac{1}{2h}\int_{t-h}^{t+h}d\tau\int_\Omega D_\tau u(\tau,\ x)v(x)dx$$

$$=\lim_{m\to\infty}\frac{1}{2h}\left\{\int_\Omega u(t+h,\ x)v(x)dx-\int_\Omega u(t-h,\ x)v(x)dx\right\}$$

$$=\lim_{h\to 0}\ \lim_{m\to\infty}\frac{1}{2h}\left\{\int_\Omega u^m(t+h,\ x)v(x)dx-\int_\Omega u^m(t-h,\ x)v(x)dx\right\}$$

$$=\lim_{h\to 0}\ \lim_{m\to\infty}\frac{1}{2h}\int_{t-h}^{t+h}\int_\Omega D_\tau u^m(\tau,\ x)v(x)dx\ d\tau$$

$$\leqq\lim_{h\to 0}\frac{1}{2h}\int_{t-h}^{t+h}\left(\limsup_{m\to\infty}\|D_t u^m(\tau,\ *)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\right)d\tau$$

$$\leqq\|v\|_{L^2(\Omega)}\limsup_{m\to\infty}\|D_t u(t,\ *)\|_{L^2(\Omega)}$$

at L_1 almost all t. Therefore

$$\|v\|_{L^2(\Omega)}\leqq\limsup_{m\to\infty}\|D_t u^m(t,\ *)\|_{L^2(\Omega)}$$

at L_1-almost all t.

The energy inequality (4.8) implies that for any $T>0$, $\{D_t u^m(t, x)\}$ is bounded in $L^2((0, T)\times\Omega)$. Let $\{D_t u^{m'}(t, x)\}$ be any weakly convergent subsequence of $\{D_t u^m(t, x)\}$ and let $w(t, x)$ be its limit. Then

$$\int_\Omega u^{m'}(t, x)\psi(x)dx-\int_\Omega P_{m'}u_0(x)\psi(x)dx=\int_0^t ds\int_\Omega D_s u^{m'}(s, x)\psi(x)dx .$$

Taking the limit of this as $m'\to\infty$, we have

$$\int_\Omega u(t, x)\psi(x)dx-\int_\Omega u_0(x)\psi(x)dx=\int_0^t ds\int_\Omega w(s, x)\psi(x)dx .$$

If follows from this and (5.7) that $D_t u(t, x)=w(t, x)$ at almost every (t, x). This proves Proposition 5.4.

§ 6. Varifold solution and BV function.

In this section we discuss the relationship of the varifold solution $V(t; x, y, S)$ of § 4 and the BV-function $u(t, x)$ given in § 5. We prove that the varifold $V(t; x, y, S)$ can be identified with the graph of the function $u(t, x)$ if $u(t, x)$ satisfies the energy conservation law. For the sake of brevity we denote $\{m''\}$ by $\{m\}$.

DEFINITION 6.1. As in Allard [1], we define the weight measure $\|V(t)\|$ of the varifold $V(t; x, y, S)$ by the equality

$$\int_{\Omega\times\boldsymbol{R}}\psi(x, y)d\|V(t)\|=\int_{\Omega\times\boldsymbol{R}\times G}\psi(x, y)dV(t; x, y, S) \tag{6.1}$$

for any $\psi(x, y)$ in $C_0(\Omega\times\boldsymbol{R})$. Similarly, for $j=1, 2, \cdots, n+1$, we define the measure $\|V(t)\llcorner\nu_j\|$ by the equality

$$\int_{\Omega\times\boldsymbol{R}}\psi(x, y)d\|V(t)\llcorner\nu_j\|=\int_{\Omega\times\boldsymbol{R}\times G}\psi(x, y)\nu_j(S)dV(t; x, y, S) . \tag{6.2}$$

As in § 5 we denote by $E(t)$ and $E_m(t)$ the subgraphs of $u(t, x)$ and $u^m(t, x)$, respectively. And we denote by χ_t and χ_t^m the characteristic functions of $E(t)$ and $E_m(t)$, respectively. Then

PROPOSITION 6.2. (i) *For each* $\phi\in L^1(\boldsymbol{R})$ *and for any* $\psi(x, y)\in C_0(\Omega\times\boldsymbol{R})$, *we have*

$$\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\psi(x, y)d\|V(t)\|=\lim_{m\to\infty}\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\psi(x, y)|D\chi_t^m| . \tag{6.3}$$

(ii) *There exists a subset* $\boldsymbol{R}_3$ *of* $\boldsymbol{R}$ *with the following properties*: $L_1(\boldsymbol{R}\setminus\boldsymbol{R}_3)=0$ *and for any* $t\in\boldsymbol{R}_3$ *and* $\psi\in C_0(\Omega\times\boldsymbol{R})$, *we have*

(6.4)
$$\liminf_{m\to\infty}\int_{\Omega\times\boldsymbol{R}}\phi(x, y)|D\chi_t^m|$$
$$\leqq\int_{\Omega\times\boldsymbol{R}}\phi(x, y)d\|V(t)\|\leqq\limsup_{m\to\infty}\int_{\Omega\times\boldsymbol{R}}\phi(x, y)|D\chi_t^m|.$$

(iii) *For any open subset* $B\subset\Omega\times\boldsymbol{R}$ *and any compact set* $K\subset B$, *we have*

(6.5)
$$\limsup_{m\to\infty}|D\chi_t^m|(B)\geqq\|V(t)\|(B)\geqq\liminf_{m\to\infty}|D\chi_t^m|(K)$$

for $t\in\boldsymbol{R}_3$.

(iv) *Assume that* B *is a bounded open subset of* $\Omega\times\boldsymbol{R}$. *Assume further that for some* $t\in\boldsymbol{R}_3$

(6.6)
$$\lim_{m\to\infty}|D\chi_t^m|(B)\quad \textit{exists}$$

and

(6.7)
$$\|V(t)\|(\partial B)=0\,.$$

Then

(6.8)
$$\|V(t)\|(B)=\lim_{m\to\infty}|D\chi_t^m|(B)\,.$$

Proof. (i) Using Proposition 4.3, we have

(6.9)
$$\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\phi(x, y)d\|V(t)\|$$
$$=\lim_{m\to\infty}\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\phi(x, y)dV^m(t; x, y, S)$$
$$=\lim_{m\to\infty}\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\phi(x, y)|D\chi_t^m|.$$

This proves (i).

Proof of (ii). Let $\{\xi_k(x, y)\}_{k=1}^{\infty}$ be a countable dense subset of $C_0(\Omega\times\boldsymbol{R})$. We have from Proposition 4.2 that

$$\int\xi_k(x, y)|D\chi_t^m|\geqq-\max|\xi_k(x, y)|M\,.$$

The right hand side is independent of m. Take $\phi\in L^1(\boldsymbol{R})$ so that $\phi(t)\geqq0$. Then Fatou's lemma gives

(6.10)
$$\int_{\boldsymbol{R}}\phi(t)dt\Big(\liminf_{m\to\infty}\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)|D\chi_t^m|\Big)$$
$$\leqq\liminf_{m\to\infty}\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)|D\chi_t^m|$$

$$\leqq \liminf_{m\to\infty} \int_{\boldsymbol{R}} \phi(t)dt \int_{\Omega\times\boldsymbol{R}} \xi_k(x, y)dV^m(t; x, y, S)$$

$$= \int_{\boldsymbol{R}} \phi(t)dt \int_{\Omega\times\boldsymbol{R}} \xi_k(x, y)d\|V(t)\| .$$

Similarly we can prove

$$\int_{\boldsymbol{R}} \phi(t)dt \int_{\Omega\times\boldsymbol{R}} \xi_k(x, y)d\|V(t)\| \leqq \int_{\boldsymbol{R}} \phi(t)dt \limsup_{m\to\infty} \int_{\Omega\times\boldsymbol{R}} \xi_k(x, y)|D\chi_t^m|. \tag{6.11}$$

As a consequence of (6.10) and (6.11), there exists a subset $\boldsymbol{R}_3 \subset \boldsymbol{R}$ with the following properties: $L_1(\boldsymbol{R} \setminus \boldsymbol{R}_3)=0$ and we have

$$\liminf_{m\to\infty} \int_{\Omega\times\boldsymbol{R}} \xi_k(x, y)|D\chi_t^m| \leqq \int_{\Omega\times\boldsymbol{R}} \xi_k(x, y)d\|V(t)\| \tag{6.12}$$

$$\leqq \limsup_{m\to\infty} \int_{\Omega\times\boldsymbol{R}} \xi_k(x, y)|D\chi_t^m| ,$$

for each $t\in\boldsymbol{R}_3$ and for all $k=1, 2, \cdots,$. Since $\{\xi_k\}_k$ is dense in $C_0(\Omega\times\boldsymbol{R})$, (6.4) holds for any $\phi\in C_0(\Omega\times\boldsymbol{R})$ and $t\in\boldsymbol{R}_3$.

(iii) Let $\psi\in C_0(B)$ be a function such that $0\leqq\psi(x, y)\leqq 1$ and $\psi(x, y)=1$ on K. Let $t\in\boldsymbol{R}_3$. Then we have from (6.4) that

$$\liminf_{m\to\infty} |D\chi_t^m|(K) \leqq \liminf_{m\to\infty} \int_{\Omega\times\boldsymbol{R}} \psi(x, y)|D\chi_t^m| . \tag{6.13}$$

$$\leqq \int_{\Omega\times\boldsymbol{R}} \psi(x, y)d\|V(t)\|$$

$$\leqq \|V(t)\|(B) .$$

Similarly, we show that

$$\int_{\Omega\times\boldsymbol{R}} \psi(x, y)d\|V(t)\| \leqq \limsup_{m\to\infty} \int_{\Omega\times\boldsymbol{R}} \psi(x, y)|D\chi_t^m| \tag{6.14}$$

$$\leqq \limsup_{m\to\infty} |D\chi_t^m|(B) .$$

(6.13) and (6.14) proves (iii).

(iv) Let $B_1, B_2, \cdots$ be a sequence of open subsets of $\Omega\times\boldsymbol{R}$ satisfying $\bigcap_{k=1}^\infty B_k=\bar{B}$. Then we have from (iii) and (6.6) that

$$\|V(t)\|(B_k) \geqq \liminf_{m\to\infty} |D\chi_t^m|(\bar{B}) \geqq \limsup_{m\to\infty} |D\chi_t^m|(B) \geqq \|V(t)\|(B) , \tag{6.15}$$

for $k=1, 2, \cdots,$. As a consequence of the assumption (6.7), we see that $\lim_{k\to\infty}\|V(t)\|(B_k)=\|V(t)\|(B)$. It follows from this and (6.15) that $\lim_{m\to\infty}|D\chi_t^m|(B) =\|V(t)\|(B)$. (iv) is proved.

PROPOSITION 6.3. *If $t\in\boldsymbol{R}_3$, then*

$$|D\chi_t|(B) \leqq \|V(t)\|(B) \tag{6.16}$$

for any open subset $B\subset\Omega\times\boldsymbol{R}$. *If for some* $B\subset\Omega\times\boldsymbol{R}$

$$\lim_{m\to\infty}|D\chi_t^m|(B)=|D\chi_t|(B)\,, \tag{6.17}$$

then

$$\|V(t)\|(B)=|D\chi_t|(B)\,. \tag{6.18}$$

Proof. Assume that $\phi(x, y)\in C_0(B;\boldsymbol{R}^{n+1})$ and $|\phi(x, y)|\leqq 1$. Then from Proposition 6.2, we have

$$\begin{aligned}\left|\int_B\phi(x, y)D\chi_t\right| &=\liminf_{m\to\infty}\left|\int_B\phi(x, y)D\chi_t^m\right| \\ &\leqq\liminf_{m\to\infty}\int_B|\phi(x, y)|\,|D\chi_t^m| \\ &\leqq\int_B|\phi(x, y)|\,d\|V(t)\| \\ &\leqq\|V(t)\|(B)\,.\end{aligned} \tag{6.19}$$

Taking supremum with respect to ϕ, we have (6.16).

If (6.17) holds, then

$$|D\chi_t|(B)\leqq\|V(t)\|(B)\leqq\limsup_{m\to\infty}|D\chi_t^m|(B)=|D\chi_t|(B)\,.$$

(6.18) holds in this case.

PROPOSITION 6.4. *There exists a subset* $\boldsymbol{R}_4\subset\boldsymbol{R}$ *such that* $L_1(\boldsymbol{R}\setminus\boldsymbol{R}_4)=0$ *and*

$$\int_{\Omega\times\boldsymbol{R}}\phi(x, y)D_{n+1}\chi_t=\int_{\Omega\times\boldsymbol{R}}\phi(x, y)d\|V(t)\llcorner\nu_{n+1}\| \tag{6.20}$$

for $t\in\boldsymbol{R}_4$ *and* $\phi\in C_0(\Omega\times\boldsymbol{R})$. *In particular, for any* $t\in\boldsymbol{R}_4$ *and* $\phi\in C_0(\Omega)$, *we have*

$$-\int_\Omega\phi(x)dx=\int_\Omega\phi(x)d\|V(t)\llcorner\nu_{n+1}\|\,. \tag{6.21}$$

Proof. Let $\{\xi_k(x, y)\}_{k=1}^\infty$ be a countable dense subset of $C_0(\Omega\times\boldsymbol{R})$. Then Proposition 5.1 asserts that for any ξ_k and $t\in\boldsymbol{R}$

$$\lim_{m\to\infty}\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)D_{n+1}\chi_t^m=\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)D_{n+1}\chi_t\,. \tag{6.22}$$

Let $\phi\in L^1(\boldsymbol{R})$. Then multiplying (6.22) by $\phi(t)$ and integrating with respect to t, we have

$$\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)D_{n+1}\chi_t \tag{6.23}$$

$$=\lim_{m\to\infty}\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)D_{n+1}\chi_t^m$$

$$=\lim_{m\to\infty}\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\xi_k(x, y)\nu_{n+1}(S)dV^m(t; x, y, S)\,.$$

Applying Proposition 4.3 to the right hand side of (6.23), we have

$$\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)D_{n+1}\chi_t$$

$$=\int_{\boldsymbol{R}}\phi(t)dt\int_{\Omega\times\boldsymbol{R}\times G}\xi_k(x, y)\nu_{n+1}(S)dV(t; x, y, S)\,.$$

Therefore there exists a subset $\boldsymbol{R}_4\subset\boldsymbol{R}$ such that $L_1(\boldsymbol{R}\setminus\boldsymbol{R}_4)=0$ and

$$\int_{\Omega\times\boldsymbol{R}}\xi_k(x, y)D_{n+1}\chi_t=\int_{\Omega\times\boldsymbol{R}\times G}\xi_k(x, y)\nu_{n+1}(S)dV(t; x, y, S)$$

for $k=1, 2, \cdots$, and $t\in\boldsymbol{R}_4$. Since $\{\xi_k\}_k$ is dense in $C_0(\Omega\times\boldsymbol{R})$, this proves (6.20) for any $\psi\in C_0(\Omega\times\boldsymbol{R})$.

If $\phi\in C_0(\Omega)$, then

$$\int_{\Omega}\phi(x)D_{n+1}\chi_t=\lim_{m\to\infty}\int_{\Omega}\phi(x)D_{n+1}\chi_t^m=-\int_{\Omega}\phi(x)dx\,.$$

This together with (6.20) proves (6.21).

PROPOSITION 6.5. *Let (t_0, t_1) be an open interval and B be an open subset of $\Omega\times\boldsymbol{R}$. Assume that*

$$\int_{B\times\mathrm{irr}(G)}dV(t; x, y, S)=0 \qquad \textit{for all} \quad t\in(t_0, t_1)\,. \tag{6.24}$$

Then there exists a subset $N\subset(t_0, t_1)$ such that $L_1(N)=0$ and

$$\int_{\Omega\times\boldsymbol{R}}\psi(x, y)D_j\chi_t=\int_{\Omega\times\boldsymbol{R}}\psi(x, y)d\|V(t)\llcorner\nu_j\|\,, \qquad j=1, 2, \cdots, n\,, \tag{6.25}$$

for any $t\in(t_0, t_1)\setminus N$ and $\psi\in C_0(B)$.

Proof. Let $\{\xi_k\}_k$ be a countable dense subset of $C_0(B)$. Let

$$I_{jk}(t)=\int_B\xi_k(x, y)d\|V(t)\llcorner\nu_j\|\,.$$

Then (6.24) implies that

$$I_{jk}(t)=\lim_{\varepsilon\to 0}I_{jk}^{\varepsilon}(t)\,,$$

where

$$I_{jk}^{\varepsilon}(t)=\int_{B}\xi_{k}(x, y)\nu_{j}(S)\zeta_{\varepsilon}(\nu_{n+1}(S))dV(t; x, y, S)$$

and $\zeta_{\varepsilon}(\tau)=1$ for $|\tau|\geqq\varepsilon$ and $\zeta_{\varepsilon}(\tau)=\varepsilon^{-1}|\tau|$ for $\varepsilon\geqq|\tau|\geqq 0$. Since $\nu_{j}(S)\zeta_{\varepsilon}(\nu_{n+1}(S))$ is a continuous function of S, we can apply Proposition 4.3 to $I_{jk}^{\varepsilon}(t)$. Hence for any $\phi\in L^{1}(t_{0}, t_{1})$ we have

$$(6.26)\qquad \int_{t_0}^{t_1}\phi(t)I_{jk}(t)dt=\lim_{\varepsilon\to 0}\lim_{m\to\infty}\int_{t_0}^{t_1}\phi(t)dt\int_{B\times G}\xi_{k}(x, y)\nu_{j}(S)\zeta_{\varepsilon}(\nu_{n+1}(S))dV^{m}(t; x, y, S)$$

$$=\lim_{m\to\infty}\int_{t_0}^{t_1}\phi(t)dt\int_{B\times G}\xi_{k}(x, y)\nu_{j}(S)dV^{m}(t; x, y, S)+\lim_{\varepsilon\to 0}\lim_{m\to\infty}J_{jkm}^{\varepsilon},$$

where

$$(6.27)\qquad J_{jkm}^{\varepsilon}=\int_{t_0}^{t_1}\phi(t)dt\int_{B\times G}\xi_{k}(x, y)\nu_{j}(S)\{\zeta_{\varepsilon}(\nu_{n+1}(S))-1\}dV^{m}(t; x, y, S).$$

Using Proposition 5.1, we have

$$(6.28)\qquad \lim_{m\to\infty}\int_{t_0}^{t_1}\phi(t)dt\int_{B\times G}\xi_{k}(x, y)\nu_{j}(S)dV^{m}(t; x, y, S)$$

$$=\lim_{m\to\infty}\int_{t_0}^{t_1}\phi(t)dt\int_{B}\xi_{k}(x, y)D_{j}\chi_{t}^{m}$$

$$=\int_{t_0}^{t_1}\phi(t)dt\int_{B}\xi_{k}(x, y)D_{j}\chi_{t}.$$

On the other hand,

$$(6.29)\qquad \lim_{m\to\infty}|J_{jkm}^{\varepsilon}|\leqq\lim_{m\to\infty}\int_{t_0}^{t_1}|\phi(t)|dt\int_{B\times G}|\xi_{k}(x, y)|\{1-\zeta_{\varepsilon}(\nu_{n+1}(S))\}dV^{m}(t; x, y, S)$$

$$\leqq\int_{t_0}^{t_1}|\phi(t)|dt\int_{B\times G}|\xi_{k}(x, y)|\{1-\zeta_{\varepsilon}(\nu_{n+1}(S))\}dV(t; x, y, S).$$

Therefore using (6.26), (6.27) and (6.28), we have

$$\left|\int_{t_0}^{t_1}\phi(t)I_{jk}(t)dt-\int_{t_0}^{t_1}\phi(t)dt\int_{B}\xi_{k}(x, y)D_{j}\chi_{t}\right|$$

$$\leqq\limsup_{\varepsilon\to\infty}\int_{t_0}^{t_1}|\phi(t)|dt\int_{B\times G}|\xi_{k}(x, y)|\{1-\zeta_{\varepsilon}(\nu_{n+1}(S))\}dV(t; x, y, S)$$

$$=0$$

because of (6.24). Since $\phi(t)$ is arbitrary, there exists a subset N_{k} of $\boldsymbol{R}$ of L_{1} measure 0 such that

$$\int_{B}\xi_{k}(x, y)d\|V(t)\llcorner\nu_{j}\|=\int_{B}\xi_{k}(x, y)D_{j}\chi_{t},\qquad k=1, 2, \cdots, n,$$

for any $t\in(t_0, t_1)\setminus N_k$. Since $\{\xi_k\}_k$ is dense in $C_0(B)$, we have (6.25) for any $\phi\in C_0(B)$ and for $t\in(t_0, t_1)\setminus\bigcup_k N_k$.

We can state relationship of spt $V(t; x, y, S)$ and the graph of $u(t, x)$. Let $\pi:\Omega\times\boldsymbol{R}\times G\to\Omega\times\boldsymbol{R}$ be the projection. We call the set

$$\mathrm{irr}(V(t))=\pi(\mathrm{spt}\,V(t)\cap\Omega\times\boldsymbol{R}\times\mathrm{irr}(G))$$

the set of irregularity of $V(t)$. In this terminology we can see from propositions above the following

THEOREM 2. $\pi(\mathrm{spt}\,V(t))\setminus\mathrm{irr}(V(t))\subset\mathrm{spt}\,|D\chi_t|\subset\partial^*E(t)$.

Proof of this Theorem is clear from Proposition 6.4 and 6.5.

The next proposition gives the direct relationship of the varifold $V(t; x, y, S)$ and the graph of $u(t, x)$. We denote by $B(x, \rho)$ the open ball of radius $\rho>0$ centered at x in Ω.

PROPOSITION 6.6. *Let $V(t; x, y, S)$ be the varifold solution of* §4 *and $u(t, x)$ be as above. Then at L_{n+1}-almost all $(t, x)\in\boldsymbol{R}\times\Omega$,*

$$u(t, x)=\lim_{\rho\to 0} u_\rho(t, x)\,, \tag{6.30}$$

where

$$u_\rho(t, x)=\frac{-\int_{B(x,\rho)\times\boldsymbol{R}} y\,d\|V(t)\llcorner\nu_{n+1}\|}{L_n(B(x, \rho))}\,. \tag{6.31}$$

Proof. Let μ be the Radon measure on Ω defined by the equality

$$\mu(B)=\int_B u(t, x)dx\,.$$

Then we have, at L_n-almost all x,

$$u(t, x)=\lim_{\rho\to 0}\mu(B(x, \rho))/L_n(B(x, \rho))\,. \tag{6.32}$$

On the other hand, for any $\phi(x)\in C_0(\Omega)$, we have from Proposition 6.4 that

$$\begin{aligned}\int_\Omega u(t, x)\phi(x)dx&=\lim_{m\to\infty}\int_\Omega u^m(t, x)\phi(x)dx\\ &=-\lim_{m\to\infty}\int_{\Omega\times\boldsymbol{R}} y\phi(x)D_{n+1}\chi_t^m\\ &=-\int_{\Omega\times\boldsymbol{R}} y\phi(x)D_{n+1}\chi_t\\ &=-\int_{\Omega\times\boldsymbol{R}} y\phi(x)d\|V(t)\llcorner\nu_{n+1}\|\,.\end{aligned}$$

This means that

$$\mu(B(x, \rho))=-\int_{B(x,\rho)\times \boldsymbol{R}} y\, d\|V(t)\llcorner\nu_{n+1}\| . \tag{6.33}$$

Combining (6.32) and (6.33), we have (6.30) and (6.31).

As a consequence of Proposition 6.6 we may think that $u(t, x)$ represents the position of the membrane described by the varifold $V(t; x, y, S)$. Therefore

$$\frac{1}{2}\int_{\Omega}|D_t u(t, x)|^2 dx \tag{6.34}$$

represents the energy of motion. Similarly we can consider

$$\int_{\Omega\times \boldsymbol{R}\times G} dV(t; x, y, S)-|\Omega| \tag{6.35}$$

as the potential energy.

THEOREM 3. (*Energy inequality*). *Let* $u(t, x)$ *be as in Proposition* 5.1. *Then* $D_t u(t, *)\in L^2(\Omega)$ *for* L_1*-almost every* t *and we have*

$$\frac{1}{2}\int_{\Omega}|D_t u(t, x)|^2 dx+\int_{\Omega\times \boldsymbol{R}\times G} dV(t; x, y, S)\leqq M, \tag{6.36}$$

where M *is as in Proposition* 4.9. *If* $u_0\in W^{2+n/2,2}(\Omega)\cap W_0^{1,2}(\Omega)$, *then we have*

$$\begin{aligned}\frac{1}{2}\int_{\Omega}|D_t u(t, x)|^2 dx&+\int_{\Omega\times \boldsymbol{R}\times G} dV(t; x, y, S)\\ &\leqq \frac{1}{2}\int_{\Omega}|u_1(x)|^2 dx+\int_{\Omega}(1+|Du_0(x)|^2)^{1/2} dx .\end{aligned} \tag{6.37}$$

Proof. Using (4.8), we have

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}|D_t u(t, x)|^2 dx+\|V(t)\|(\Omega\times \boldsymbol{R})\\ &\leqq \frac{1}{2}\int_{\Omega}|D_t u^m(t, x)|^2 dx+\limsup_{m\to\infty}|D\chi_t^m|(\Omega\times \boldsymbol{R})\\ &\leqq M .\end{aligned} \tag{6.38}$$

If $u_0(x)$ is of class $W^{2+n/2,2}(\Omega)\cap W_0^{1,2}(\Omega)$, then Sobolev's imbeding theorem asserts that $DP_m u_0(x)$ converges to $Du_0(x)$ uniformly. This yields that

$$\lim_{m\to\infty}\int_{\Omega}(1+|DP_m u_0(x)|^2)^{1/2} dx=\int_{\Omega}(1+|Du_0(x)|^2)^{1/2} dx . \tag{6.39}$$

Applying this to (4.7), we can prove (6.37).

Next we prove

LEMMA 6.7. *Let B be an open subset of $\Omega\times\boldsymbol{R}$. Assume that for L_1-almost all $t\in(t_0, t_1)$*

$$|D\chi_t|(B)=\lim_{m\to\infty}|D\chi_t^m|(B). \tag{6.40}$$

Assume further that

$$\int_{B\times \mathrm{irr}(G)} dV(t; x, y, S)=0 \qquad \text{for almost all} \quad t\in(t_0, t_1). \tag{6.41}$$

Then at almost all $t\in(t_0, t_1)$, the varifold $V(t; x, y, S)$ is canonically identified with the function $u(t, x)$ in B. Let ω be an open subset of Ω. Assume that (6.40) *and* (6.41) *hold for $B=\omega\times\boldsymbol{R}$. Then $u(t, x)$ is a BV-solution of* (1.1) *in $(t_0, t_1)\times\omega$.*

Proof. We have only to prove the first part of the Proposition. We put

$$B(x, y; \rho)=\{(w, z)\in\Omega\times\boldsymbol{R}: |w-x|^2+|z-y|^2<\rho^2\}.$$

For any continuous function $\alpha(S)$ of $S\in G$, we consider

$$V_t^{x,y}(\alpha)=\lim_{\rho\to 0}\int_{B(x,y,\rho)}\alpha(S)dV(t; x, y, S)/\|V(t)\|(B(x, y; \rho)) \tag{6.42}$$

for almost all t. This exists at $\|V(t)\|$-almost every (x, y). (cf. 3.3 of Allard [1].)

The mapping $C(G)\ni\alpha\to V_t^{x,y}(\alpha)\in\boldsymbol{R}$ defines a positive Radon measure $V_t^{x,y}(S)$ on G, that is,

$$V_t^{x,y}(\alpha)=\int_G\alpha(S)dV_t^{x,y}(S). \tag{6.43}$$

It is clear from the definition that

$$\int_G dV_t^{x,y}(S)=1 \tag{6.44}$$

and that for any $\phi\in C_0(\Omega\times\boldsymbol{R})$

$$\int_{\Omega\times\boldsymbol{R}\times G}\phi(x, y)\alpha(S)dV(t; x, y, S) \tag{6.45}$$

$$=\int_{\Omega\times\boldsymbol{R}}\phi(x, y)\Big(\int_G\alpha(S)dV_t^{x,y}(S)\Big)d\|V(t)\|.$$

We cannot apply (6.45) to $\alpha(S)=\nu_j(S)$, $j=1, 2, \cdots, n$, because $\nu_j(S)$ is not continuous on G. We claim that if $\operatorname{spt}\phi$ is contained in B, then equality

$$\int_{B\times G}\phi(x, y)\nu_j(S)dV(t; x, y, S)=\int_B\phi(x, y)\Big\{\int_G\tilde{\nu}_j(S)dV_t^{x,y}(S)\Big\}d\|V(t)\| \tag{6.46}$$

holds, where $\tilde{\nu}_j(S)=\nu_j(S)$ for $S\in G\setminus\mathrm{irr}(G)$ and $\tilde{\nu}_j(S)=0$ for $S\in\mathrm{irr}(G)$.

We prove the claim. Let ε be an arbitrary positive number and $\zeta_\varepsilon(t)$ be the function used in the proof of Proposition 6.5. Then

$$(6.47)\qquad \int_{B\times G}\phi(x, y)\nu_j(S)dV(t; x, y, S)$$
$$=\int_{B\times G}\phi(x, y)\tilde{\nu}_j(S)dV(t; x, y, S)$$
$$=\lim_{\varepsilon\to 0}\int_{B\times G}\phi(x, y)\nu_j(S)\zeta_\varepsilon(\nu_{n+1}(S))dV(t; x, y, S).$$

Since $\nu_j(S)\zeta_\varepsilon(\nu_{n+1}(S))$ is a continuous function of S, we can apply (6.45) to the right hand side of (6.47). Thus we have

$$\int_{B\times G}\phi(x, y)\nu_j(S)dV(t; x, y, S)$$
$$=\lim_{\varepsilon\to 0}\int_B\phi(x, y)\Big(\int_G\nu_j(S)\zeta_\varepsilon(\nu_{n+1}(S))dV_t^{x,y}(S)\Big)d\|V(t)\|$$
$$=\int_B\phi(x, y)\Big(\int_G\tilde{\nu}_j(S)dV_t^{x,y}(S)\Big)d\|V(t)\|.$$

We have proved the claim (6.46).

Next we wish to prove that

$$(6.48)\qquad \nu_j(t; x, y)=\int_G\tilde{\nu}_j(S)dV_t^{x,y}(S), \qquad j=1, 2, \cdots, n+1,$$

for almost all t and $\|V(t)\|$-almost every $(x, y)\in B$. In fact combining Proposition 6.5 and (6.46), we have

$$(6.49)\qquad \int_B\phi(x, y)\nu_j(t; x, y)|D\chi_t|=\int_B\phi(x, y)D_j\chi_t$$
$$=\int_B\phi(x, y)d\|V(t)\llcorner\nu_j\|$$
$$=\int_{B\times G}\phi(x, y)\nu_j(S)dV(t; x, y, S)$$
$$=\int_B\phi(x, y)\Big\{\int_G\tilde{\nu}_j(S)dV_t^{x,y}(S)\Big\}d\|V(t)\|.$$

As a consequence of (6.49), for any $(x, y)\in B$ and for sufficiently small $\rho>0$, we have

(6.50)
$$\int_{B(x,y;\rho)} \nu_j(t; x, y)|D\chi_t| = \int_{B(x,y;\rho)} \left(\int_G \tilde{\nu}_j(S) dV_t^{x,y}(S)\right) d\|V(t)\|.$$

For each $(x, y) \in B$ and almost all t, we can choose a sequence of positive numbers $\{\rho_k\}_{k=1}^{\infty}$, such that

(6.51)
$$\lim_{k\to\infty} \rho_k = 0$$

and

(6.52)
$$\|V(t)\|(\partial B(x, y; \rho_k)) = 0, \qquad k = 1, 2, \cdots, .$$

By virtue of Proposition 6.3 and (6.52), we have

$$|D\chi_t|(\partial B(x, y; \rho_k)) = 0, \qquad k = 1, 2, \cdots, .$$

This and assumption (6.40) imply that

$$|D\chi_t|(B(x, y; \rho_k)) = \lim_{m\to\infty} |D\chi_t^m|(B(x, y; \rho_k))$$

(cf. Giusti [6]). Using Proposition 6.2 (iv), we see that

(6.53)
$$|D\chi_t|(B(x, y; \rho_k)) = \|V(t)\|(B(x, y; \rho_k)), \qquad k = 1, 2, \cdots, .$$

This together with (6.50) yields that

(6.54)
$$\int_{B(x,y;\rho_k)} \nu_j(t; x, y)|D\chi_t| / |D\chi_t|(B(x, y; \rho_k)) = \int_{B(x,y,\rho_k)} \left(\int_G \tilde{\nu}_j(S) dV_t^{x,y}(S)\right) d\|V(t)\| / \|V(t)\|(B(x, y; \rho_k)).$$

Let k tend to ∞ and take the limit of (6.54). Then (6.51) and Besicovitch's theorem (cf. [3] or [5]) give (6.48).

Applying the next Lemma 6.8 to (6.48), we conclude that

(6.55)
$$\tilde{\nu}_j(S) = \nu_j(t; x, y)$$

at $V_t^{x,y}$-almost all $S \in G$. If $S \neq S'$ then $\tilde{\nu}_j(S) \neq \tilde{\nu}_j(S')$. Thus (6.55) implies that

$$\operatorname{spt} V_t^{x,y} = \text{one point} = \operatorname{Tan}_{x,y} \partial^* E(t).$$

And for each $\alpha \in C(G)$, we have

(6.56)
$$\int_G \alpha(S) dV_t^{x,y}(S) = \alpha(\operatorname{Tan}_{x,y} \partial^* E(t)).$$

It follows from (6.56), (6.45), (6.53) and Besicovitch's theorem that for any $\phi \in C_0(B \times G)$, we have

$$\int_{B\times G}\phi(x,\ y,\ S)dV(t\,;\,x,\ y,\ S) \tag{6.57}$$

$$=\int_{B\times G}\phi(x,\ y,\ \mathrm{Tan}_{x,y}\partial^{*}E(t))d\|V(t)\|$$

$$=\int_{B\times G}\phi(x,\ y,\ \mathrm{Tan}_{x,y}\partial^{*}E(t))\,|D\chi_{t}|.$$

Therefore $V(t\,;\,x,\ y,\ S)$ is canonically identified with the graph of $u(t,\ x)$. Lemma 6.7 has been proved upto the following Lemma 6.8.

LEMMA 6.8. *Let P be a probability measure on a space X. Let $v(x)$ be an $\boldsymbol{R}^{n}$-valued function which is integrable with respect to P. Let*

$$v=\int_{X}v(x)dP(x)\,.$$

Assume that $|v(x)|\leqq 1$ and $|v|=1$. Then $v=v(x)$ at P-almost every x.

Proof is clear.

THEOREM 4. *Assume that $u_{1}\in L^{2}(\Omega)$ and $u_{0}\in W^{2+n/2,2}(\Omega)$. Assume further that the function $u(t,\ x)$ of Proposition 5.1 satisfies the energy conservation law for $t\in(t_{0},\ t_{1})$, i.e.,*

$$\frac{1}{2}\int|D_{t}u(t,\ x)|^{2}dx+\int_{\Omega\times\boldsymbol{R}}|D\chi_{t}| \tag{6.58}$$

$$=\frac{1}{2}\int_{\Omega}|u_{1}(x)|^{2}dx+\int_{\Omega}(1+|Du_{0}(x)|^{2})^{1/2}dx\,.$$

Let ω be any open subset of Ω such that

$$\int_{\omega\times\boldsymbol{R}\times\mathrm{1rr}(G)}dV(t\,;\,x,\ y,\ S)=0 \tag{6.59}$$

for atmost all $t\in(t_{0},\ t_{1})$. Then at L_{1}-almost all $t\in(t_{0},\ t_{1})$, the varifold solution $V(t\,;\,x,\ y,\ S)$ is canonically identified with the graph of the function $u(t,\ x)$ at $\mathcal{H}_{n+1}$-almost every $(x,\ y)\in\omega\times\boldsymbol{R}$ and $u(t,\ x)$ is the solution of (1.1) in $(t_{0},\ t_{1})\times\omega$.

Proof. Let

$$M_{m}=\frac{1}{2}\int_{\Omega}|P_{m}u_{1}(x)|^{2}dx+\int_{\Omega}(1+|DP_{m}u_{0}(x)|^{2})^{1/2}dx\,.$$

Then the proof of (6.37) asserts that

$$\lim_{m\to\infty}M_{m}=\frac{1}{2}\int_{\Omega}|u_{1}(x)|^{2}dx+\int_{\Omega}(1+|Du_{0}(x)|^{2})^{1/2}dx\,. \tag{6.60}$$

We have from (4.7)

$$M_m=\frac{1}{2}\int_\Omega |D_t u^m(t, x)|^2 dx+\int_{\Omega\times\boldsymbol{R}} |D\chi_t^m|. \tag{6.61}$$

The assumption (6.58) means that

$$\frac{1}{2}\int_\Omega |D_t u(t, x)|^2 dx+\int_{\Omega\times\boldsymbol{R}} |D\chi_t| \tag{6.62}$$
$$=\lim_{m\to\infty}\left\{\frac{1}{2}\int_\Omega |D_t u^m(t, x)|^2 dx+\int_{\Omega\times\boldsymbol{R}} |D\chi_t^m|\right\}.$$

Since

$$\int_\Omega |D_t u(t, x)|^2 dx \leqq \limsup_{m\to\infty}\int_\Omega |D_t u^m(t, x)|^2 dx \tag{6.63}$$

$$\int_{\Omega\times\boldsymbol{R}} |D\chi_t| \leqq \limsup_{m\to\infty}\int_{\Omega\times\boldsymbol{R}} |D\chi_t^m|, \tag{6.64}$$

the equality (6.62) asserts that equalities hold in both (6.63) and (6.64), namely, we have

$$\int_\Omega |D_t u(t, x)|^2 dx=\lim_{m\to\infty}\int_\Omega |D_t u^m(t, x)|^2 dx \tag{6.65}$$

and

$$\int_{\Omega\times\boldsymbol{R}} |D\chi_t|=\lim\int_{\Omega\times\boldsymbol{R}} |D\chi_t^m|. \tag{6.66}$$

Therefore, the set $\Omega\times\boldsymbol{R}$ itself satisfies the condition (6.40) of Lemma 6.7. As the consequence of Lemma 6.7, we can prove Theorem 4.

§ 7. Generalized Hamilton's principle.

So far we have treated the special varifold solution $V(t; x, y, S)$ constructed in § 4. In the present section we treat any varifold solution $W(t; x, y, S)$ of (1.1) satisfying additional conditions which will be given below. And we prove that a generalized Hamilton's principle holds for such a good varifold solution.

We define measures $\|W(t)\llcorner\nu_j\|$, $j=1, 2, \cdots, n+1$, on $\Omega\times\boldsymbol{R}$ by the following formula: For any Borel set $A\subset\Omega\times\boldsymbol{R}$

$$\|W(t)\llcorner\nu_j\|(A)=\int_{A\times G}\nu_j(S)dW(t; x, y, S) \tag{7.1}$$

in just the same way as in § 6. In analogy with Proposition 6.6, we put, for $x\in\Omega$ and $t\in\boldsymbol{R}$,

$$w(t, x)=\lim_{\rho\to 0} w_\rho(t, x), \tag{7.2}$$

where

$$w_\rho(t, x)=\int_{B(x,\rho)\times R} y\, d\|W(t)\llcorner \nu_{n+1}\| \Big/ \int_{B(x,\rho)\times R} d\|W(t)\llcorner \nu_{n+1}\| . \tag{7.3}$$

We call $w(t, x)$ the position of the membrane. It follows from Besicovitch's theorem that $w(t, x)$ exists almost every x with respect to the measure $\|W(t)\llcorner \nu_{n+1}\|$.

We call

$$\frac{1}{2}\int_\Omega |D_t w(t, x)|^2 dx$$

the energy of motion if it is finite. Similarly, we may call

$$\int_{\Omega\times R\times G} dW(t; x, y, S)-|\Omega|=\int_{\Omega\times R} d\|W(t)\|-|\Omega|$$

the potential energy.

We assume that the following conditions hold for the varifold solution $W(t; x, y, S)$:

(A1) The position function $w(t, x)$ is a function of bounded variation in Ω for a fixed $t\in \boldsymbol{R}$ and $\mathrm{spt}\|W(t)\llcorner \nu_{n+1}\| \subset \partial^* F(t)$, where $F(t)$ is the subgraph of the function $w(t, x)$.

(A2) $D_t w(t, x)\in L^2(\Omega)$ for each t and

$$\int_0^T dt\int_\Omega \frac{1}{2}|D_t w(t, x)|^2 dx+\int_0^T dt\int_{\Omega\times R\times G} dW(t; x, y, S)<\infty . \tag{7.4}$$

(A3) For each $\phi(x)\in C_0(\Omega)$

$$-\int_{\Omega\times R} \phi(x) d\|W(t)\llcorner \nu_{n+1}\|=\int_\Omega \phi(x)dx . \tag{7.5}$$

The last equality expresses a generalization of the law of conservation of mass. As we have proved in §6, the varifold solution $V(t; x, y, S)$ constructed in §4 has all these properties.

If $W(t; x, y, S)$ satisfies all of these conditions, then we consider the action

$$A(W)=\int_0^T dt\int_\Omega \frac{1}{2}|D_t w(t, x)|^2 dx-\int_0^T dt\left\{\int_{\Omega\times R\times G} dW(t; x, y, S)-|\Omega|\right\}, \tag{7.6}$$

and we shall show that W is a critical point of this action functional, i. e.,

$$\delta A(W)=0 . \tag{7.7}$$

To state this fact more precisely we introduce admissible functions $\phi(t, x) \in C^2(\boldsymbol{R}\times\Omega)$ such that

$$\phi(0, x)=D_t\phi(0, x)=0 , \qquad \phi(T, x)=D_t\phi(T, x)=0$$

and $\phi(t, x)|_{\partial\Omega}=0$. Then for each $\sigma\in \boldsymbol{R}$ we can define a diffeomorphism

$$\eta(\sigma):\ \Omega\times\boldsymbol{R}\ni(x, y)\to(x, y+\sigma\phi(t, x))\in\Omega\times\boldsymbol{R} . \tag{7.8}$$

This induces a map $\eta(\sigma)_{\#}$ of varifolds, which is defined by the equality

$$(7.9)\qquad \langle \eta(\sigma)_{\#}W(t),\ \phi\rangle = \int_{\Omega\times\boldsymbol{R}\times G}\phi(x,\ y+\sigma\psi(t,\ x),\ D\eta(\sigma)S)|\wedge^n D\eta(\sigma)|dW(t;\ x,\ y,\ S).$$

(cf. Allard [1], § 3.2), where $D\eta(\sigma)$ is the differential of the map $\eta(\sigma)$ and $\wedge^n D\eta(\sigma)$ is its n-exterior product. The precise formulation of the generalized Hamilton's principle is

THEOREM 5. *Assume that* $W(t;\ x,\ y,\ S)$ *is a varifold solution of the equations* (1.1) *and* (1.2) *and that it satisfies the assumptions* (A1), (A2) *and* (A3). *Then*

$$(7.10)\qquad \frac{d}{d\sigma}A(\eta(\sigma)_{\#}W)|_{\sigma=0}=0.$$

Proof. We first calculate the position $w^{\sigma}(t,\ x)$ corresponding to the varifold $\eta(\sigma)_{\#}W(t;\ x,\ y,\ S)$, that is,

$$(7.11)\qquad w^{\sigma}(t,\ x)=\lim_{\rho\to 0} w_{\rho}^{\sigma}(t,\ x),$$

where

$$(7.12)\qquad w_{\rho}^{\sigma}(t,\ x)=\frac{\int_{B(x,\ \rho)\times\boldsymbol{R}\times G} y\nu_{n+1}(S)d(\eta(\sigma)_{\#}W(t;\ z,\ y,\ S))}{\int_{B(x,\ \rho)\times\boldsymbol{R}\times G}\nu_{n+1}(S)d(\eta(\sigma)_{\#}W(t;\ z,\ y,\ S))}.$$

We have for any $x\in\Omega$ and $\rho>0$,

$$(7.13)\qquad \int_{B(x,\ \rho)\times\boldsymbol{R}\times G} y\nu_{n+1}(S)d(\eta(\sigma)_{\#}W(t;\ z,\ y,\ S)) = \int_{B(x,\ \rho)\times\boldsymbol{R}\times G}(y+\sigma\psi(t,\ z))\nu_{n+1}(D\eta(\sigma)S)|\wedge^n D\eta(\sigma)|dW(t;\ z,\ y,\ S).$$

Using assumptions (A1) and (A3), we see that this is equal to

$$(7.14)\qquad \int_{B(x,\ \rho)\times\boldsymbol{R}\times G}(w(t,\ z)+\sigma\psi(t,\ z))\nu_{n+1}(D\eta(\sigma)S)|\wedge^n D\eta(\sigma)|dW(t;\ z,\ y,\ S).$$

Similarly, we have

$$(7.15)\qquad \int_{B(x,\ \rho)\times\boldsymbol{R}\times G}\nu_{n+1}(S)d(\eta(\sigma)_{\#}W(t;\ z,\ y,\ S)) = \int_{B(x,\ \rho)\times\boldsymbol{R}\times G}\nu_{n+1}(D\eta(\sigma)S)|\wedge^n D\eta(\sigma)|dW(t;\ z,\ y,\ S).$$

It follows from (7.11), (7.12), (7.13), (7.14) and Besicovitch's theorem that

$$(7.16)\qquad w^{\sigma}(t,\ x)=w(t,\ x)+\sigma\psi(t,\ x),$$

at almost every $x\in\Omega$ with respect to the measure μ such that for any Borel subset $B\subset\Omega$

$$\mu(B)=\int_{B\times R\times G}\nu_{n+1}(D\eta(\sigma)S)|\wedge^n D\eta(\sigma)|dW(t;x,y,S).$$

We claim that (7.16) holds at L_n-almost all x in Ω. To prove this we shall show that an n-dimensional vector subspace $S\in G$ satisfies $\nu_{n+1}(D\eta(\sigma)S)=0$ if and only if $\nu_{n+1}(S)=0$. Assume that $\nu_{n+1}(S)\neq 0$. Then we can choose a basis $v_1, v_2, \cdots, v_n$ of S so that $v_1=e_1+\beta_1 e_{n+1}$, $v_2=e_2+\beta_2 e_{n+1}$, $\cdots$, $v_n=e_n+\beta_n e_{n+1}$, where e_i, $i=1, 2, \cdots, n$, is the unit vector parallel to the x_i-axis and e_{n+1} is the unit vector parallel to the y-axis. Since $D\eta(\sigma)v_i=e_i+\left(\beta_i+\sigma\frac{\partial}{\partial x_i}\phi(t, x)\right)e_{n+1}$, we have

$$v_1\wedge v_2\wedge\cdots\wedge v_n=e_1\wedge e_2\cdots\wedge e_n+g\wedge e_{n+1}$$

with some $g\in\wedge^{n-1}\boldsymbol{R}^n$. This implies that $\nu_{n+1}(D\eta(\sigma)S)\neq 0$. Similarly we can prove that $\nu_{n+1}(D\eta(\sigma)S)=0$ if $\nu_{n+1}(S)=0$.

Since $|\wedge^n D\eta(\sigma)|$ never vanishes, we see that

$$0=\mu(B)=\int_{B\times R\times G}\nu_{n+1}(D\eta(\sigma)S)|\wedge^n D\eta(\sigma)|dW(t;x,y,S)$$

if and only if

$$L_n(B)=\int_{B\times R\times G}\nu_{n+1}(S)dW(t;x,y,S)=0.$$

This proves that (7.16) holds for L_n-almost every x in Ω. Thus we have

$$A(\eta(\sigma)_{\#}W)=\frac{1}{2}\int_0^T dt\int_\Omega |D_t w(t,x)+\sigma D_t\phi(t,x)|^2 dx \\ -\int_0^T dt\int_{\Omega\times R\times G} d(\eta(\sigma)_{\#}W(t;x,y,S))+|\Omega|. \tag{7.17}$$

We now calculate the variation $\frac{d}{d\sigma}A(\eta(\sigma)_{\#}W)|_{\sigma=0}$. First we have

$$\begin{aligned}&\frac{d}{d\sigma}\int_0^T dt\int_\Omega\frac{1}{2}|D_t w^\sigma(t,x)|^2 dx\Big|_{\sigma=0} \\ &=\frac{d}{d\sigma}\int_0^T dt\int_\Omega\frac{1}{2}|D_t w(t,x)+\sigma\phi(t,x)|^2 dx\Big|_{\sigma=0} \\ &=\int_0^T dt\int_{\Omega\times R\times G} w(t,x)D_t^2\phi(t,x)\nu_{n+1}(S)dW(t;x,y,S) \\ &=\int_0^T dt\int_{\Omega\times R\times G} D_t^2\phi(t,x)y\nu_{n+1}(S)dW(t;x,y,S).\end{aligned}\tag{7.18}$$

Next we describe the variation of the second term of the right hand side of

(7.17). Let $\dot{\eta}(x, y)=D_\sigma\eta(\sigma)(x, y)|_{\sigma=0}=(0, 0, \cdots, \phi(t, x))$ be the vector field which is the tangent at $\sigma=0$ to the 1-parameter family of diffeomorphisms $\eta(\sigma)$. We know that (cf. Allard [1], §3.3)

$$\frac{d}{d\sigma}\int_{\Omega\times\boldsymbol{R}\times G} d(\eta(\sigma)_\# W)(t; x, y, S)|_{\sigma=0} = \int_{\Omega\times\boldsymbol{R}\times G} \sum_{k=1}^{n} D_k\phi(t, x)\nu_k(S)\nu_{n+1}(S)dW(t; x, y, S). \tag{7.19}$$

Consequently

$$\frac{d}{d\sigma}(A(\eta(\sigma)_\# W))|_{\sigma=0} = \int_0^T dt\int_{\Omega\times\boldsymbol{R}\times G} D_t^2\phi(t, x)y\nu_{n+1}(S)dW(t; x, y, S) + \int_0^T dt\int_{\Omega\times\boldsymbol{R}\times G} \sum_{k=1}^{n} D_k\phi(t, x)\nu_k(S)\nu_{n+1}(S)dW(t; x, y, S). \tag{7.20}$$

Since $W(t; x, y, S)$ is a varifold solution of (1.1), (1.3) and $\phi(0, x)=D_t\phi(0, x)=0$, the right hand side vanishes by virtute of (3.3). We have

$$\frac{d}{d\sigma}A(\eta(\sigma)_\# W)|_{\sigma=0}=0.$$

Theorem 6 is proved.

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