S. SAITOH
KODAI MATH. J.
9 (1986), 50—57

ON THE CONVOLUTION OF L_2 FUNCTIONS

BY SABUROU SAITOH

1. Introduction.

For the convolution $F*G$ of $F \in L_p(-\infty, \infty)(p \geqq 1)$ and $G \in L_1(-\infty, \infty)$, we know the fundamental inequality

$$\|F*G\|_p \leqq \|F\|_p \|G\|_1. \tag{1.1}$$

See, for example, [8, p. 3]. Note that for $F, G \in L_2(-\infty, \infty)$, in general, $F*G \notin L_2(-\infty, \infty)$. In this paper, we will give an identification of a Hilbert space spanned by the convolutions $F*G$ and establish fundamental inequalities in the convolution. Note that when the space is $L_2(0, \infty)$, the results are very simple and quite different from the present case $L_2(-\infty, \infty)$. See [7].

2. The case of functions with compact supports.

We first consider the case of the convolution $F*G$ of $F \in L_2(a, b)$ and $G \in L_2(c, d)$. Without loss of generality we assume that $a+d \leqq b+c$. Of course, in the convolution we regard F and G as zero in the outsides of the intervals $[a, b]$ and $[c, d]$, respectively. We consider the integral transform, for $F \in L_2(a, b)$ and $z = x+iy \in \boldsymbol{C}$

$$f(z) = \frac{1}{2\pi}\int_a^b F(t)e^{-izt}dt. \tag{2.1}$$

As we see from the general theory [5, 6] of integral transforms, the images $f(z)$ form the Hilbert space $H_{(a,b)}$ admitting the reproducing kernel on $\boldsymbol{C}$

$$K_{(a,b)}(z, \bar{u}) = \frac{1}{2\pi}\int_a^b e^{-izt}e^{i\bar{u}t}dt. \tag{2.2}$$

Since the family $\{e^{-izt}; z \in \boldsymbol{C}\}$ is complete in $L_2(a, b)$, we further have the isometrical identity

$$\|f\|_{H_{(a,b)}}^2 = \frac{1}{2\pi}\int_a^b |F(t)|^2 dt. \tag{2.3}$$

Hence, by using the Fourier transform for (2.1) in the framework of the L_2 space, we have

Received March 16, 1985

(2.4) $$\|f\|^2_{H(a,b)}=\frac{1}{2\pi}\int_a^d \left|\underset{N\to\infty}{\text{l.i.m.}}\int_{-N}^N f(x)e^{ixt}dx\right|^2 dt.$$

We consider similarly the integral transform, for $G\in L_2(c, d)$

(2.5) $$g(z)=\frac{1}{2\pi}\int_c^d G(t)e^{-izt}dt$$

and the Hilbert space $H_{(c,d)}$ admitting the reproducing kernel

$$K_{(c,d)}(z, \bar{u})=\frac{1}{2\pi}\int_c^d e^{-izt}e^{i\bar{u}t}dt.$$

Then, we have

(2.6) $$f(z)g(z)=\frac{1}{4\pi^2}\int_{a+c}^{b+d}(F*G)(t)e^{-izt}dt,$$

where

$$(F*G)(t)=\begin{cases}\displaystyle\int_a^{t-c}F(t_1)G(t-t_1)dt_1 & \text{for } a+c\leqq t\leqq a+d\\ \displaystyle\int_{t-d}^{t-c}F(t_1)G(t-t_1)dt_1 & \text{for } a+d\leqq t\leqq b+c\\ \displaystyle\int_{t-d}^{b}F(t_1)G(t-t_1)dt_1 & \text{for } b+c\leqq t\leqq b+d.\end{cases}$$

The product $f(z)g(z)$ belongs to the Hilbert space $[H_{(a,b)}\otimes H_{(c,d)}]_R$ which is the restriction of the tensor product $H_{(a,b)}\otimes H_{(c,d)}$ to the diagonal set $\boldsymbol{C}$ of $\boldsymbol{C}\times\boldsymbol{C}$. Here the norm is given by

(2.7) $$\|fg\|^2_{[H(a,b)\otimes H(c,d)]_R}=\min\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(f_j, f_k)_{H(a,b)}(g_j, g_k)_{H(c,d)}.$$

The minimum is taken over all functions $\sum_{j=1}^{\infty}f_j(z_1)g_j(z_2)$ on $\boldsymbol{C}\times\boldsymbol{C}$ satisfying

(2.8) $$f(z)g(z)=\sum_{j=1}^{\infty}f_j(z)g_j(z) \quad \text{on } \boldsymbol{C}$$

for $f_j\in H_{(a,b)}$ and $g_j\in H_{(c,d)}$. Moreover, the Hilbert space $[H_{(a,b)}\otimes H_{(c,d)}]_R$ admits the reproducing kernel $K_{(a,b)}(z, \bar{u})K_{(c,d)}(z, \bar{u})$ and is characterized by this property ([1, pp. 357-362 and p. 344]).

In order to realize the norm in $[H_{(a,b)}\otimes H_{(c,d)}]_R$, we compute the kernel $K_{(a,b)}(z, \bar{u})K_{(c,d)}(z, \bar{u})$ in a reduced form; that is,

(2.9) $$K_{(a,b)}(z, \bar{u})K_{(c,d)}(z, \bar{u})$$

$$=\frac{1}{4\pi^2}\int_a^b\int_c^d e^{-izt_1}e^{iut_1}e^{-izt_2}e^{iut_2}dt_1dt_2$$

$$=\frac{1}{4\pi^2}\int_{a+c}^{a+d}\{t-(a+c)\}e^{-izt}e^{iut}dt$$

$$+\frac{1}{4\pi^2}\int_{a+d}^{b+c}(d-c)e^{-izt}e^{iut}dt$$

$$+\frac{1}{4\pi^2}\int_{b+c}^{b+d}\{(b+d)-t\}e^{-izt}e^{iut}dt.$$

We denote, in general, the characteristic function of $[a, b]$ by $\chi(t;[a, b])$ such that

$$\chi(t;[a, b])=\begin{cases}1 & \text{for } t\in[a, b]\\ 0 & \text{for } t<a,\text{ or } b<t.\end{cases}$$

We set

$$V(t)=\{t-(a+c)\}\chi(t;[a+c, a+d])+(d-c)\chi(t;[a+d, b+c])$$
$$+(b+d-t)\chi(t;[b+c, b+d]).$$

Then, any member $\phi(z)$ of $[H_{(a,b)}\otimes H_{(c,d)}]_R$ is expressible in the form

$$\phi(z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\Phi(t)e^{-izt}V(t)dt, \tag{2.10}$$

for a uniquely determined function Φ satisfying

$$\int_{-\infty}^{\infty}|\Phi(t)|^2V(t)dt<\infty. \tag{2.11}$$

Moreover, the norm is given by, as in (2.4)

$$\|\phi\|^2_{[H_{(a,b)}\otimes H_{(c,d)}]_R}=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}|\Phi(t)|^2V(t)dt \tag{2.12}$$

$$=\int_{\infty}^{\infty}\left|\underset{N\to\infty}{\text{l.i.m.}}\int_{-N}^{N}\phi(x)e^{ixt}dx\right|^2W(t)dt,$$

where

$$W(t)=\frac{\chi(t;[a+c, a+d])}{t-(a+c)}+\frac{\chi(t;[a+d, b+c])}{d-c}+\frac{\chi(t;[b+c, b+d])}{b+d-t}.$$

See [5, 6]. From the property of (2.7), we, in particular, obtain the following inequalities.

THEOREM 2.2. *For any $f\in H_{(a,b)}$ and $g\in H_{(c,d)}$, we have the inequality*

$$\int_{a+c}^{b+d}\left|\underset{N\to\infty}{\text{l.i.m.}}\int_{-N}^{N}f(x)g(x)e^{ixt}dx\right|^2W(t)dt \tag{2.13}$$

$$\leq\frac{1}{2\pi}\int_{a}^{b}\left|\underset{N\to\infty}{\text{l.i.m.}}\int_{-N}^{N}f(x)e^{ixt}dx\right|^2dt$$

$$\cdot\frac{1}{2\pi}\int_{c}^{d}\left|\underset{N\to\infty}{\text{l.i.m.}}\int_{-N}^{N}g(x)e^{ixt}dx\right|^2dt$$

or, for any $F\in L_2(a, b)$ *and* $G\in L_2(c, d)$

$$\int_{a+c}^{b+d}|(F*G)(t)|^2W(t)dt\leqq\int_a^b|F(t)|^2dt\int_c^d|G(t)|^2dt. \tag{2.14}$$

As a property of the convolution $F*G$, we have

COROLLARY 2.1. *The convolution* $F*G$ *of* $F\in L_2(a, b)$ *and* $G\in L_2(c, d)$ *is expressible in the form*

$$(F*G)(t)=\Phi(t)V(t) \tag{2.15}$$

for a function Φ *satisfying*

$$\int_{-\infty}^{\infty}|\Phi(t)|^2V(t)dt<\infty. \tag{2.16}$$

Conversely, for any Φ *satisfying* (2.16), *the right hand in* (2.15) *is expressible in the form, for* $F_j\in L_2(a, b)$ *and* $G_j\in L_2(c, d)$

$$\Phi(t)V(t)=\sum_{j=1}^{\infty}(F_j*G_j)(t)$$

in the sense of the strong convergence in the norm (2.16).

Further, when $G\equiv 1$ on $[0, d]$, we have

COROLLARY 2.2. *For any* $F\in L_2(a, b)$ *and for any* d *such that* $a+d\leqq b$, *we have the inequality*

$$\int_a^{a+d}\frac{1}{t-a}\left|\int_a^t F(t_1)dt_1\right|^2dt+\frac{1}{d}\int_{a+d}^b\left|\int_{t-d}^t F(t_1)dt_1\right|^2dt$$
$$+\int_b^{b+d}\frac{1}{b+d-t}\left|\int_{t-d}^b F(t_1)dt_1\right|^2dt\leqq d\int_a^b|F(t)|^2dt. \tag{2.17}$$

Further, when $a=c=0$ *and* $b=d>0$, *we have*

$$\int_0^b\frac{1}{t}\left|\int_0^t F(t_1)dt_1\right|^2dt+\int_b^{2b}\frac{1}{2b-t}\left|\int_{t-b}^b F(t_1)dt_1\right|^2dt\leqq b\int_0^b|F(t)|^2dt. \tag{2.18}$$

Corollay 2.2 will give a natural relationship between the magnitudes of the integrals

$$\left|\int_a^t F(t_1)dt_1\right|^2 \quad\text{and}\quad \int_a^b|F(t)|^2dt$$

in a sense. Cf. Hardy-Littlewood-Polya [3, pp. 239-246].

In particular, when $(a, b)=(-a, a)$, we have

$$K_{(-a,a)}(z, \bar{u})=\frac{\sin(az-a\bar{u})}{\pi(z-\bar{u})}$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-a}^{a} |F(t)|^2 dt.$$

See, for example, de Branges [2, pp. 46-48]. Hence, we have

COROLLARY 2.3. *For any f and $g \in H_{(-a,a)}$, we have the inequality*

$$\int_{-2a}^{2a} \frac{1}{2a-|t|} \left| \underset{N\to\infty}{\mathrm{l.i.m.}} \int_{-N}^{N} f(x)g(x)e^{ixt}dx \right|^2 dt \leqq \int_{-\infty}^{\infty} |f(x)|^2 dx \int_{-\infty}^{\infty} |g(x)|^2 dx. \tag{2.19}$$

Further, for any F and $G \in L_2(-\infty, \infty)$ and for any $a>0$, we have the inequality

$$\int_{-2a}^{2a} \frac{1}{2a-|t|} |(F*G)(t)|^2 dt \leqq \int_{-a}^{a} |F(t)|^2 dt \int_{-a}^{a} |G(t)|^2 dt. \tag{2.20}$$

3. Equality problems.

We will consider the equality problems for the inequalities obtained in § 2. Note that there does, in general, not exist a general treatment for the equality problem in (2.7). See [4] for some general discussions for this equality problem. But, in the present case we obtain directly

THEOREM 3.1. *In the inequality* (2.14), *equality holds for $F \in L_2(a, b)$ and $G \in L_2(c, d)$ if and only if F and G are expressible in the form*

$$F(t) = C_1 e^{i\bar{u}t} \quad \text{on } [a, b] \quad \text{and} \quad G(t) = C_2 e^{i\bar{u}t} \quad \text{on } [c, d] \tag{3.1}$$

for some constants C_1 and C_2, and for some point $u \in \boldsymbol{C}$.

Hence, further, equality holds in (2.13) *for $f \in H_{(a,b)}$ and $g \in H_{(c,d)}$ if and only if f and g are expressible in the from*

$$f(z) = C_1 K_{(a,b)}(z, \bar{u}) \quad \text{and} \quad g(z) = C_2 K_{(c,d)}(z, \bar{u}). \tag{3.2}$$

Proof. We will consider the equality problem in the inequality (2.14). Note that the inequality (2.14) is directly derived as follows:

$$\begin{aligned} \int_{a+c}^{b+d} |(F*G)(t)|^2 W(t)dt &= \int_{a+c}^{a+d} \frac{1}{t-(a+c)} \left| \int_{a}^{t-c} F(t_1)G(t-t_1)dt_1 \right|^2 dt \\ &+ \int_{a+d}^{b+c} \frac{1}{d-c} \left| \int_{t-d}^{t-c} F(t_1)G(t-t_1)dt_1 \right|^2 dt \\ &+ \int_{b+c}^{b+d} \frac{1}{b+d-t} \left| \int_{t-d}^{b} F(t_1)G(t-t_1)dt_1 \right|^2 dt \\ &\leqq \int_{a+c}^{a+d} \left(\int_{a}^{t-c} |F(t_1)G(t-t_1)|^2 dt_1 \right) dt \end{aligned} \tag{3.3}$$

$$+\int_{a+d}^{b+c}\left(\int_{t-d}^{t-c}|F(t_1)G(t-t_1)|^2dt_1\right)dt$$

$$+\int_{b+c}^{b+d}\left(\int_{t-d}^{b}|F(t_1)G(t-t_1)|^2dt_1\right)dt$$

$$=\int_a^b|F(t)|^2dt\int_c^d|G(t)|^2dt.$$

Hence, equality holds here if and only if

$$F(t_1)G(t-t_1)=H(t)$$

or

$$F(t_1)G(t_2)=H(t_1+t_2) \quad \text{on } [a, b]\times[c, d] \tag{3.4}$$

for some function H on $[a+c, b+d]$. Hence, from this functional equation, we have the desired result (3.1).

4. The case of $L_2(-\infty, \infty)$.

Next, we will consider the case of $F, G\in L_2(-\infty, \infty)$. Then, for any $a>0$ and for the restriction of F and G to $[-a, a]$ we can consider the functions

$$f_a(z)=\frac{1}{2\pi}\int_{-a}^{a}F(t)e^{-izt}dt \quad \text{and} \quad g_a(z)=\frac{1}{2\pi}\int_{-a}^{a}G(t)e^{-izt}dt.$$

Then, we note that the norms

$$\|f_a g_a\|_{[H(-a,a)\otimes H(-a,a)]_R}$$

do not decrease for $a>0$ and so the limit

$$\lim_{a\to\infty}\int_{-2a}^{2a}\frac{1}{2a-|t|}|(F*G)(t)|^2dt=\lim_{a\to\infty}\left\{\int_{-2a}^{0}\frac{1}{t+2a}\left|\int_{-a}^{t+a}F(t_1)G(t-t_1)dt_1\right|^2dt\right.$$
$$\left.+\int_0^{2a}\frac{1}{2a-t}\left|\int_{t-a}^{a}F(t_1)G(t-t_1)dt_1\right|^2dt\right\}$$

exists.

In order to show this fact, we consider the expression, for any $0<a<b$

$$f_b(z)=\frac{1}{2\pi}\int_{-a}^{a}F(t)e^{-izt}dt+\frac{1}{2\pi}\int_{-b}^{-a}F(t)e^{-izt}dt+\frac{1}{2\pi}\int_{a}^{b}F(t)e^{-izt}dt \tag{4.1}$$
$$:=f_a(z)+f_{(-b,-a)}(z)+f_{(a,b)}(z)$$

and the corresponding reproducing kernels

$$K_{(-b,b)}(z, \bar{u})=K_{(-a,a)}(z, \bar{u})+K_{(-b,-a)}(z, \bar{u})+K_{(a,b)}(z, \bar{u}). \tag{4.2}$$

These mean that

$$H_{(-b,b)}=H_{(-a,a)}\oplus H_{(-b,-a)}\oplus H_{(a,b)} \tag{4.3}$$

and

$$\|f_b\|^2_{H(-b,b)}=\|f_a\|^2_{H(-a,a)}+\|f_{(-b,-a)}\|^2_{H(-b,-a)}+\|f_{(a,b)}\|^2_{H(a,b)}. \tag{4.4}$$

Note that in this case the sum is a direct sum. See [1, pp. 352-354]. From (4.2), we have the identity

$$\begin{aligned} K_{(-b,b)}(z,\bar{u})^2&=(K_{(-a,a)}(z,\bar{u})+K_{(-b,-a)}(z,\bar{u})+K_{(a,b)}(z,\bar{u}))^2 \\ &=K_{(-a,a)}(z,\bar{u})K_{(-a,a)}(z,\bar{u})+\cdots+K_{(a,b)}(z,\bar{u})K_{(a,b)}(z,\bar{u}) \end{aligned} \tag{4.5}$$

and the corresponding expression

$$\begin{aligned} f_b(z)g_b(z)&=(f_a(z)+f_{(-b,-a)}(z)+f_{(a,b)}(z))(g_a(z)+g_{(-b,-a)}(z)+g_{(a,b)}(z)) \\ &=f_a(z)g_a(z)+\cdots+f_{(a,b)}(z)g_{(a,b)}(z). \end{aligned} \tag{4.6}$$

From these identities we obtain conversely the corresponding identities to (4.3) and (4.4).

$$[H_{(-b,b)}\otimes H_{(-b,b)}]_R=[H_{(-a,a)}\otimes H_{(-a,a)}]_R\oplus\cdots\oplus[H_{(a,b)}\otimes H_{(a,b)}]_R. \tag{4.7}$$

and

$$\begin{aligned} &\|f_bg_b\|^2_{[H(-b,b)\otimes H(-b,b)]_R} \\ &=\|f_ag_a\|^2_{[H(-a,a)\otimes H(-a,a)]_R}+\cdots+\|f_{(a,b)}g_{(a,b)}\|^2_{[H(a,b)\otimes H(a,b)]_R}. \end{aligned} \tag{4.8}$$

Hence, in particular, we obtain the desired result

$$\|f_ag_a\|_{[H(-a,a)\otimes H(-a,a)]_R}\leqq\|f_bg_b\|_{[H(-b,b)\otimes H(-b,b)]_R}. \tag{4.9}$$

Hence, in the inequality (2.20), we obtain the fundamental

THEOREM 4.1. *For any F and $G\in L_2(-\infty,\infty)$, we have the inequality*

$$\lim_{a\to\infty}\int_{-2a}^{2a}\frac{1}{2a-|t|}|(F*G)(t)|^2dt\leqq\int_{-\infty}^{\infty}|F(t)|^2dt\int_{-\infty}^{\infty}|G(t)|^2dt. \tag{4.10}$$

Equality does not hold here for F, $G\neq0$ as functions of $L_2(\infty,\infty)$.

The equality statement in this theorem follows from the proof of Theorem 3.1. Of course, we can obtain the corresponding results for iterated convolutions by a similar method, but the results are more complicated than the case of $L_2(0,\infty)$. See [7].

REFERENCES

[1] N. ARONSZAJN, The theory of reproducing kernels, Trans. Amer. Math. Soc., 68 (1950), 337-404.

[2] L. DE BRANGES, Hilbert spaces of entire functions, Prentice-Hall, Englewood Cliffs. New Jersey 1968.

[3] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, Inequalities, Cambridge University Press. 1967.

[4] S. SAITOH, Reproducing kernels of the direct product of two Hilbert spaces, Riazi J. Kar. Math. Assoc., **4** (1982), 1-20.

[5] S. SAITOH, Integral transforms in Hilbert spaces, Proc. Japan Acad., **58** (1982), 361-364.

[6] S. SAITOH, Hilbert spaces induced by Hilbert space valued functions, Proc. Amer. Math. Soc., **89** (1983), 74-78.

[7] S. SAITOH, A fundamental inequality in the convolution of L_2 functions on the half line, Proc. Amer. Math. Soc., **91** (1984), 285-286.

[8] E.M. STEIN AND G. WEISS, Introduction to Fourier analysis on Euclidean spaces, Princeton University Press. Princeton, N.J. 1971.

DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING
GUNMA UNIVERSITY
KIRYU 376, JAPAN